Limits of real valued functions Examples

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Limit of a real valued function

Example 1. limit of a constant function
For every $k \in \mathbb{R}$, $\lim_{x\to a} k = k$. (limit of a constant k is k)

Proof. Let $k \in \mathbb{R}$ be arbitrary.
Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = k$ for all $x \in \mathbb{R}$.
Let a be an accumulation point of $\mathbb{R}$.
Let $\epsilon > 0$ be given.
Let $\delta = 1$.
Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$.
Since $|f(x) - k| = |k - k| = 0 < \epsilon$, then the conditional if $0 < |x - a| < \delta$, then $|k - k| < \epsilon$ is trivially true.
Therefore, $\lim_{x\to a} k = k$. □

Example 2. limit of the identity function
For all $a \in \mathbb{R}$, $\lim_{x\to a} x = a$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x$ for all $x \in \mathbb{R}$.
Let $a \in \mathbb{R}$ be given.
Since every real number is an accumulation point of $\mathbb{R}$, then a is an accumulation point of $\mathbb{R}$, the domain of f.
Let $\epsilon > 0$ be given.
Let $\delta = \epsilon$.
Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$.
Then $|f(x) - a| = |x - a| < \delta = \epsilon$, so $|f(x) - a| < \epsilon$.
Therefore, $\lim_{x\to a} x = a$. □

Example 3. limit of the square function
For all $a \in \mathbb{R}$, $\lim_{x\to a} x^2 = a^2$.

Proof. Let a be an arbitrary real number.
Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$ for all $x \in \mathbb{R}$.
Since every real number is an accumulation point of $\mathbb{R}$, then a is an accumulation point of $\mathbb{R}$, the domain of f.
We must prove $\lim_{x\to a} x^2 = a^2$.

Let $\epsilon > 0$ be given.
Let $\delta = \min\{1, \frac{\epsilon}{1+2|a|}\}$.
Then $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{1+2|a|}$.
Since $|a| \geq 0$, then $2|a| \geq 0$, so $1 + 2|a| \geq 1 > 0$.
Hence, $1 + 2|a| > 0$, so $\frac{\epsilon}{1+2|a|} > 0$.
Thus, $\delta > 0$.
Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$.
Since $0 < |x - a| < \delta$, then $|x - a| < \delta$.
Since

$$\begin{aligned} |x+a| &= |x-a+2a| \\ &\leq |x-a| + |2a| \\ &= |x-a| + 2|a| \\ &< \delta + 2|a| \\ &\leq 1 + 2|a|, \end{aligned}$$

then $0 \leq |x + a| < 1 + 2|a|$.

Hence,

$$\begin{aligned} |x^2 - a^2| &= |(x-a)(x+a)| \\ &= |x-a||x+a| \\ &< \delta(1+2|a|) \\ &\leq \frac{\epsilon}{1+2|a|} \cdot (1+2|a|) \\ &= \epsilon. \end{aligned}$$

Therefore, $|x^2 - a^2| < \epsilon$, as desired. □

Example 4. limit of the cube function

For all $a \in \mathbb{R}$, $\lim_{x \to a} x^3 = a^3$.

Proof. Let a be an arbitrary real number.
Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^3$ for all $x \in \mathbb{R}$.
Since every real number is an accumulation point of $\mathbb{R}$, then a is an accumulation point of $\mathbb{R}$, the domain of f.
We must prove $\lim_{x \to a} x^3 = a^3$.
Let $\epsilon > 0$ be given.
Let $\delta = \min\{1, \frac{\epsilon}{1+3|a|+3|a|^2}\}$.
Then $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{1+3|a|+3|a|^2}$.
Since $|a| \geq 0$, then $\frac{\epsilon}{1+3|a|+3|a|^2} > 0$, so $\delta > 0$.
Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$.
Since

$$\begin{aligned} |x| &= |(x-a)+a| \\ &\leq |x-a| + |a| \\ &< \delta + |a| \\ &\leq 1 + |a|, \end{aligned}$$

then $|x| < 1 + |a|$.

Since

$$\begin{aligned} |x^2 + ax + a^2| &\leq |x^2 + ax| + |a|^2 \\ &\leq |x|^2 + |a||x| + |a|^2 \\ &< (1 + |a|)^2 + |a|(1 + |a|) + |a|^2 \\ &= 1 + 3|a| + 3|a|^2, \end{aligned}$$

then $0 \leq |x^2 + ax + a^2| < 1 + 3|a| + 3|a|^2$.

Hence,

$$\begin{aligned} |x^3 - a^3| &= |(x - a)(x^2 + ax + a^2)| \\ &= |x - a||x^2 + ax + a^2| \\ &< \delta \cdot (1 + 3|a| + 3|a|^2) \\ &\leq \frac{\epsilon}{1 + 3|a| + 3|a|^2} \cdot (1 + 3|a| + 3|a|^2) \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x) - a^3| < \epsilon$, as desired. $\square$

Example 5. limit of the reciprocal function

For all positive real a, $\lim_{x \to a} \frac{1}{x} = \frac{1}{a}$.

Proof. Let a be a positive real number.

Then $a \in \mathbb{R}$ and $a > 0$.

Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = \frac{1}{x}$ for all $x \in \mathbb{R}^*$.

We first show that a is an accumulation point of $\mathbb{R}^*$, the domain of f.

Since $a > 0$, then $a \in (0, \infty)$.

Since $(0, \infty)$ is an interval, then a is an accumulation point of $(0, \infty)$.

Since $(0, \infty) \subset (-\infty, 0) \cup (0, \infty) = \mathbb{R} - \{0\} = \mathbb{R}^*$, then a is an accumulation point of $\mathbb{R}^*$.

To prove $\lim_{x \to a} \frac{1}{x} = \frac{1}{a}$, let $\epsilon > 0$ be given.

Let $\delta = \min\{\frac{a}{2}, \frac{a^2\epsilon}{2}\}$.

Then $\delta \leq \frac{a}{2}$ and $\delta \leq \frac{a^2\epsilon}{2}$ and $\delta > 0$.

Let $x \in \mathbb{R}^*$ such that $0 < |x - a| < \delta$.

Then $0 < |x - a|$ and $|x - a| < \delta$.

Since $x \in \mathbb{R}^*$, then $x \in \mathbb{R}$ and $x \neq 0$, so $|x| > 0$.

Since $|x - a| < \delta \leq \frac{a}{2}$, then $|x - a| < \frac{a}{2}$.

Since $\frac{a}{2} > |x - a| \geq |a| - |x| = a - |x|$, then $\frac{a}{2} > a - |x|$, so $|x| > \frac{a}{2} > 0$.

Thus, $0 < \frac{a}{2} < |x|$, so $0 < \frac{1}{|x|} < \frac{2}{a}$.

Observe that

$$\begin{aligned}
|\frac{1}{x} - \frac{1}{a}| &= |\frac{1}{a} - \frac{1}{x}| \\
&= |\frac{x-a}{ax}| \\
&= \frac{1}{a} \cdot \frac{1}{|x|} \cdot |x-a| \\
&< \frac{1}{a} \cdot \frac{2}{a} \cdot \delta \\
&= \frac{2}{a^2} \cdot \delta \\
&\leq \frac{2}{a^2} \cdot \frac{a^2\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|f(x) - \frac{1}{a}| < \epsilon$, as desired. ☐

Example 6. limit of the absolute value function

For all $a \in \mathbb{R}$, $\lim_{x \to a} |x| = |a|$.

Proof. Let a be an arbitrary real number.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x|$ for all $x \in \mathbb{R}$.

Since every real number is an accumulation point of $\mathbb{R}$, then a is an accumulation point of $\mathbb{R}$, the domain of f.

To prove $\lim_{x \to a} |x| = |a|$, let $\epsilon > 0$ be given.

Let $\delta = \epsilon$.

Then $\delta > 0$.

Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$.

Then $||x| - |a|| \leq |x - a| < \delta = \epsilon$, so $||x| - |a|| < \epsilon$, as desired. ☐

Example 7. limit of the square root function

For all $a \geq 0$, $\lim_{x \to a} \sqrt{x} = \sqrt{a}$.

Proof. Let $f : [0, \infty) \to \mathbb{R}$ be a function defined by $f(x) = \sqrt{x}$ for all $x \geq 0$.

Let $a \geq 0$ be given.

Either $a > 0$ or $a = 0$.

We consider each case separately.

Case 1: Suppose $a = 0$.

Observe that 0 is an accumulation point of the set $[0, \infty)$, the domain of f.

We must prove $\lim_{x \to 0} \sqrt{x} = 0$.

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon^2$.

Then $\delta > 0$.

Let $x \geq 0$ such that $0 < |x| < \delta$.

Then $0 < x < \delta$, so $0 < \sqrt{x} < \sqrt{\delta}$.

Thus, $|\sqrt{x}| = \sqrt{x} < \sqrt{\delta} = \sqrt{\epsilon^2} = |\epsilon| = \epsilon$, so $|\sqrt{x}| < \epsilon$, as desired.

Case 2: Suppose $a > 0$.

Observe that a is an accumulation point of the set $[0, \infty)$, the domain of f.
We must prove $\lim_{x\to a} \sqrt{x} = \sqrt{a}$.
Let $\epsilon > 0$ be given.
Let $\delta = \epsilon\sqrt{a}$.
Since $a > 0$, then $\sqrt{a} > 0$.
Since $\epsilon > 0$ and $\sqrt{a} > 0$, then $\delta > 0$.
Let $x \in [0, \infty)$ such that $0 < |x - a| < \delta$.
Since $x \in [0, \infty)$, then $x \geq 0$, so $\sqrt{x} \geq 0$.
Since $\sqrt{x} \geq 0$ and $\sqrt{a} > 0$, then $\sqrt{x} + \sqrt{a} \geq \sqrt{a} > 0$, so $\frac{1}{\sqrt{a}} \geq \frac{1}{\sqrt{x}+\sqrt{a}} > 0$.
Thus, $0 < \frac{1}{\sqrt{x}+\sqrt{a}} \leq \frac{1}{\sqrt{a}}$.
Hence,

$$\begin{aligned}
|\sqrt{x} - \sqrt{a}| &= |(\sqrt{x} - \sqrt{a}) \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}| \\
&= |\frac{x - a}{\sqrt{x} + \sqrt{a}}| \\
&= |x - a| \cdot \frac{1}{\sqrt{x} + \sqrt{a}} \\
&< \delta \cdot \frac{1}{\sqrt{a}} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|\sqrt{x} - \sqrt{a}| < \epsilon$, as desired. □

Example 8. limit of f at a need not equal $f(a)$, function with a removable discontinuity

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ 5 & \text{if } x = 1 \end{cases}$$

Then $\lim_{x\to 1} f(x) = 2$ and $\lim_{x\to 1} f(x) \neq f(1)$.

Proof. Since every real number is an accumulation point of $\mathbb{R}$, then 1 is an accumulation point of $\mathbb{R}$, the domain of f.
We prove $\lim_{x\to 1} f(x) = 2$.
Let $\epsilon > 0$ be given.
Let $\delta = \epsilon$.
Then $\delta > 0$.
Let $x \in \mathbb{R}$ such that $0 < |x - 1| < \delta$.
Then $0 < |x - 1|$ and $|x - 1| < \delta$.
Since $|x - 1| > 0$, then $x - 1 \neq 0$, so $x \neq 1$.
Thus,

$$\begin{aligned} |f(x) - 2| &= |(x + 1) - 2| \\ &= |x - 1| \\ &< \delta \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x) - 2| < \epsilon$, as desired.

Since $\lim_{x\to 1} f(x) = 2 \neq 5 = f(1)$, then $\lim_{x\to 1} f(x) \neq f(1)$. □

Example 9. function with a jump discontinuity

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then $\lim_{x\to 0} f(x)$ does not exist in $\mathbb{R}$.

Proof. Observe that 0 is an accumulation point of $\mathbb{R}$, the domain of f.

We prove there is no real number L such that $\lim_{x\to 0} f(x) = L$ by contradiction.

Suppose there is a real number L such that $\lim_{x\to 0} f(x) = L$.

Then for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x| < \delta$, then $|f(x) - L| < \epsilon$.

Let $\epsilon = \frac{1}{3}$.

Then there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x| < \delta$, then $|f(x) - L| < \frac{1}{3}$.

Let $x_1 = \frac{\delta}{2}$.

Since $\delta > 0$, then $0 < \frac{\delta}{2} = |\frac{\delta}{2}| < \delta$, so $|f(\frac{\delta}{2}) - L| < \frac{1}{3}$.

Hence, $|1 - L| < \frac{1}{3}$.

Let $x_2 = \frac{-\delta}{2}$.

Since $0 < \frac{\delta}{2} = |\frac{\delta}{2}| = |\frac{-\delta}{2}| < \delta$, then $0 < |\frac{-\delta}{2}| < \delta$, so $|f(\frac{-\delta}{2}) - L| < \frac{1}{3}$.

Since $\frac{\delta}{2} > 0$, then $\frac{-\delta}{2} < 0$, so $|0 - L| < \frac{1}{3}$.

Thus, $|L| < \frac{1}{3}$.

Observe that

$$\begin{aligned} 1 &= |(1 - L) + L| \\ &\leq |1 - L| + |L| \\ &< \frac{1}{3} + \frac{1}{3} \\ &= \frac{2}{3}. \end{aligned}$$

Therefore, $1 < \frac{2}{3}$, a contradiction.

Thus, there is no real number L such that $\lim_{x\to 0} f(x) = L$. □

Proof. Observe that 0 is an accumulation point of $\mathbb{R}$, the domain of f.

To prove there is no real number L such that $\lim_{x\to 0} f(x) = L$, we prove for every real number L, $\lim_{x\to 0} f(x) \neq L$.

Let L be an arbitrary real number.

To prove $\lim_{x\to 0} f(x) \neq L$ using the sequential characterization of a function limit, we must prove there exists a sequence (x_n) of points in $\mathbb{R} - \{0\} = \mathbb{R}^*$ such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} f(x_n) \neq L$.

Let (x_n) be a sequence defined by $x_n = \frac{(-1)^{n+1}}{n}$ for all $n \in \mathbb{N}$.

We first prove $x_n \neq 0$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Then $|x_n| = |\frac{(-1)^{n+1}}{n}| = \frac{1}{n} > 0$, so $|x_n| > 0$.

Thus, $x_n \neq 0$, so $x_n \neq 0$ for all $n \in \mathbb{N}$.

Therefore, (x_n) is a sequence of real numbers in $\mathbb{R} - \{0\} = \mathbb{R}^*$.

We next prove $\lim_{n\to\infty} x_n = 0$.

Since $-|x_n| \leq x_n \leq |x_n|$ for all $n \in \mathbb{N}$, then $-\frac{1}{n} \leq x_n \leq \frac{1}{n}$ for all $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$, then by the squeeze rule for sequences, $\lim_{n\to\infty} x_n = 0$.

We next prove $\lim_{n\to\infty} f(x_n) \neq L$.

Let $n \in \mathbb{N}$ be given.

Then either n is even or n is odd.

If n is even, then $f(x_n) = f(\frac{(-1)^{n+1}}{n}) = f(\frac{-1}{n}) = 0$.

If n is odd, then $f(x_n) = f(\frac{(-1)^{n+1}}{n}) = f(\frac{1}{n}) = 1$.

Thus, the sequence $(f(x_n))$ consists of the terms $1, 0, 1, 0, 1, 0, ...$

The even subsequence is the constant sequence with terms $0, 0, 0, 0, ...$, so the even subsequence converges to 0.

The odd subsequence is the constant sequence with terms $1, 1, 1, 1, ...$, so the odd subsequence converges to 1.

Therefore, the sequence $(f(x_n))$ is divergent, so $(f(x_n))$ is not convergent.

Hence, there is no real number L such that $\lim_{n\to\infty} f(x_n) = L$.

Therefore, for every real number L, $\lim_{n\to\infty} f(x_n) \neq L$.

Since L is an arbitrary real number, then we conclude $\lim_{n\to\infty} f(x_n) \neq L$. □

Example 10. unbounded function, infinite discontinuity

Show that $\lim_{x\to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$.

Proof. Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = \frac{1}{x}$ for all $x \in \mathbb{R}^*$.

Observe that 0 is an accumulation point of $\mathbb{R}^* = \mathbb{R} - \{0\}$, the domain of f.

We prove there is no real number L such that $\lim_{x\to 0} \frac{1}{x} = L$.

Observe that

$$\neg(\exists L \in \mathbb{R})(\lim_{x\to 0} f(x) = L) \quad \Leftrightarrow$$
$$\neg(\exists L \in \mathbb{R})(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R}^*)(0 < |x| < \delta \to |f(x) - L| < \epsilon) \quad \Leftrightarrow$$
$$(\forall L \in \mathbb{R})(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R}^*)(0 < |x| < \delta \wedge |f(x) - L| \geq \epsilon).$$

Thus, we prove $(\forall L \in \mathbb{R})(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \neq 0)(0 < |x| < \delta \wedge |\frac{1}{x} - L| \geq \epsilon)$.
Let L be an arbitrary real number.
Let $\epsilon = |L| + 1$.
Since $|L| \geq 0$, then $\epsilon = |L| + 1 \geq 1 > 0$, so $\epsilon > 0$.
Let $\delta > 0$ be given.
We must prove there exists $x \neq 0$ such that $0 < |x| < \delta$ and $|\frac{1}{x} - L| \geq \epsilon$.

Let $x = \min\{\frac{\delta}{2}, \frac{1}{|L|+\epsilon}\}$.
Then $x \leq \frac{\delta}{2}$ and $x \leq \frac{1}{|L|+\epsilon}$ and either $x = \frac{\delta}{2}$ or $x = \frac{1}{|L|+\epsilon}$.
Since $|L| \geq 0$ and $\epsilon > 0$, then $|L| + \epsilon > 0$, so $\frac{1}{|L|+\epsilon} > 0$.
Since $\delta > 0$, then $\frac{\delta}{2} > 0$.
Since either $x = \frac{\delta}{2}$ or $x = \frac{1}{|L|+\epsilon}$ and $\frac{\delta}{2} > 0$ and $\frac{1}{|L|+\epsilon} > 0$, then $x > 0$, so $x \neq 0$.
Since $\delta > 0$ and $0 < x = |x| \leq \frac{\delta}{2} < \delta$, then $0 < |x| < \delta$.
Since $x \leq \frac{1}{|L|+\epsilon}$ and $|L| + \epsilon > 0$, then $x(|L| + \epsilon) \leq 1$.
Since $x > 0$, then $|L| + \epsilon \leq \frac{1}{x} = \frac{1}{|x|} = |\frac{1}{x}|$.
Thus, $\epsilon \leq |\frac{1}{x}| - |L|$.
Therefore, $|\frac{1}{x} - L| \geq |\frac{1}{x}| - |L| \geq \epsilon$, so $|\frac{1}{x} - L| \geq \epsilon$.
Thus, there exists $x \neq 0$ such that $0 < |x| < \delta$ and $|\frac{1}{x} - L| \geq \epsilon$, as desired. ☐

Example 11. oscillating function
Show that $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist in $\mathbb{R}$.

Proof. Observe that 0 is an accumulation point of the set $\mathbb{R} - \{0\}$, the domain of f.
Suppose $\lim_{x\to 0} f(x)$ does exist in $\mathbb{R}$.
Then there is a real number L such that $\lim_{x\to 0} \sin(\frac{1}{x}) = L$.
Thus, for every $\epsilon > 0$, there is $\delta > 0$ such that for every $x \neq 0$, if $0 < |x| < \delta$, then $|\sin(\frac{1}{x}) - L| < \epsilon$.
Let $\epsilon = \frac{1}{2}$.
Then there is $\delta > 0$ such that for every $x \neq 0$, if $0 < |x| < \delta$, then $|\sin(\frac{1}{x}) - L| < \frac{1}{2}$.
Let $M = \max\{\frac{1}{2\pi\delta} - \frac{1}{4}, \frac{1}{2\pi\delta} - \frac{3}{4}\}$.
Since $\delta > 0$, then $M \in \mathbb{R}$, so by the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > M$.
Let $x_1 = \frac{2}{\pi(4n+1)}$.
Since $n \in \mathbb{N}$, then $n > 0$, so $4n + 1 > 0$.
Thus, $x_1 > 0$, so $x_1 \neq 0$.

Since $n > M$ and $M \geq \frac{1}{2\pi\delta} - \frac{1}{4}$, then $n > \frac{1}{2\pi\delta} - \frac{1}{4}$.
Thus, $4n > \frac{2}{\pi\delta} - 1$, so $4n + 1 > \frac{2}{\pi\delta}$.
Hence, $\delta(4n+1) > \frac{2}{\pi}$, so $\delta > \frac{2}{\pi(4n+1)} > 0$.
Therefore, $0 < \frac{2}{\pi(4n+1)} = |\frac{2}{\pi(4n+1)}| = |x_1| < \delta$.
Since $x_1 \neq 0$ and $0 < |x_1| < \delta$, then $|\sin(\frac{1}{x_1}) - L| < \frac{1}{2}$.

Let $x_2 = \frac{2}{\pi(4n+3)}$.
Since $n \in \mathbb{N}$, then $n > 0$, so $4n + 3 > 0$.
Thus, $x_2 > 0$, so $x_2 \neq 0$.
Since $n > M$ and $M \geq \frac{1}{2\pi\delta} - \frac{3}{4}$, then $n > \frac{1}{2\pi\delta} - \frac{3}{4}$.
Thus, $4n > \frac{2}{\pi\delta} - 3$, so $4n + 3 > \frac{2}{\pi\delta}$.
Hence, $\delta(4n+3) > \frac{2}{\pi}$, so $\delta > \frac{2}{\pi(4n+3)} > 0$.
Therefore, $0 < \frac{2}{\pi(4n+3)} = |\frac{2}{\pi(4n+3)}| = |x_2| < \delta$.
Since $x_2 \neq 0$ and $0 < |x_2| < \delta$, then $|\sin(\frac{1}{x_2}) - L| < \frac{1}{2}$.

Observe that

$$\begin{aligned}
2 &= |1 - (-1)| \\
&= |\sin(\frac{\pi}{2} + 2\pi n) - \sin(\frac{3\pi}{2} + 2\pi n)| \\
&= |\sin(\frac{\pi + 4\pi n}{2}) - \sin(\frac{3\pi + 4\pi n}{2})| \\
&= |\sin(\frac{\pi}{2}(1 + 4n)) - \sin(\frac{\pi}{2}(3 + 4n))| \\
&= |\sin(\frac{\pi(4n+1)}{2}) - \sin(\frac{\pi(4n+3)}{2})| \\
&= |\sin(\frac{1}{\frac{2}{\pi(4n+1)}}) - \sin(\frac{1}{\frac{2}{\pi(4n+3)}})| \\
&= |\sin(\frac{1}{x_1}) - \sin(\frac{1}{x_2})| \\
&= |\sin(\frac{1}{x_1}) - L + L - \sin(\frac{1}{x_2})| \\
&\leq |\sin(\frac{1}{x_1}) - L| + |L - \sin(\frac{1}{x_2})| \\
&= |\sin(\frac{1}{x_1}) - L| + |\sin(\frac{1}{x_2}) - L| \\
&< \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}$$

Thus, we have $2 < 1$, a contradiction.
Therefore, $\lim_{x \to 0} f(x)$ does not exist in $\mathbb{R}$. $\square$

Proof. Let $f : \mathbb{R}^* \to \mathbb{R}$ be a function defined by $f(x) = \sin(\frac{1}{x})$ for all $x \neq 0$.
Observe that 0 is an accumulation point of $\mathbb{R}^* = \mathbb{R} - \{0\}$, the domain of f.

To prove $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist in $\mathbb{R}$, we must prove $\lim_{x\to 0} f(x)$ does not exist in $\mathbb{R}$, so we must prove there is no real number L such that $\lim_{x\to 0} f(x) = L$.

Thus, we prove for every real number L, $\lim_{x\to 0} f(x) \neq L$.

Let L be an arbitrary real number.

To prove $\lim_{x\to 0} f(x) \neq L$, we use the sequential criterion for a function limit.

Thus, we must prove there exists a sequence (x_n) of points in $domf - \{0\}$ such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} f(x_n) \neq L$.

Let (x_n) be a sequence in $\mathbb{R}$ defined by $x_n = \frac{2}{(2n-1)\pi}$ for all $n \in \mathbb{N}$.

We first prove $x_n \neq 0$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1$ and $x_n = \frac{2}{(2n-1)\pi}$.

Since $n \geq 1$, then $2n \geq 2$, so $2n - 1 \geq 1 > 0$.

Hence, $1 \geq \frac{1}{2n-1} > 0$, so $\frac{1}{2n-1} > 0$.

Thus, $\frac{2}{(2n-1)\pi} > 0$, so $\frac{2}{(2n-1)\pi} \neq 0$.

Therefore, $x_n \neq 0$, so $x_n \neq 0$ for all $n \in \mathbb{N}$.

Thus, $x_n \in \mathbb{R} - \{0\} = \mathbb{R}^* = \mathbb{R}^* - \{0\}$ for each $n \in \mathbb{N}$, so (x_n) is a sequence of points in $domf - \{0\}$.

We next prove $\lim_{n\to\infty} x_n = 0$.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1$ and $x_n = \frac{2}{(2n-1)\pi}$ and $\frac{1}{2n-1} > 0$.

Since $n \geq 1$, then $2n \geq n + 1$, so $2n - 1 \geq n \geq 1 > 0$.

Thus, $2n - 1 \geq n > 0$, so $\frac{1}{n} \geq \frac{1}{2n-1} > 0$.

Hence, $\frac{2}{n\pi} \geq \frac{2}{(2n-1)\pi} > 0$, so $0 < \frac{2}{(2n-1)\pi} \leq \frac{2}{n\pi}$.

Therefore, $0 < \frac{2}{(2n-1)\pi} \leq \frac{2}{n\pi}$ for all $n \in \mathbb{N}$, so $0 < x_n \leq \frac{2}{n\pi}$ for all $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} \frac{2}{n\pi} = \frac{2}{\pi}(\lim_{n\to\infty} \frac{1}{n}) = \frac{2}{\pi} \cdot 0 = 0 = \lim_{n\to\infty} 0$ and $0 < x_n \leq \frac{2}{n\pi}$ for all $n \in \mathbb{N}$, then by the squeeze rule for convergent sequences, $\lim_{n\to\infty} x_n = 0$.

Lastly, we prove $\lim_{n\to\infty} f(x_n) \neq L$.

Let (a_n) be the sequence defined by $a_n = f(x_n)$ for all $n \in \mathbb{N}$.

We must prove $\lim_{n\to\infty} a_n \neq L$.

Let $n \in \mathbb{N}$.

Then $a_n = f(x_n) = \sin(\frac{1}{x_n}) = \sin(\frac{(2n-1)\pi}{2})$.

Thus, $a_n = 1$ if n is odd and $a_n = -1$ if n is even.

Let (b_n) be the odd subsequence given by $b_n = a_{2n-1}$.

Then $b_n = 1$ and (b_n) converges to 1.

Let (c_n) be the even subsequence given by $c_n = a_{2n}$.

Then $c_n = -1$ and (c_n) converges to -1.

Since (b_n) and (c_n) are convergent subsequences of (a_n) and $\lim_{n\to\infty} b_n = 1 \neq -1 = \lim_{n\to\infty} c_n$, then we conclude that (a_n) is divergent.

Therefore, (a_n) is not convergent, so (a_n) cannot converge to L. $\square$