Limits of real valued functions Theory

Jason Sass

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Limit of a real valued function

Proposition 1. *Let f be a real valued function.*
Let a be an accumulation point of $dom f$.
If L is a real number, then $\lim_{x\to a} f(x) = L$ iff $\lim_{x\to a} |f(x) - L| = 0$.

Proof. Suppose L is a real number.
Then

$$\begin{aligned}
\lim_{x\to a} f(x) = L &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in dom f)(0 < |x-a| < \delta \to |f(x) - L| < \epsilon) &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in dom f)(0 < |x-a| < \delta \to ||f(x) - L|| < \epsilon) &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in dom f)(0 < |x-a| < \delta \to ||f(x) - L| - 0| < \epsilon) &\Leftrightarrow \\
\lim_{x\to a} |f(x) - L| = 0. &
\end{aligned}$$

□

Proposition 2. *Let $E \subset \mathbb{R}$.*
If a is an accumulation point of E, then a is an accumulation point of $E - \{a\}$.

Proof. Suppose a is an accumulation point of E.
Let $\epsilon > 0$ be given.
Since a is an accumulation point of E, then there exists $x \in E$ such that $x \in N(a;\epsilon)$ and $x \neq a$.
Since $x \in E$ and $x \neq a$, then $x \in E - \{a\}$.
Thus, there exists $x \in E - \{a\}$ such that $x \in N(a;\epsilon)$ and $x \neq a$.
Therefore, a is an accumulation point of the set $E - \{a\}$. □

Proposition 3. *Let $E \subset \mathbb{R}$.*
A point a is an accumulation point of E iff there is a sequence (x_n) of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$.

Proof. We first prove if a is an accumulation point of E, then there is a sequence (x_n) of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$.

Suppose a is an accumulation point of E.

Then for every $\delta > 0$ there exists $x \in E$ such that $x \in N'(a;\delta)$.

Let $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$.

Then for each $n \in \mathbb{N}$, there exists $x \in E$ such that $x \in N'(a;\frac{1}{n})$.

Thus, there exists a function $f : \mathbb{N} \to E$ such that $f(n) \in N'(a;\frac{1}{n})$ for each $n \in \mathbb{N}$.

Hence, there exists a sequence (x_n) in E such that $x_n \in N'(a;\frac{1}{n})$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Then $x_n \in E$ and $x_n \in N'(a;\frac{1}{n})$.

Since $x_n \in N'(a;\frac{1}{n})$, then $x_n \in N(a;\frac{1}{n})$ and $x_n \neq a$.

Since $x_n \in E$ and $x_n \neq a$, then $x_n \in E - \{a\}$.

Since $x_n \in N(a;\frac{1}{n})$, then $|x_n - a| < \frac{1}{n}$.

Thus, $x_n \in E - \{a\}$ and $|x_n - a| < \frac{1}{n}$ for each $n \in \mathbb{N}$.

Since $x_n \in E - \{a\}$ for each $n \in \mathbb{N}$, then $\{x_n : n \in \mathbb{N}\} \subset E - \{a\}$, so there is a sequence (x_n) of points in the set $E - \{a\}$.

We prove the sequence (x_n) converges to a.

Let $\epsilon > 0$ be given.

Then $\frac{1}{\epsilon} > 0$.

Since $\frac{1}{\epsilon} \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $n > N > \frac{1}{\epsilon}$, so $n > \frac{1}{\epsilon}$.

Hence, $\epsilon > \frac{1}{n}$, so $\frac{1}{n} < \epsilon$.

Since $|x_n - a| < \frac{1}{n}$ for each $n \in \mathbb{N}$, then $|x_n - a| < \frac{1}{n} < \epsilon$.

Hence, $|x_n - a| < \epsilon$, so $\lim_{n\to\infty} x_n = a$, as desired. □

Proof. Conversely, we prove if a is a point and there is a sequence (x_n) of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$, then a is an accumulation point of E.

Suppose a is a point and there is a sequence (x_n) of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$.

We must prove a is an accumulation point of E.

Let $\delta > 0$ be given.

Since (x_n) is a sequence of points in $E - \{a\}$, then $\{x_n : n \in \mathbb{N}\} \subset E - \{a\}$, so $x_n \in E - \{a\}$ for all $n \in \mathbb{N}$.

Hence, $x_n \in E$ and $x_n \neq a$ for all $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} x_n = a$ and $\delta > 0$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|x_n - a| < \delta$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|x_n - a| < \delta$, so $x_n \in N(a;\delta)$.

Since $n \in \mathbb{N}$, then $x_n \in E$ and $x_n \neq a$.

Since $x_n \in N(a;\delta)$ and $x_n \neq a$, then $x_n \in N'(a;\delta)$.

Thus, there exists $x_n \in E$ such that $x_n \in N'(a;\delta)$, so a is an accumulation point of E. □

Theorem 4. ***sequential characterization of a function limit***

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ *be a function.*

Let a *be an accumulation point of* E.

Let $L \in \mathbb{R}$.

Then $\lim_{x\to a} f(x) = L$ *iff for every sequence* (x_n) *of points in* $E - \{a\}$ *such that* $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} f(x_n) = L$.

Proof. We prove if $\lim_{x\to a} f(x) = L$, then for every sequence (x_n) of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} f(x_n) = L$.

Suppose $\lim_{x\to a} f(x) = L$.

Since a is an accumulation point of E, then there is a sequence (x_n) of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$.

Let (x_n) be an arbitrary sequence of points in $E-\{a\}$ such that $\lim_{n\to\infty} x_n = a$.

We must prove $\lim_{n\to\infty} f(x_n) = L$.

Let $\epsilon > 0$ be given.

Since $\lim_{x\to a} f(x) = L$, then there exists $\delta > 0$ such that for all $x \in E$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Since $\lim_{n\to\infty} x_n = a$ and $\delta > 0$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|x_n - a| < \delta$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|x_n - a| < \delta$.

Since (x_n) is a sequence of points in $E - \{a\}$, then $x_n \in E - \{a\}$ for all $n \in \mathbb{N}$, so $x_n \in E$ and $x_n \neq a$ for all $n \in \mathbb{N}$.

Since $n \in \mathbb{N}$, then $x_n \in E$ and $x_n \neq a$.

Since $x_n \neq a$, then $|x_n - a| > 0$.

Thus, $0 < |x_n - a| < \delta$.

Since $x_n \in E$ and $0 < |x_n - a| < \delta$, then we conclude $|f(x_n) - L| < \epsilon$.

Therefore, $\lim_{n\to\infty} f(x_n) = L$, as desired. □

Proof. Conversely, we prove if for every sequence (x_n) of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$ implies $\lim_{n\to\infty} f(x_n) = L$, then $\lim_{x\to a} f(x) = L$.

We prove by contrapositive.

Suppose $\lim_{x\to a} f(x) \neq L$.

Then there exists $\epsilon_0 > 0$ such that for each $\delta > 0$ there corresponds $x \in E$ such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon_0$.

Thus, there exists $\epsilon_0 > 0$ such that for each $\delta > 0$ there corresponds $x \in E$ such that $x \neq a$ and $|x - a| < \delta$ and $|f(x) - L| \geq \epsilon_0$, so there exists $\epsilon_0 > 0$ such that for each $\delta > 0$ there corresponds $x \in E - \{a\}$ such that $|x - a| < \delta$ and $|f(x) - L| \geq \epsilon_0$.

Let $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$.

Then for each $n \in \mathbb{N}$, there corresponds $x \in E - \{a\}$ such that $|x - a| < \frac{1}{n}$ and $|f(x) - L| \geq \epsilon_0$.

Thus, there exists a function $g : \mathbb{N} \to \mathbb{R}$ such that $g(n) \in E - \{a\}$ and $|g(n) - a| < \frac{1}{n}$ and $|f(g(n)) - L| \geq \epsilon_0$ for each $n \in \mathbb{N}$, so there exists a sequence (x_n) in $\mathbb{R}$ such that $x_n \in E - \{a\}$ and $|x_n - a| < \frac{1}{n}$ and $|f(x_n) - L| \geq \epsilon_0$ for each $n \in \mathbb{N}$.

Since $x_n \in E - \{a\}$ for each $n \in \mathbb{N}$, then (x_n) is a sequence of points in $E - \{a\}$.

We prove $\lim_{n\to\infty} x_n = a$.

Let $\epsilon > 0$ be given.

Then $\epsilon \neq 0$, so $\frac{1}{\epsilon} \in \mathbb{R}$.

Hence, by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $n > N > \frac{1}{\epsilon}$, so $n > \frac{1}{\epsilon}$.

Hence, $\epsilon > \frac{1}{n}$, so $\frac{1}{n} < \epsilon$.

Since $n \in \mathbb{N}$ and $|x_n - a| < \frac{1}{n}$ for each $n \in \mathbb{N}$, then $|x_n - a| < \frac{1}{n}$.

Thus, $|x_n - a| < \frac{1}{n} < \epsilon$, so $|x_n - a| < \epsilon$.

Therefore, $\lim_{n\to\infty} x_n = a$, as desired.

We prove $\lim_{n\to\infty} f(x_n) \neq L$.

Let $N \in \mathbb{N}$ be given.

Let $n = N + 1$.

Then $n \in \mathbb{N}$ and $n > N$.

Since $n \in \mathbb{N}$ and $|f(x_n) - L| \geq \epsilon_0$ for each $n \in \mathbb{N}$, then $|f(x_n) - L| \geq \epsilon_0$.

Thus, there exists $\epsilon_0 > 0$ such that for each $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ for which $n > N$ and $|f(x_n) - L| \geq \epsilon_0$.

Therefore, $\lim_{n\to\infty} f(x_n) \neq L$.

Hence, we have shown there exists a sequence (x_n) of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} f(x_n) \neq L$, as desired. □

Proposition 5. ***limit of an absolute value equals absolute value of a limit***

Let f be a real valued function.

Let a be an accumulation point of $dom f$.

If the limit of f at a exists, then $\lim_{x\to a} |f(x)| = |\lim_{x\to a} f(x)|$.

Proof. Suppose the limit of f at a exists.

Then there exists a real number L such that $\lim_{x\to a} f(x) = L$.

We must prove $\lim_{x\to a} |f(x)| = |L|$.

Let $\epsilon > 0$ be given.

Since $\lim_{x\to a} f(x) = L$, then there exists $\delta > 0$ such that for all $x \in dom f$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Let $x \in dom f$ such that $0 < |x - a| < \delta$.

Then $|f(x) - L| < \epsilon$.

Hence, $||f(x)| - |L|| \leq |f(x) - L| < \epsilon$, so $||f(x)| - |L|| < \epsilon$.

Therefore, $\lim_{x\to a} |f(x)| = |L|$. ☐

Lemma 6. *Let* $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ *be a function.*

Let a *be a point.*

If the limit of f *at* a *exists and is positive, then there exists* $\delta > 0$ *such that* $f(x) > 0$ *for all* $x \in N'(a;\delta) \cap E$.

Proof. Suppose the limit of f at a exists and is positive.

Then there is a real number L such that $\lim_{x\to a} f(x) = L$ and $L > 0$.

Since $\lim_{x\to a} f(x) = L$ and $L > 0$, then there exists $\delta > 0$ such that for all $x \in E$, if $0 < |x - a| < \delta$, then $|f(x) - L| < L$.

Since $\lim_{x\to a} f(x) = L$, then a is an accumulation point of E, so there exists $x \in N'(a;\delta) \cap E$.

Let $x \in N'(a;\delta) \cap E$ be arbitrary.

Then $x \in N'(a;\delta)$ and $x \in E$.

Since $x \in N'(a;\delta)$, then $x \in N(a;\delta)$ and $x \neq a$.

Since $x \in N(a;\delta)$, then $|x - a| < \delta$.

Since $x \neq a$, then $x - a \neq 0$, so $|x - a| > 0$.

Since $0 < |x - a|$ and $|x - a| < \delta$, then $0 < |x - a| < \delta$.

Since $x \in E$ and $0 < |x - a| < \delta$, then $|f(x) - L| < L$.

Thus, $-L < f(x) - L < L$, so $-L < f(x) - L$.

Therefore, $0 < f(x)$, so $f(x) > 0$, as desired. ☐

Proposition 7. ***limit of a square root equals square root of a limit***

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ *be a function.*

If $\lim_{x\to a} f(x)$ *exists and is positive, then* $\lim_{x\to a} \sqrt{f(x)} = \sqrt{\lim_{x\to a} f(x)}$.

Proof. Suppose $\lim_{x\to a} f(x)$ exists and is positive.

Then there exists $L \in \mathbb{R}$ such that $\lim_{x\to a} f(x) = L$ and $L > 0$.

Let $g(x) = \sqrt{f(x)}$.

Then g is a function and $dom g = \{x \in E : g(x) \in \mathbb{R}\} = \{x \in E : \sqrt{f(x)} \in \mathbb{R}\} = \{x \in E : f(x) \geq 0\}$.

We must prove $\lim_{x\to a} g(x) = \sqrt{L}$.

We first prove a is an accumulation point of $dom g$.

Let $\delta > 0$ be given.

Since $\lim_{x\to a} f(x) = L$ and $L > 0$, then by the previous lemma, there exists $\delta_1 > 0$ such that $f(x) > 0$ for all $x \in N'(a;\delta_1) \cap E$.

Since $\lim_{x\to a} f(x) = L$, then a is an accumulation point of E.

Either $\delta_1 \geq \delta$ or $\delta_1 < \delta$.

We consider these cases separately.

Case 1: Suppose $\delta_1 \geq \delta$.

Since a is an accumulation point of E and $\delta > 0$, then there exists $x \in E$ such that $x \in N'(a;\delta)$.

Since $x \in N'(a;\delta)$, then $x \in N(a;\delta)$ and $x \neq a$.

Since $0 < \delta \leq \delta_1$, then $N(a;\delta) \subset N(a;\delta_1)$.
Since $x \in N(a;\delta)$ and $N(a;\delta) \subset N(a;\delta_1)$, then $x \in N(a;\delta_1)$.
Since $x \in N(a;\delta_1)$ and $x \neq a$, then $x \in N'(a;\delta_1)$.
Since $x \in N'(a;\delta_1)$ and $x \in E$, then $x \in N'(a;\delta_1) \cap E$, so $f(x) > 0$.
Since $x \in E$ and $f(x) > 0$, then $x \in domg$.
Therefore, there exists $x \in domg$ such that $x \in N'(a;\delta)$.
Case 2: Suppose $\delta_1 < \delta$.
Since a is an accumulation point of E and $\delta_1 > 0$, then there exists $x \in E$ such that $x \in N'(a;\delta_1)$.
Since $x \in N'(a;\delta_1)$ and $x \in E$, then $x \in N'(a;\delta_1) \cap E$, so $f(x) > 0$.
Since $x \in E$ and $f(x) > 0$, then $x \in domg$.
Since $x \in N'(a;\delta_1)$, then $x \in N(a;\delta_1)$ and $x \neq a$.
Since $0 < \delta_1 < \delta$, then $N(a;\delta_1) \subset N(a;\delta)$.
Since $x \in N(a;\delta_1)$ and $N(a;\delta_1) \subset N(a;\delta)$, then $x \in N(a;\delta)$.
Since $x \in N(a;\delta)$ and $x \neq a$, then $x \in N'(a;\delta)$.
Therefore, there exists $x \in domg$ such that $x \in N'(a;\delta)$.
In all cases, there exists $x \in domg$ such that $x \in N'(a;\delta)$, so a is an accumulation point of $domg$.

We next prove $\lim_{x\to a} g(x) = \sqrt{L}$.
Since a is an accumulation point of $domg$, then there exists a sequence in $domg - \{a\}$ that converges to a.
Let (x_n) be an arbitrary sequence in $domg - \{a\}$ such that $\lim_{n\to\infty} x_n = a$.

Let $n \in \mathbb{N}$ be given.
Then $x_n \in domg - \{a\}$, so $x_n \in domg$ and $x_n \neq a$.
Since $x_n \in domg$ and $domg \subset E$, then $x_n \in E$.
Since $x_n \in E$ and $x_n \neq a$, then $x_n \in E - \{a\}$.
Thus, $x_n \in E - \{a\}$ for all $n \in \mathbb{N}$, so (x_n) is a sequence of points in $E - \{a\}$.

Since a is an accumulation point of E and $\lim_{x\to a} f(x) = L$ and (x_n) is an arbitrary sequence of points in $E - \{a\}$ and $\lim_{n\to\infty} x_n = a$, then by the sequential characterization of a function limit, we have $\lim_{n\to\infty} f(x_n) = L$.
Since $\lim_{n\to\infty} f(x_n) = L$ and $L > 0$ and the limit of a square root of a convergent sequence equals the square root of the limit, then $\lim_{n\to\infty} \sqrt{f(x_n)} = \sqrt{\lim_{n\to\infty} f(x_n)}$.
Hence, $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} \sqrt{f(x_n)} = \sqrt{\lim_{n\to\infty} f(x_n)} = \sqrt{L}$.
Since a is an accumulation point of $domg$ and (x_n) is an arbitrary sequence of points in $domg - \{a\}$ such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} g(x_n) = \sqrt{L}$, then by the sequential characterization of a function limit, we have $\lim_{x\to a} g(x) = \sqrt{L}$, as desired. $\square$

Algebraic properties of function limits

Theorem 8. ***scalar multiple rule for limits***

Let f be a real valued function.

Let a be a point.

If the limit of f at a exists and is a real number, then for every $\lambda \in \mathbb{R}$, the limit of λf exists and $\lim_{x\to a} \lambda f(x) = \lambda \lim_{x\to a} f(x)$.

Proof. Suppose the limit of f at a exists and is a real number.

Then a is an accumulation point of $domf$ and there exists $L \in \mathbb{R}$ such that $\lim_{x\to a} f(x) = L$.

Let $\lambda \in \mathbb{R}$ be given.

To prove $\lim_{x\to a} \lambda f(x) = \lambda \lim_{x\to a} f(x)$, we must prove $\lim_{x\to a}(\lambda f)(x) = \lambda L$.

Since $dom(\lambda f) = domf$ and a is an accumulation point of $domf$, then a is an accumulation point of $dom(\lambda f)$.

Either $\lambda = 0$ or $\lambda \neq 0$.

We consider these cases separately.

Case 1: Suppose $\lambda = 0$.

Observe that

$$\begin{aligned}
\lim_{x\to a}(0f)(x) &= \lim_{x\to a}(0f(x)) \\
&= \lim_{x\to a} 0 \\
&= 0 \\
&= 0L.
\end{aligned}$$

Therefore, $\lim_{x\to a}(0f)(x) = 0L$, as desired.

Case 2: Suppose $\lambda \neq 0$.

Let $\epsilon > 0$.

Since $|\lambda| \geq 0$ and $\lambda \neq 0$, then $|\lambda| > 0$.

Hence, $\frac{\epsilon}{|\lambda|} > 0$.

Since $\lim_{x\to a} f(x) = L$, then there exists $\delta > 0$ such that for all $x \in domf$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \frac{\epsilon}{|\lambda|}$.

Let $x \in dom(\lambda f)$ such that $0 < |x - a| < \delta$.

Then $x \in domf$ and $0 < |x - a| < \delta$, so $|f(x) - L| < \frac{\epsilon}{|\lambda|}$.

Observe that

$$\begin{aligned}
|(\lambda f)(x) - \lambda L| &= |\lambda f(x) - \lambda L| \\
&= |\lambda(f(x) - L)| \\
&= |\lambda||f(x) - L| \\
&< |\lambda|\frac{\epsilon}{|\lambda|} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|(\lambda f)(x) - \lambda L| < \epsilon$, so $\lim_{x\to a}(\lambda f)(x) = \lambda L$, as desired. $\square$

Theorem 9. ***limit of a sum equals sum of limits***

Let f and g be real valued functions.

Let a be an accumulation point of $dom f \cap dom g$.

If the limit of f at a exists and is a real number and the limit of g at a exists and is a real number, then the limit of $f+g$ exists and

$\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

Proof. Suppose the limit of f at a exists and is a real number and the limit of g at a exists and is a real number.

Then there exist real numbers L and M such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$.

To prove $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$, we must prove $\lim_{x \to a}(f+g)(x) = L + M$.

Since $dom(f+g) = dom f \cap dom g$ and a is an accumulation point of $dom f \cap dom g$, then a is an accumulation point of $dom(f+g)$.

Let $\epsilon > 0$ be given.

Then $\frac{\epsilon}{2} > 0$.

Since $\lim_{x \to a} f(x) = L$, then there exists $\delta_1 > 0$ such that for all $x \in dom f$, if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \frac{\epsilon}{2}$.

Since $\lim_{x \to a} g(x) = M$, then there exists $\delta_2 > 0$ such that for all $x \in dom g$, if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \frac{\epsilon}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.

Let $x \in dom(f+g)$ such that $0 < |x - a| < \delta$.

Since $x \in dom(f+g)$, then $x \in dom f \cap dom g$, so $x \in dom f$ and $x \in dom g$.

Since $0 < |x - a| < \delta \leq \delta_1$, then $0 < |x - a| < \delta_1$.

Since $x \in dom f$ and $0 < |x - a| < \delta_1$, then $|f(x) - L| < \frac{\epsilon}{2}$.

Since $0 < |x - a| < \delta \leq \delta_2$, then $0 < |x - a| < \delta_2$.

Since $x \in dom g$ and $0 < |x - a| < \delta_2$, then $|g(x) - M| < \frac{\epsilon}{2}$.

Observe that

$$\begin{aligned} |(f+g)(x) - (L+M)| &= |f(x) + g(x) - L - M| \\ &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $|(f+g)(x) - (L+M)| < \epsilon$, so $\lim_{x \to a}(f+g)(x) = L + M$, as desired. $\square$

Corollary 10. ***limit of a difference equals difference of limits***

Let f and g be real valued functions.

Let a be an accumulation point of $dom f \cap dom g$.

If the limit of f at a exists and is a real number and the limit of g at a exists and is a real number, then the limit of $f-g$ exists and

$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$.

Proof. Suppose the limit of f at a exists and is a real number and the limit of g at a exists and is a real number.

Then there exist real numbers L and M such that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

To prove $\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$, we must prove $\lim_{x\to a}(f - g)(x) = L - M$.

Since a is an accumulation point of $dom f \cap dom g$ and $dom(-g) = dom g$, then a is an accumulation point of $dom f \cap dom(-g)$.

Observe that

$$\begin{aligned} L - M &= \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \\ &= \lim_{x\to a} f(x) + (-\lim_{x\to a} g(x)) \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} -g(x) \\ &= \lim_{x\to a} [f(x) + (-g(x))] \\ &= \lim_{x\to a} (f(x) - g(x)) \\ &= \lim_{x\to a} (f - g)(x). \end{aligned}$$

Therefore, $\lim_{x\to a}(f - g)(x) = L - M$, as desired. ☐

Corollary 11. ***limit of a finite sum equals finite sum of limits***

Let $n \in \mathbb{N}$ *with* $n \geq 2$.

Let a *be an accumulation point of* $\bigcap_{i=1}^{n} dom f_i$.

Let $f_1, f_2, ..., f_n$ *be real valued functions.*

Then $\lim_{x\to a}[f_1(x) + f_2(x) + ... + f_n(x)] = \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x) + ... + \lim_{x\to a} f_n(x)$.

Proof. We prove by induction.

Let $S = \{n \in \mathbb{N} : n \geq 2 \wedge \lim_{x\to a}[f_1(x)+f_2(x)+...+f_n(x)] = \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x) + ... + \lim_{x\to a} f_n(x)\}$.

Basis: Let $n = 2$.

Since a is an accumulation point of $\bigcap_{i=1}^{2} dom f_i = dom f_1 \cap dom f_2$ and $\lim_{x\to a}[f_1(x) + f_2(x)] = \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x)$, then $2 \in S$.

Induction:

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $k \geq 2$ and $\lim_{x\to a}[f_1(x)+f_2(x)+...+f_k(x)] = \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x) + ... + \lim_{x\to a} f_k(x)$.

Since a is an accumulation point of $\bigcap_{i=1}^{k} dom f_i$ and $\bigcap_{i=1}^{k} dom f_i$ is a subset of $dom f_i$ for each $i \in \{1, 2, ..., k\}$, then a is an accumulation point of $dom f_i$ for each $i \in \{1, 2, ..., k\}$.

Since $k \in \mathbb{N}$, then $k + 1 \in \mathbb{N}$ and $k + 1 \geq 3 > 2$, so $k + 1 > 2$.

Observe that

$$\begin{aligned}\lim_{x\to a}[f_1(x)+f_2(x)+...+f_{k+1}(x)] &= \lim_{x\to a}[f_1(x)+f_2(x)+...+f_k(x)+f_{k+1}(x)] \\ &= \lim_{x\to a}[f_1(x)+f_2(x)+...+f_k(x)]+\lim_{x\to a} f_{k+1}(x) \\ &= \lim_{x\to a} f_1(x)+\lim_{x\to a} f_2(x)+...+\lim_{x\to a} f_k(x)+\lim_{x\to a} f_{k+1}(x).\end{aligned}$$

Thus, $k+1 \in S$, so $k \in S$ implies $k+1 \in S$ for all natural numbers $k \geq 2$.

Since $2 \in S$ and $k \in S$ implies $k+1 \in S$ for all natural numbers $k \geq 2$, then by PMI, $\lim_{x\to a}[f_1(x)+f_2(x)+...+f_n(x)] = \lim_{x\to a} f_1(x)+\lim_{x\to a} f_2(x)+ ... + \lim_{x\to a} f_n(x)$ for all natural numbers $n \geq 2$. □

Lemma 12. ***local boundedness of a function limit***

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

Let a be a point.

If the limit of f at a exists, then there exist $\delta > 0$ and $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in N(a;\delta) \cap E$.

Proof. Suppose the limit of f at a exists.

Then a is an accumulation point of E and there exists $L \in \mathbb{R}$ such that $\lim_{x\to a} f(x) = L$.

Let $\epsilon = 1$.

Then there exists $\delta > 0$ such that $|f(x) - L| < 1$ for all $x \in N'(a;\delta) \cap E$.

Since $\delta > 0$ and a is an accumulation point of E, then there exists $c \in N'(a;\delta) \cap E$, so $c \in N'(a;\delta)$ and $c \in E$.

Since $c \in N'(a;\delta)$ and $N'(a;\delta) \subset N(a;\delta)$, then $c \in N(a;\delta)$.

Hence, $c \in N(a;\delta) \cap E$, so $N(a;\delta) \cap E \neq \emptyset$.

Either $a \in E$ or $a \notin E$.

We consider each case separately.

Case 1: Suppose $a \notin E$.

Let $M = 1 + |L|$.

Let $x \in N(a;\delta) \cap E$ be arbitrary.

Then $x \in N(a;\delta)$ and $x \in E$.

Since $x \in E$ and $a \notin E$, then $x \neq a$.

Since $x \in N(a;\delta)$ and $x \neq a$, then $x \in N'(a;\delta)$.

Since $x \in N'(a;\delta)$ and $x \in E$, then $x \in N'(a;\delta) \cap E$, so $|f(x) - L| < 1$.

Observe that

$$\begin{aligned}|f(x)| &= |f(x) - L + L| \\ &\leq |f(x) - L| + |L| \\ &< 1 + |L| \\ &= M.\end{aligned}$$

Therefore, $|f(x)| < M$.

Case 2: Suppose $a \in E$.

Let $M = \max\{1 + |L|, |f(a)|\}$.
Then $1 + |L| \leq M$ and $|f(a)| \leq M$.
Let $x \in N(a; \delta) \cap E$ be arbitrary.
Then $x \in N(a; \delta)$ and $x \in E$.
Either $x = a$ or $x \neq a$.
We consider each case separately.
Case 2a: Suppose $x = a$.
Then $|f(x)| = |f(a)| \leq M$, so $|f(x)| \leq M$.
Case 2b: Suppose $x \neq a$.
Since $x \in N(a; \delta)$ and $x \neq a$, then $x \in N'(a; \delta)$.
Since $x \in N'(a; \delta)$ and $x \in E$, then $x \in N('a; \delta) \cap E$, so $|f(x) - L| < 1$.
Observe that

$$\begin{aligned} |f(x)| &= |f(x) - L + L| \\ &\leq |f(x) - L| + |L| \\ &< 1 + |L| \\ &\leq M. \end{aligned}$$

Therefore, $|f(x)| < M$.
Hence, in all cases, $|f(x)| \leq M$. □

Theorem 13. ***limit of a product equals product of limits***
Let f and g be real valued functions.
Let a be an accumulation point of $domf \cap domg$.
If the limit of f at a exists and is a real number and the limit of g at a exists and is a real number, then the limit of fg exists and
$\lim_{x\to a}(f(x)g(x)) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$.

Proof. Suppose the limit of f at a exists and is a real number and the limit of g at a exists and is a real number.

Then there exist real numbers L and M such that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

To prove $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$, we must prove $\lim_{x\to a}(fg)(x) = LM$.

Since $dom(fg) = domf \cap domg$ and a is an accumulation point of $domf \cap domg$, then a is an accumulation point of $dom(fg)$.

Let $\epsilon > 0$ be given.

Since the limit of g exists at a, then g is locally bounded near a.

Hence, there exist $\delta_1 > 0$ and $b > 0$ such that $|g(x)| < b$ for all $x \in N(a; \delta_1) \cap domg$.

Let $e' = \frac{\epsilon}{b+|L|}$.

Since $b > 0$ and $|L| \geq 0$, then $b + |L| > 0$, so $e' > 0$.

Since $\lim_{x\to a} f(x) = L$ and $e' > 0$, then there exists $\delta_2 > 0$ such that for all $x \in domf$, if $0 < |x - a| < \delta_2$, then $|f(x) - L| < e'$.

Since $\lim_{x\to a} g(x) = M$ and $e' > 0$, then there exists $\delta_3 > 0$ such that for all $x \in domg$, if $0 < |x - a| < \delta_3$, then $|g(x) - M| < e'$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.
Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta \leq \delta_3$ and $\delta > 0$.
Let $x \in dom(fg)$ such that $0 < |x - a| < \delta$.
Since $x \in dom(fg)$, then $x \in dom f \cap dom g$, so $x \in dom f$ and $x \in dom g$.
Since $0 < |x - a| < \delta \leq \delta_1$, then $0 < |x - a| < \delta_1$, so $|x - a| < \delta_1$.
Thus, $x \in N(a; \delta_1)$.
Since $x \in N(a; \delta_1)$ and $x \in dom g$, then $x \in N(a; \delta_1) \cap dom g$, so $|g(x)| \leq b$.
Since $0 < |x - a| < \delta \leq \delta_2$, then $0 < |x - a| < \delta_2$.
Since $x \in dom f$ and $0 < |x - a| < \delta_2$, then $|f(x) - L| < e'$.
Since $0 < |x - a| < \delta \leq \delta_3$, then $0 < |x - a| < \delta_3$.
Since $x \in dom g$ and $0 < |x - a| < \delta_3$, then $|g(x) - M| < e'$.
Since $0 \leq |f(x) - L| < e'$ and $0 \leq |g(x)| < b$, then $|f(x) - L||g(x)| < e'b$.
Since $0 \leq |g(x) - M| < e'$ and $|L| \geq 0$, then $|L||g(x) - M| \leq |L|e'$.
Observe that

$$\begin{aligned}
|(fg)(x) - LM| &= |f(x)g(x) - LM| \\
&= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&\leq |f(x)g(x) - Lg(x)| + |Lg(x) - LM| \\
&= |(f(x) - L)g(x)| + |L(g(x) - M)| \\
&= |f(x) - L||g(x)| + |L||g(x) - M| \\
&< e'b + |L|e' \\
&= e'(b + |L|) \\
&= \epsilon.
\end{aligned}$$

Therefore, $|(fg)(x) - LM| < \epsilon$, so $\lim_{x \to a}(fg)(x) = LM$, as desired. □

Lemma 14. ***boundedness away from zero***
Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Let a be a point.
If there is a real number L such that $\lim_{x \to a} f(x) = L$ and $L \neq 0$, then there exists $\delta > 0$ such that $|f(x)| > \frac{|L|}{2}$ for all $x \in N'(a; \delta) \cap E$.

Proof. Suppose there is a real number L such that $\lim_{x \to a} f(x) = L$ and $L \neq 0$.
Since $L \neq 0$, then $|L| > 0$, so $\frac{|L|}{2} > 0$.
Since $\lim_{x \to a} f(x) = L$, then a is an accumulation point of E and there exists $\delta > 0$ such that $|f(x) - L| < \frac{|L|}{2}$ for all $x \in N'(a; \delta) \cap E$.
Since a is an accumulation point of E and $\delta > 0$, then $N'(a; \delta) \cap E \neq \emptyset$.
Let $x \in N'(a; \delta) \cap E$ be arbitrary.
Then $|f(x) - L| < \frac{|L|}{2}$.
Since $\frac{|L|}{2} > |f(x) - L| \geq |L| - |f(x)|$, then $\frac{|L|}{2} > |L| - |f(x)|$.
Therefore, $|f(x)| > \frac{|L|}{2}$, as desired. □

Lemma 15. *Let f be a real valued function.*

Let a be a point.

If the limit of f at a exists and is a nonzero real number, then the limit of $\frac{1}{f}$ exists and $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{\lim_{x\to a} f(x)}$.

Proof. Suppose the limit of f at a exists and is a nonzero real number.

Then a is a point of accumulation of $dom f$ and there exists a real number L such that $\lim_{x\to a} f(x) = L$ and $L \neq 0$.

To prove $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{\lim_{x\to a} f(x)}$, we must prove $\lim_{x\to a} \frac{1}{f}(x) = \frac{1}{L}$.

We first prove a is an accumulation point of $dom \frac{1}{f}$.

Let $\delta > 0$ be given.

Since $\lim_{x\to a} f(x) = L$ and $L \neq 0$, then f is bounded away from zero.

Hence, there exists $\delta_1 > 0$ such that $|f(x)| > \frac{|L|}{2}$ for all $x \in N'(a; \delta_1) \cap dom f$.

Let $\delta_2 = \min\{\delta, \delta_1\}$.

Then $\delta_2 \leq \delta$ and $\delta_2 \leq \delta_1$ and $\delta_2 > 0$.

Since a is an accumulation point of $dom f$ and $\delta_2 > 0$, then there exists $c \in dom f$ such that $c \in N'(a; \delta_2)$.

Since $c \in N'(a; \delta_2)$, then $c \in N(a; \delta_2)$ and $c \neq a$.

Since $0 < \delta_2 \leq \delta_1$, then $N(a; \delta_2) \subset N(a; \delta_1)$.

Since $c \in N(a; \delta_2)$ and $N(a; \delta_2) \subset N(a; \delta_1)$, then $c \in N(a; \delta_1)$.

Since $c \in N(a; \delta_1)$ and $c \neq a$ and $c \in dom f$, then $c \in N'(a; \delta_1) \cap dom f$, so $|f(c)| > \frac{|L|}{2}$.

Since $L \neq 0$, then $|L| > 0$, so $\frac{|L|}{2} > 0$.

Thus, $|f(c)| > \frac{|L|}{2} > 0$, so $|f(c)| > 0$.

Hence $f(c) \neq 0$.

Since $c \in dom f$ and $f(c) \neq 0$, then $c \in dom \frac{1}{f}$.

Since $0 < \delta_2 \leq \delta$, then $N(a; \delta_2) \subset N(a; \delta)$.

Since $c \in N(a; \delta_2)$ and $N(a; \delta_2) \subset N(a; \delta)$, then $c \in N(a; \delta)$.

Since $c \neq a$, then $c \in N'(a; \delta)$.

Hence, there exists $c \in dom \frac{1}{f}$ such that $c \in N'(a; \delta)$.

Therefore, a is an accumulation point of $dom \frac{1}{f}$. □

Proof. To prove $\lim_{x\to a} \frac{1}{f}(x) = \frac{1}{L}$, let $\epsilon > 0$ be given.

Since $L \neq 0$, then $\frac{1}{L} \in \mathbb{R}$ and $|L| > 0$.

Since $\epsilon > 0$ and $|L|^2 > 0$, then $\frac{\epsilon |L|^2}{2} > 0$.

Since $\lim_{x\to a} f(x) = L$, then there exists $\delta_1 > 0$ such that for all $x \in dom f$, if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \frac{\epsilon |L|^2}{2}$.

Since $\lim_{x\to a} f(x) = L$ and $L \neq 0$, then f is bounded away from zero.

Hence, there exists $\delta_2 > 0$ such that $|f(x)| > \frac{|L|}{2}$ for all $x \in N'(a; \delta_2) \cap dom f$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.

Let $x \in dom \frac{1}{f}$ such that $0 < |x - a| < \delta$.

Since $x \in dom \frac{1}{f}$, then $x \in dom f$ and $f(x) \neq 0$.

Since $0 < |x - a| < \delta \leq \delta_1$, then $0 < |x - a| < \delta_1$.

Thus, $x \in domf$ and $0 < |x-a| < \delta_1$, so $|f(x) - L| < \frac{\epsilon |L|^2}{2}$.
Hence, $0 \leq |f(x) - L| < \frac{\epsilon |L|^2}{2}$.
Since $0 < |x-a| < \delta \leq \delta_2$, then $0 < |x-a| < \delta_2$, so $x \in N'(a; \delta_2)$.
Since $x \in N'(a; \delta_2)$ and $x \in domf$, then $x \in N'(a; \delta_2) \cap domf$, so $|f(x)| > \frac{|L|}{2}$.
Since $|f(x)| > \frac{|L|}{2} > 0$, then $\frac{2}{|L|} > \frac{1}{|f(x)|} > 0$, so $0 < \frac{1}{|f(x)|} < \frac{2}{|L|}$.
Observe that

$$\begin{aligned}
|\frac{1}{f}(x) - \frac{1}{L}| &= |\frac{1}{f(x)} - \frac{1}{L}| \\
&= |\frac{L - f(x)}{f(x)L}| \\
&= |\frac{f(x) - L}{f(x)L}| \\
&= |f(x) - L| \cdot \frac{1}{|f(x)|} \cdot \frac{1}{|L|} \\
&< \frac{\epsilon |L|^2}{2} \cdot \frac{2}{|L|} \cdot \frac{1}{|L|} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|\frac{1}{f}(x) - \frac{1}{L}| < \epsilon$, so $\lim_{x \to a} \frac{1}{f}(x) = \frac{1}{L}$, as desired. □

Theorem 16. ***limit of a quotient equals quotient of limits***
Let f and g be real valued functions.
Let a be an accumulation point of $domf \cap domg$.
If the limit of f at a exists and is a real number and the limit of g at a exists and is a nonzero real number, then the limit of $\frac{f}{g}$ exists and
$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$.

Proof. Suppose the limit of f at a exists and is a real number and the limit of g at a exists and is a nonzero real number.

Then there exist real numbers L and M such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ and $M \neq 0$.

We must prove $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

We first prove a is an accumulation point of $domf \cap dom\frac{1}{g}$.
Let $\delta > 0$ be given.
Since $\lim_{x \to a} g(x) = M$ and $M \neq 0$, then g is bounded away from zero, so there exists $\delta_1 > 0$ such that $|g(x)| > \frac{|M|}{2}$ for all $x \in N'(a; \delta_1) \cap domg$.
Let $\delta_2 = \min\{\delta, \delta_1\}$.
Then $\delta_2 \leq \delta$ and $\delta_2 \leq \delta_1$ and $\delta_2 > 0$.
Since a is an accumulation point of $domf \cap domg$ and $\delta_2 > 0$, then there exists $c \in domf \cap domg$ such that $c \in N'(a; \delta_2)$.
Since $c \in domf \cap domg$, then $c \in domf$ and $c \in domg$.
Since $c \in N'(a; \delta_2)$, then $c \in N(a; \delta_2)$ and $c \neq a$.

Since $0 < \delta_2 \leq \delta_1$, then $N(a;\delta_2) \subset N(a;\delta_1)$.
Since $c \in N(a;\delta_2)$ and $N(a;\delta_2) \subset N(a;\delta_1)$, then $c \in N(a;\delta_1)$.
Since $c \in N(a;\delta_1)$ and $c \neq a$ and $c \in dom g$, then $c \in N'(a;\delta_1) \cap dom g$, so $|g(c)| > \frac{|M|}{2}$.
Since $M \neq 0$, then $|M| > 0$, so $\frac{|M|}{2} > 0$.
Thus, $|g(c)| > \frac{|M|}{2} > 0$, so $|g(c)| > 0$.
Hence $g(c) \neq 0$.
Since $c \in dom g$ and $g(c) \neq 0$, then $c \in dom \frac{1}{g}$.
Since $c \in dom f$ and $c \in dom \frac{1}{g}$, then $c \in dom f \cap dom \frac{1}{g}$.
Since $0 < \delta_2 \leq \delta$, then $N(a;\delta_2) \subset N(a;\delta)$.
Since $c \in N(a;\delta_2)$ and $N(a;\delta_2) \subset N(a;\delta)$, then $c \in N(a;\delta)$.
Since $c \neq a$, then $c \in N'(a;\delta)$.
Hence, there exists $c \in dom f \cap dom \frac{1}{g}$ such that $c \in N'(a;\delta)$.
Therefore, a is an accumulation point of $dom f \cap dom \frac{1}{g}$.
Since $\lim_{x \to a} g(x) = M$ and $M \neq 0$, then by the previous lemma, $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$.
Since a is an accumulation point of $dom f \cap dom g$, then

$$\begin{aligned}
\frac{L}{M} &= L \cdot \frac{1}{M} \\
&= (\lim_{x \to a} f(x))(\lim_{x \to a} \frac{1}{g(x)}) \\
&= \lim_{x \to a} (f(x) \cdot \frac{1}{g(x)}) \\
&= \lim_{x \to a} \frac{f(x)}{g(x)}.
\end{aligned}$$

Therefore, $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, as desired. □

Lemma 17. *For all $n \in \mathbb{N}$ and all $a \in \mathbb{R}$, $\lim_{x \to a} x^n = a^n$.*

Proof. Let $a \in \mathbb{R}$ be arbitrary.
We prove $\lim_{x \to a} x^n = a^n$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : \lim_{x \to a} x^n = a^n\}$.
Basis:
Since $\lim_{x \to a} x = a$, then $1 \in S$.
Induction:
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $\lim_{x \to a} x^k = a^k$.
Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^k$.
Let $g : \mathbb{R} \to \mathbb{R}$ be the function given by $g(x) = x$.
Then f and g are polynomial functions and $dom f = dom g = \mathbb{R}$.
Since $dom f \cap dom g = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$ and a is an accumulation point of $\mathbb{R}$, then a is an accumulation point of $dom f \cap dom g$.

Observe that

$$\begin{aligned} a^{k+1} &= a^k \cdot a \\ &= (\lim_{x\to a} x^k)(\lim_{x\to a} x) \\ &= \lim_{x\to a}(x^k \cdot x) \\ &= \lim_{x\to a}(x^{k+1}). \end{aligned}$$

Therefore, $\lim_{x\to a}(x^{k+1}) = a^{k+1}$, so $k+1 \in S$.

Hence, $k \in S$ implies $k+1 \in S$ for all $k \in \mathbb{N}$.

Since $1 \in S$ and $k \in S$ implies $k+1 \in S$ for all $k \in \mathbb{N}$, then by induction, $S = \mathbb{N}$.

Therefore, $\lim_{x\to a} x^n = a^n$ for all $n \in \mathbb{N}$. □

Theorem 18. ***limit of a polynomial function***

If p is a polynomial function and $c \in \mathbb{R}$, then $\lim_{x\to c} p(x) = p(c)$.

Proof. Suppose p is a polynomial function and $c \in \mathbb{R}$.

Since p is a polynomial function, then there exist a nonnegative integer n and real numbers $a_0, a_1, ..., a_n$ such that $p(x) = a_n x^n + ... + a_1 x + a_0$.

Since n is a nonnegative integer, then $n \in \mathbb{Z}$ and $n \geq 0$, so either $n > 0$ or $n = 0$.

We consider these cases separately.

Case 1: Suppose $n = 0$.

Then $p(x) = a_0$ for all x, so $\lim_{x\to c} p(x) = \lim_{x\to c} a_0 = a_0 = p(c)$.

Case 2: Suppose $n > 0$.

Then

$$\begin{aligned} p(c) &= a_n c^n + ... + a_2 c^2 + a_1 c + a_0 \\ &= a_n \lim_{x\to c} x^n + ... + a_2 \lim_{x\to c} x^2 + a_1 \lim_{x\to c} x + \lim_{x\to c} a_0 \\ &= \lim_{x\to c} a_n x^n + ... + \lim_{x\to c} a_2 x^2 + \lim_{x\to c} a_1 x + \lim_{x\to c} a_0 \\ &= \lim_{x\to c}(a_n x^n + ... + a_2 x^2 + a_1 x + a_0) \\ &= \lim_{x\to c} p(x). \end{aligned}$$

Therefore, $\lim_{x\to c} p(x) = p(c)$, as desired. □

Theorem 19. ***limit of a rational function***

Let r be a rational function defined by $r(x) = \frac{p(x)}{q(x)}$ such that p and q are polynomial functions.

If $c \in \mathbb{R}$ such that $q(c) \neq 0$, then $\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}$.

Proof. Let $c \in \mathbb{R}$ such that $q(c) \neq 0$.

Observe that c is an accumulation point of $\mathbb{R} = \mathbb{R} \cap \mathbb{R} = dom p \cap dom q$.

Since p is a polynomial function and $c \in \mathbb{R}$, then $\lim_{x\to c} p(x) = p(c)$.

Since q is a polynomial function and $c \in \mathbb{R}$, then $\lim_{x \to c} q(x) = q(c)$.
Since $q(c) \neq 0$, then $\lim_{x \to c} q(x) \neq 0$, so

$$\begin{aligned} r(c) &= \frac{p(c)}{q(c)} \\ &= \frac{\lim_{x \to c} p(x)}{\lim_{x \to c} q(x)} \\ &= \lim_{x \to c} \frac{p(x)}{q(x)} \\ &= \lim_{x \to c} r(x). \end{aligned}$$

Therefore, $\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}$. □

Theorem 20. ***a limit preserves a non strict inequality***
Let f and g be real valued functions such that the limit of f at a exists and the limit of g at a exists and a is an accumulation point of $dom f \cap dom g$.
If $f(x) \leq g(x)$ for all $x \in dom f \cap dom g$, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

Proof. Suppose $f(x) \leq g(x)$ for all $x \in dom f \cap dom g$.
Since the limit of f at a exists, then there exists $L \in \mathbb{R}$ such that $\lim_{x \to a} f(x) = L$.
Since the limit of g at a exists, then there exists $M \in \mathbb{R}$ such that $\lim_{x \to a} g(x) = M$.
We must prove $L \leq M$.
Suppose for the sake of contradiction $L > M$.
Then $L - M > 0$, so $\frac{L-M}{2} > 0$.
Let $\epsilon = \frac{L-M}{2}$.
Then $\epsilon > 0$.
Since $\lim_{x \to a} f(x) = L$, then there exists $\delta_1 > 0$ such that for all $x \in dom f$, if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \epsilon$.
Since $\lim_{x \to a} g(x) = M$, then there exists exists $\delta_2 > 0$ such that for all $x \in dom g$, if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \epsilon$.
Let $\delta = \min\{\delta_1, \delta_2\}$.
Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.
Since a is an accumulation point of $dom f \cap dom g$ and $\delta > 0$, then there exists $x \in dom f \cap dom g$ such that $x \in N'(a; \delta)$.
Since $x \in dom f \cap dom g$, then $x \in dom f$ and $x \in dom g$.
Since $x \in N'(a; \delta)$, then $x \in N(a; \delta)$ and $x \neq a$, so $|x - a| < \delta$ and $|x - a| > 0$.
Thus, $0 < |x - a| < \delta$.
Since $0 < |x - a| < \delta \leq \delta_1$, then $0 < |x - a| < \delta_1$.
Since $x \in dom f$ and $0 < |x - a| < \delta_1$, then $|f(x) - L| < \epsilon$.
Thus, $-\epsilon < f(x) - L < \epsilon$, so $L - \epsilon < f(x) < L + \epsilon$.
Since $0 < |x - a| < \delta \leq \delta_2$, then $0 < |x - a| < \delta_2$.
Since $x \in dom g$ and $0 < |x - a| < \delta_2$, then $|g(x) - M| < \epsilon$.

Thus, $-\epsilon < g(x) - M < \epsilon$, so $M - \epsilon < g(x) < M + \epsilon$.

Since $x \in domf \cap domg$, then $f(x) \leq g(x)$.

Since $\epsilon = \frac{L-M}{2}$, then $2\epsilon = L - M$, so $\epsilon + \epsilon = L - M$.

Thus, $M + \epsilon = L - \epsilon$.

Therefore, $g(x) < M + \epsilon = L - \epsilon < f(x) \leq g(x)$, so $g(x) < g(x)$, a contradiction.

Hence, $L \leq M$, as desired. □

Corollary 21. *Let f be a real valued function such that $\lim_{x \to a} f(x)$ exists.*

1. *If $M \in \mathbb{R}$ is an upper bound of $rngf$, then $\lim_{x \to a} f(x) \leq M$.*
2. *If $m \in \mathbb{R}$ is a lower bound of $rngf$, then $m \leq \lim_{x \to a} f(x)$.*

Proof. We prove 1.

Since $\lim_{x \to a} f(x)$ exists, then a is an accumulation point of $domf$ and there exists a real number L such that $\lim_{x \to a} f(x) = L$.

Suppose $M \in \mathbb{R}$ is an upper bound of $rngf$.

Then $f(x) \leq M$ for all $x \in domf$.

We must prove $L \leq M$.

Let $g : \mathbb{R} \to \mathbb{R}$ be the constant function defined by $g(x) = M$ for all $x \in \mathbb{R}$.

Then $\lim_{x \to a} g(x) = \lim_{x \to a} M = M$.

Since $domf \subset \mathbb{R}$, then $domf \cap domg = domf \cap \mathbb{R} = domf$.

Since a is an accumulation point of $domf$ and $domf = domf \cap domg$, then a is an accumulation point of $domf \cap domg$.

Let $x \in domf \cap domg$.

Then $x \in domf$, so $f(x) \leq M$.

Hence, $f(x) \leq M$ for all $x \in domf \cap domg$.

Therefore, by the inequality rule for function limits, $\lim_{x \to a} f(x) \leq \lim_{x \to a} M$, so $L \leq M$, as desired. □

Proof. We prove 2.

Since $\lim_{x \to a} f(x)$ exists, then a is an accumulation point of $domf$ and there exists a real number L such that $\lim_{x \to a} f(x) = L$.

Suppose $m \in \mathbb{R}$ is a lower bound of $rngf$.

Then $m \leq f(x)$ for all $x \in domf$.

We must prove $m \leq L$.

Let $g : \mathbb{R} \to \mathbb{R}$ be the constant function defined by $g(x) = m$ for all $x \in \mathbb{R}$.

Then $\lim_{x \to a} g(x) = \lim_{x \to a} m = m$.

Since $domf \subset \mathbb{R}$, then $domf \cap domg = domf \cap \mathbb{R} = domf$.

Since a is an accumulation point of $domf$ and $domf = domf \cap domg$, then a is an accumulation point of $domf \cap domg$.

Let $x \in domf \cap domg$.

Then $x \in domf$, so $m \leq f(x)$.

Hence, $m \leq f(x)$ for all $x \in domf \cap domg$.

Therefore, by the inequality rule for function limits, $\lim_{x \to a} m \leq \lim_{x \to a} f(x)$, so $m \leq L$, as desired. □

Corollary 22. ***limit of a function is between any upper and lower bound of the range of a function***

Let f be a real valued function.

If $\lim_{x\to a} f(x)$ exists and there exist real numbers m and M such that $m \leq f(x) \leq M$ for all $x \in domf$, then $m \leq \lim_{x\to a} f(x) \leq M$.

Proof. Let f be a real valued function.

Suppose $\lim_{x\to a} f(x)$ exists and there exist real numbers m and M such that $m \leq f(x) \leq M$ for all $x \in domf$.

Then $m \leq f(x)$ for all $x \in domf$ and $f(x) \leq M$ for all $x \in domf$.

Since $\lim_{x\to a} f(x)$ exists, then there exists $L \in \mathbb{R}$ such that $\lim_{x\to a} f(x) = L$.

We must prove $m \leq L \leq M$.

Since $f(x) \leq M$ for all $x \in domf$, then M is an upper bound of $rngf$.

Hence, by the previous corollary, $L \leq M$.

Since $m \leq f(x)$ for all $x \in domf$, then m is a lower bound of $rngf$.

Hence, by the previous corollary, $m \leq L$.

Therefore, $m \leq L \leq M$, as desired. □

Theorem 23. ***squeeze rule for function limits***

Let f, g, h be real valued functions with common domain E.

Let a be an accumulation point of E.

If $f(x) \leq h(x) \leq g(x)$ for all $x \in E$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, then $\lim_{x\to a} h(x) = \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

Proof. Suppose $f(x) \leq h(x) \leq g(x)$ for all $x \in E$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

Since $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, then there is a real number L such that $L = \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

We must prove $\lim_{x\to a} h(x) = L$.

Let $\epsilon > 0$ be given.

Since $\lim_{x\to a} f(x) = L$, then there exists $\delta_1 > 0$ such that for all $x \in E$, if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \epsilon$.

Since $\lim_{x\to a} g(x) = L$, then there exists $\delta_2 > 0$ such that for all $x \in E$, if $0 < |x - a| < \delta_2$, then $|g(x) - L| < \epsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.

Let $x \in E$ such that $0 < |x - a| < \delta$.

Since $x \in E$, then $f(x) \leq h(x) \leq g(x)$.

Therefore, $f(x) \leq h(x)$ and $h(x) \leq g(x)$.

Since $0 < |x - a| < \delta \leq \delta_1$, then $0 < |x - a| < \delta_1$.

Since $x \in E$ and $0 < |x - a| < \delta_1$, then $|f(x) - L| < \epsilon$.

Since $0 < |x - a| < \delta \leq \delta_2$, then $0 < |x - a| < \delta_2$.

Since $x \in E$ and $0 < |x - a| < \delta_2$, then $|g(x) - L| < \epsilon$.

Observe that

$$\begin{aligned} |f(x) - L| < \epsilon &\Leftrightarrow -\epsilon < f(x) - L < \epsilon \\ &\Rightarrow -\epsilon < f(x) - L \\ &\Leftrightarrow L - \epsilon < f(x). \end{aligned}$$

Since $L - \epsilon < f(x)$ and $f(x) \leq h(x)$, then $L - \epsilon < h(x)$, so $-\epsilon < h(x) - L$.

Observe that

$$\begin{aligned} |g(x) - L| < \epsilon & \Leftrightarrow & -\epsilon < g(x) - L < \epsilon \\ & \Rightarrow & g(x) - L < \epsilon \\ & \Leftrightarrow & g(x) < L + \epsilon. \end{aligned}$$

Since $h(x) \leq g(x)$ and $g(x) < L + \epsilon$, then $h(x) < L + \epsilon$, so $h(x) - L < \epsilon$.

Since $-\epsilon < h(x) - L$ and $h(x) - L < \epsilon$, then $-\epsilon < h(x) - L < \epsilon$, so $|h(x) - L| < \epsilon$.

Therefore, $\lim_{x\to a} h(x) = L$, as desired. □

Proof. Suppose $f(x) \leq h(x) \leq g(x)$ for all $x \in E$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

Since $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, then there is a real number L such that $L = \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

We must prove $\lim_{x\to a} h(x) = L$.

Since a is an accumulation point of the set E, then there exists a sequence (x_n) of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$.

Let (x_n) be an arbitrary sequence of points in $E-\{a\}$ such that $\lim_{n\to\infty} x_n = a$.

Since (x_n) is a sequence of points in $E - \{a\}$, then $x_n \in E - \{a\}$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Then $x_n \in E - \{a\}$, so $x_n \in E$ and $x_n \neq a$.

Since $x_n \in E$, then $f(x_n) \leq h(x_n) \leq g(x_n)$.

Hence, $f(x_n) \leq h(x_n) \leq g(x_n)$ for all $n \in \mathbb{N}$.

Since a is an accumulation point of E and $\lim_{x\to a} f(x) = L$ and (x_n) is a sequence of points in $E - \{a\}$ and $\lim_{n\to\infty} x_n = a$, then by the sequential characterization, $(f(x_n))$ is a sequence and $\lim_{n\to\infty} f(x_n) = L$.

Since a is an accumulation point of E and $\lim_{x\to a} g(x) = L$ and (x_n) is a sequence of points in $E - \{a\}$ and $\lim_{n\to\infty} x_n = a$, then by the sequential characterization, $(g(x_n))$ is a sequence and $\lim_{n\to\infty} g(x_n) = L$.

Since $(f(x_n))$ and $(g(x_n))$ and $(h(x_n))$ are sequences and $f(x_n) \leq h(x_n) \leq g(x_n)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f(x_n) = L = \lim_{n\to\infty} g(x_n)$, then by the squeeze rule for convergent sequences, $\lim_{n\to\infty} h(x_n) = L$.

Since a is an accumulation point of E and (x_n) is an arbitrary sequence of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} h(x_n) = L$, then by the sequential characterization, $\lim_{x\to a} h(x) = L$, as desired. □