Limits of real valued functions Exercises

Jason Sass

June 29, 2021

Limit of a real valued function

Exercise 1. Given $\lim_{x\to 2}(x^2-3)=1$ and $\epsilon = 0.01$, find $\delta > 0$ that satisfies the definition of limit of a function.

Solution. Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by $f(x)=x^2-3$.

Observe that 2 is an accumulation point of the set $\mathbb{R}$, the domain of f.

Since $\lim_{x\to 2} f(x) = 1$ and $\epsilon = 0.01$, we must find $\delta > 0$ so that for all $x\in\mathbb{R}$, if $0<|x-2|<\delta$, then $|f(x)-1|<0.01$.

Let $\delta = \frac{0.01}{5}$.

Then $0<\delta = 0.002 < 1$, so $0<\delta$ and $\delta < 1$.

Let $x\in\mathbb{R}$ such that $0<|x-2|<\delta$.

Since $0<|x-2|<\delta$, then $0<|x-2|$ and $|x-2|<\delta$.

Since $|x-2|<\delta$ and $\delta<1$, then $|x-2|<1$, so $-1<x-2<1$.

Hence, $3<x+2<5$.

Since $0<3<x+2<5$, then $0<x+2=|x+2|<5$, so $0<|x+2|<5$.

Thus,

$$\begin{aligned} |f(x)-1| &= |(x^2-3)-1| \\ &= |x^2-4| \\ &= |(x-2)(x+2)| \\ &= |x-2||x+2| \\ &< 5\delta \\ &= 0.01. \end{aligned}$$

Therefore, $|f(x)-1|<0.01$. $\square$

Exercise 2. Show that $\lim_{x\to -2}\frac{x^2-4}{x+2} = -4$.

Solution. Let $f:(-2,0)\to\mathbb{R}$ be a function defined by $f(x)=\frac{x^2-4}{x+2}$ for all $x\in(-2,0)$.

Observe that -2 is an accumulation point of the interval $(-2,0)$, the domain of f.

We prove $\lim_{x\to -2} f(x) = -4$.
Let $\epsilon > 0$ be given.
Let $\delta = \epsilon$.
Then $\delta > 0$.
Let $x \in (-2, 0)$ such that $0 < |x+2| < \delta$.
Since $0 < |x+2| < \delta$, then $0 < |x+2|$ and $|x+2| < \delta$.
Since $|x+2| > 0$, then $x + 2 \neq 0$.
Thus,

$$\begin{aligned} |f(x)+4| &= |\frac{x^2-4}{x+2} + 4| \\ &= |\frac{(x-2)(x+2)}{x+2} + 4| \\ &= |(x-2)+4| \\ &= |x+2| \\ &< \delta \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x)+4| < \epsilon$, as desired. □

Exercise 3. Show that $\lim_{x\to 2}(3x-2) = 4$.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = 3x - 2$ for all $x \in \mathbb{R}$.

Since every real number is an accumulation point of $\mathbb{R}$, then 2 is an accumulation point of $\mathbb{R}$.

We prove $\lim_{x\to 2} f(x) = 4$.
Let $\epsilon > 0$ be given.
Let $\delta = \frac{\epsilon}{3}$.
Then $\delta > 0$.
Let $x \in \mathbb{R}$ such that $0 < |x-2| < \delta$.
Then $|x-2| < \delta$.
Thus,

$$\begin{aligned} |f(x)-4| &= |(3x-2)-4| \\ &= |3x-6| \\ &= 3|x-2| \\ &< 3\delta \\ &= 3 \cdot \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x)-4| < \epsilon$, as desired. □

Exercise 4. Show that $\lim_{x\to 2}(x^2 + 4x) = 12$.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^2 + 4x$ for all $x \in \mathbb{R}$.

Since every real number is an accumulation point of $\mathbb{R}$, then 2 is an accumulation point of $\mathbb{R}$.

We prove $\lim_{x\to 2} f(x) = 12$.

Let $\epsilon > 0$ be given.

Let $\delta = \min\{1, \frac{\epsilon}{9}\}$.

Then $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{9}$ and $\delta > 0$.

Let $x \in \mathbb{R}$ such that $0 < |x - 2| < \delta$.

Observe that

$$\begin{aligned} |x+6| &= |(x-2)+8| \\ &\leq |x-2|+8 \\ &< \delta + 8 \\ &\leq 9. \end{aligned}$$

Hence, $0 \leq |x + 6| < 9$.

Thus,

$$\begin{aligned} |f(x) - 12| &= |(x^2+4x) - 12| \\ &= |(x-2)(x+6)| \\ &= |x-2||x+6| \\ &< \delta(9) \\ &\leq \frac{\epsilon}{9} \cdot 9 \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x) - 12| < \epsilon$, as desired. ☐

Exercise 5. Show that $\lim_{x\to 3} \frac{2x+3}{4x-9} = 3$.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \frac{2x+3}{4x-9}$ for all $x \neq \frac{9}{4}$.

Observe that 3 is an accumulation point of $\{x \in \mathbb{R} : x \neq \frac{9}{4}\}$, the domain of f.

We prove $\lim_{x\to 3} f(x) = 3$.

Let $\epsilon > 0$ be given.

Let $\delta = \min\{\frac{1}{2}, \frac{\epsilon}{10}\}$.

Then $\delta \leq \frac{1}{2}$ and $\delta \leq \frac{\epsilon}{10}$ and $\delta > 0$.

Let $x \in \mathbb{R}$ such that $x \neq \frac{9}{4}$ and $0 < |x - 3| < \delta$.

Then $0 < |x - 3| < \delta \leq \frac{1}{2}$, so $0 < |x - 3| < \frac{1}{2}$.

Hence, $\frac{-1}{2} < x - 3 < \frac{1}{2}$, so $\frac{-1}{2} < x - 3$.

Thus, $\frac{5}{2} < x$.

Observe that

$$\begin{aligned} \frac{5}{2} < x &\Leftrightarrow 0 < 10 < 4x \\ &\Leftrightarrow 0 < 1 < 4x - 9 \\ &\Leftrightarrow 0 < \frac{1}{4x-9} < 1. \end{aligned}$$

Hence, $0 < \frac{1}{4x-9} < 1$.
Thus,

$$\begin{aligned}
|f(x) - 3| &= |\frac{2x+3}{4x-9} - 3| \\
&= |\frac{-10x+30}{4x-9}| \\
&= |\frac{-10(x-3)}{4x-9}| \\
&= 10|x-3||\frac{1}{4x-9}| \\
&= 10|x-3|\frac{1}{4x-9} \\
&< 10\delta \\
&\leq 10 \cdot \frac{\epsilon}{10} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|f(x) - 3| < \epsilon$, as desired. □

Exercise 6. Show that $\lim_{x \to 6} \frac{x^2-3x}{x+3} = 2$.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \frac{x^2-3x}{x+3}$ for all $x \neq -3$.

Observe that 6 is an accumulation point of the set $\{x \in \mathbb{R} : x \neq -3\}$, the domain of f.

We prove $\lim_{x \to 6} f(x) = 2$.
Let $\epsilon > 0$ be given.
Let $\delta = \min\{1, \epsilon\}$.
Then $\delta \leq 1$ and $\delta \leq \epsilon$ and $\delta > 0$.
Let $x \in \mathbb{R}$ such that $x \neq -3$ and $0 < |x - 6| < \delta$.
Then $0 < |x - 6| < \delta \leq 1$, so $0 < |x - 6| < 1$.
Hence, $-1 < x - 6 < 1$, so $5 < x < 7$.
Since $5 < x < 7$, then $0 < 6 < x + 1 < 8$, so $0 < x + 1 < 8$.
Since $5 < x$, then $0 < 8 < x + 3$, so $0 < \frac{1}{x+3} < \frac{1}{8}$.
Thus,

$$\begin{aligned}
|f(x) - 2| &= |\frac{x^2-3x}{x+3} - 2| \\
&= |\frac{x^2-5x-6}{x+3}| \\
&= |\frac{(x-6)(x+1)}{x+3}| \\
&= |x-6| \cdot (x+1) \cdot \frac{1}{x+3} \\
&< \delta \\
&\leq \epsilon.
\end{aligned}$$

Therefore, $|f(x) - 2| < \epsilon$, as desired. $\square$

Exercise 7. Let $f : (0, 1) \to \mathbb{R}$ be a function defined by $f(x) = \frac{x^3 - x^2 + x - 1}{x - 1}$.

Then $\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x - 1} = 2$.

Solution. Observe that 1 is an accumulation point of the interval $(0, 1)$, the domain of f.

We prove $\lim_{x \to 1} f(x) = 2$.

Let $\epsilon > 0$ be given.

Let $\delta = \frac{\epsilon}{2}$.

Then $\delta > 0$.

Let $x \in (0, 1)$ such that $0 < |x - 1| < \delta$.

Since $x \in (0, 1)$, then $0 < x < 1$, so $0 < 1 < x + 1 < 2$.

Hence, $|x + 1| = x + 1 < 2$, so $|x + 1| < 2$.

Thus,

$$\begin{aligned}
|f(x) - 2| &= |\frac{x^3 - x^2 + x - 1}{x - 1} - 2| \\
&= |\frac{x^2(x - 1) + (x - 1)}{x - 1} - 2| \\
&= |\frac{(x - 1)(x^2 + 1)}{x - 1} - 2| \\
&= |(x^2 + 1) - 2| \\
&= |x^2 - 1| \\
&= |(x - 1)(x + 1)| \\
&= |(x - 1)||x + 1| \\
&< 2\delta \\
&= \epsilon.
\end{aligned}$$

Therefore, $|f(x) - 2| < \epsilon$, as desired. $\square$

Exercise 8. Show that $\lim_{x \to 4} \sqrt{x} = 2$.

Solution. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function defined by $f(x) = \sqrt{x}$ for all $x \geq 0$.

Observe that 4 is an accumulation point of the set $\{x \in \mathbb{R} : x \geq 0\} = [0, \infty)$, the domain of f.

We prove $\lim_{x \to 4} \sqrt{x} = 2$.

Let $\epsilon > 0$ be given.

Let $\delta = \min\{1, 2\epsilon\}$.

Then $\delta \leq 1$ and $\delta \leq 2\epsilon$ and $\delta > 0$.

Let $x \in \mathbb{R}$ such that $0 < |x - 4| < \delta$.

Since $|x - 4| < \delta$, then $-\delta < x - 4 < \delta$, so $-\delta < x - 4$.

Hence, $4 - \delta < x$.

Since $\delta \leq 1 < 4$, then $\delta < 4$, so $0 < 4 - \delta$.

Thus, $0 < 4 - \delta < x$.

Observe that

$$\begin{aligned} 0 < 4-\delta < x \quad & \Leftrightarrow \quad 0 < \sqrt{4-\delta} < \sqrt{x} \\ & \Leftrightarrow \quad 0 < 2 < \sqrt{4-\delta}+2 < \sqrt{x}+2 \\ & \Leftrightarrow \quad 0 < 2 < \sqrt{x}+2 \\ & \Leftrightarrow \quad 0 < \frac{1}{\sqrt{x}+2} < \frac{1}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} |\sqrt{x}-2| &= |(\sqrt{x}-2)\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}| \\ &= |\frac{x-4}{\sqrt{x}+2}| \\ &= |x-4|\cdot\frac{1}{\sqrt{x}+2} \\ &< \delta\cdot\frac{1}{2} \\ &\leq 2\epsilon\cdot\frac{1}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $|\sqrt{x}-2| < \epsilon$, as desired. □

Exercise 9. Show that $\lim_{x\to 0} \sqrt[3]{x} = 0$.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \sqrt[3]{x}$.
Observe that 0 is an accumulation point of the set $\mathbb{R}$, the domain of f.
We prove $\lim_{x\to 0} \sqrt[3]{x} = 0$.
Let $\epsilon > 0$ be given.
Let $\delta = \epsilon^3$.
Then $\delta > 0$.
Let $x \in \mathbb{R}$ such that $0 < |x| < \delta$.
Then $0 < |x| < \epsilon^3$, so $0 < |x|^{\frac{1}{3}} < (\epsilon^3)^{\frac{1}{3}}$.
Thus, $0 < |x|^{\frac{1}{3}} < \epsilon$, so $0 < |x^{\frac{1}{3}}| < \epsilon$.
Hence, $|x^{\frac{1}{3}}| < \epsilon$, so $|\sqrt[3]{x}| < \epsilon$, as desired. □

Exercise 10. Show that $\lim_{x\to \frac{1}{2}} \frac{1}{x} = 2$.

Proof. Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = \frac{1}{x}$ for all $x \in \mathbb{R}^*$.
Observe that $\frac{1}{2}$ is an accumulation point of $\mathbb{R}^*$, the domain of f.
Let $\epsilon > 0$ be given.
Let $\delta = \min\{\frac{1}{4}, \frac{\epsilon}{8}\}$.
Then $\delta \leq \frac{1}{4}$ and $\delta \leq \frac{\epsilon}{8}$ and $\delta > 0$.
Let $x \in \mathbb{R}^*$ such that $0 < |x - \frac{1}{2}| < \delta$.
Since $x \in \mathbb{R}^*$, then $x \neq 0$, so $|x| > 0$.

Since $0 < |x - \frac{1}{2}| < \delta \leq \frac{1}{4}$, then $0 < |x - \frac{1}{2}| < \frac{1}{4}$, so $\frac{1}{4} > |x - \frac{1}{2}| \geq \frac{1}{2} - |x|$.
Hence, $\frac{1}{4} > \frac{1}{2} - |x|$, so $|x| > \frac{1}{4}$.
Thus, $4 > \frac{1}{|x|} > 0$, so $0 < \frac{1}{|x|} < 4$.
Observe that

$$\begin{aligned} |f(x) - 2| &= |\frac{1}{x} - 2| \\ &= |2 - \frac{1}{x}| \\ &= |\frac{2}{x}(x - \frac{1}{2})| \\ &= |\frac{2}{x}| \cdot |x - \frac{1}{2}| \\ &= 2 \cdot |\frac{1}{x}| \cdot |x - \frac{1}{2}| \\ &= 2 \cdot \frac{1}{|x|} \cdot |x - \frac{1}{2}| \\ &< 8 \cdot |x - \frac{1}{2}| \\ &< 8\delta \\ &\leq 8 \cdot \frac{\epsilon}{8} \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x) - 2| < \epsilon$. □

Exercise 11. Show that $\lim_{x \to 2} \frac{1}{1-x} = -1$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \frac{1}{1-x}$ for all $x \neq 1$.
Observe that 2 is an accumulation point of the set $\{x \in \mathbb{R} : x \neq 1\}$, the domain of f.
Let $\epsilon > 0$ be given.
Let $\delta = \min\{\frac{1}{2}, \frac{\epsilon}{2}\}$.
Then $\delta \leq \frac{1}{2}$ and $\delta \leq \frac{\epsilon}{2}$ and $\delta > 0$.
Let $x \in \mathbb{R}$ such that $x \neq 1$ and $0 < |x - 2| < \delta$.
Since $0 < |x - 2| < \delta \leq \frac{1}{2}$, then $0 < |x - 2| < \frac{1}{2}$.
Hence, $\frac{1}{2} > |x - 2| \geq 2 - |x|$, so $\frac{1}{2} > 2 - |x|$.
Thus, $|x| > \frac{3}{2}$.
Hence, $|x - 1| \geq |x| - 1 > \frac{1}{2}$, so $|x - 1| > \frac{1}{2} > 0$.
Thus, $2 > |\frac{1}{x-1}| > 0$, so $0 < |\frac{1}{x-1}| < 2$.

Observe that

$$\begin{aligned}
|f(x) - (-1)| &= |\frac{1}{1-x} + 1| \\
&= |\frac{2-x}{1-x}| \\
&= \frac{|2-x|}{|1-x|} \\
&= \frac{|x-2|}{|x-1|} \\
&= |x-2| \cdot |\frac{1}{x-1}| \\
&< 2\delta \\
&\leq 2 \cdot \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|f(x) - (-1)| < \epsilon$. □

Exercise 12. Show that $\lim_{x \to 1} \frac{x}{1+x} = \frac{1}{2}$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \frac{x}{1+x}$ for all $x \neq -1$.

Observe that 1 is an accumulation point of the set $\{x \in \mathbb{R} : x \neq -1\}$, the domain of f.

Let $\epsilon > 0$ be given.

Let $\delta = \min\{1, 2\epsilon\}$.

Then $\delta \leq 1$ and $\delta \leq 2\epsilon$ and $\delta > 0$.

Let $x \in \mathbb{R}$ such that $x \neq -1$ and $0 < |x-1| < \delta$.

Since $0 < |x-1| < \delta \leq 1$, then $0 < |x-1| < 1$.

Thus, $-1 < x - 1 < 1$, so $-1 < x - 1$.

Hence, $0 < x$, so $0 < 1 < x + 1$.

Thus, $0 < \frac{1}{x+1} < 1$.

Observe that

$$\begin{aligned}
|f(x) - \frac{1}{2}| &= |\frac{x}{1+x} - \frac{1}{2}| \\
&= |\frac{x-1}{2(1+x)}| \\
&= \frac{1}{2} \cdot |x-1| \cdot |\frac{1}{x+1}| \\
&< \frac{1}{2}\delta \\
&\leq \frac{1}{2} \cdot 2\epsilon \\
&= \epsilon.
\end{aligned}$$

Therefore, $|f(x) - \frac{1}{2}| < \epsilon$. □

Exercise 13. Show that $\lim_{x\to -1} \frac{x+5}{2x+3} = 4$.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \frac{x+5}{2x+3}$ for all $x \neq \frac{-3}{2}$.

Observe that -1 is an accumulation point of $\{x \in \mathbb{R} : x \neq \frac{-3}{2}\}$, the domain of f.

We prove $\lim_{x\to -1} f(x) = 4$.

Let $\epsilon > 0$ be given.

Let $\delta = \min\{\frac{1}{4}, \frac{\epsilon}{14}\}$.

Then $\delta \leq \frac{1}{4}$ and $\delta \leq \frac{\epsilon}{14}$ and $\delta > 0$.

Let $x \in \mathbb{R}$ such that $x \neq \frac{-3}{2}$ and $0 < |x - (-1)| < \delta$.

Then $0 < |x+1| < \delta \leq \frac{1}{4}$, so $0 < |x+1| < \frac{1}{4}$.

Hence, $\frac{-1}{4} < x + 1 < \frac{1}{4}$, so $\frac{-1}{4} < x + 1$.

Thus, $\frac{-5}{4} < x$.

Observe that

$$\begin{aligned} \frac{-5}{4} < x \quad & \Leftrightarrow \quad \frac{-5}{2} < 2x \\ & \Leftrightarrow \quad 0 < \frac{1}{2} < 2x + 3 \\ & \Leftrightarrow \quad 0 < \frac{1}{2x+3} < 2. \end{aligned}$$

Hence, $0 < \frac{1}{2x+3} < 2$.

Thus,

$$\begin{aligned} |f(x) - 4| &= |\frac{x+5}{2x+3} - 4| \\ &= |\frac{-7x-7}{2x+3}| \\ &= 7|x+1||\frac{1}{2x+3}| \\ &= 7|x+1|\frac{1}{2x+3} \\ &< 14\delta \\ &\leq 14 \cdot \frac{\epsilon}{14} \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x) - 4| < \epsilon$, as desired. $\square$

Exercise 14. Show that $\lim_{x\to -2} \frac{2x^2+3x-2}{x+2} = -5$.

Solution. Let $f : \mathbb{R} - \{-2\} \to \mathbb{R}$ be a function defined by $f(x) = \frac{2x^2+3x-2}{x+2}$ for all $x \neq -2$.

Observe that -2 is an accumulation point of the set $\{x \in \mathbb{R} : x \neq -2\}$, the domain of f.

We prove $\lim_{x\to -2} f(x) = -5$.

Let $\epsilon > 0$ be given.
Let $\delta = \frac{\epsilon}{2}$.
Then $\delta > 0$.
Let $x \in \mathbb{R} - \{-2\}$ such that $0 < |x+2| < \delta$.
Since $0 < |x+2| < \delta$, then $0 < |x+2|$ and $|x+2| < \delta$.
Since $|x+2| > 0$, then $x + 2 \neq 0$.
Observe that

$$\begin{aligned}
|f(x) + 5| &= |\frac{2x^2 + 3x - 2}{x+2} + 5| \\
&= |\frac{(2x-1)(x+2)}{x+2} + 5| \\
&= |(2x-1) + 5| \\
&= |2x + 4| \\
&= |2(x+2)| \\
&= 2|x+2| \\
&< 2\delta \\
&= \epsilon.
\end{aligned}$$

Therefore, $|f(x) + 5| < \epsilon$, as desired. □

Exercise 15. Show that $\lim_{x\to 1} \frac{x^2-x+1}{x+1} = \frac{1}{2}$.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \frac{x^2-x+1}{x+1}$ for all $x \neq -1$.

Observe that 1 is an accumulation point of the set $\{x \in \mathbb{R} : x \neq -1\}$, the domain of f.

We prove $\lim_{x\to 1} f(x) = \frac{1}{2}$.
Let $\epsilon > 0$ be given.
Let $\delta = \min\{\frac{1}{2}, \frac{3\epsilon}{2}\}$.
Then $\delta \leq \frac{1}{2}$ and $\delta \leq \frac{3\epsilon}{2}$ and $\delta > 0$.
Let $x \in \mathbb{R}$ such that $x \neq -1$ and $0 < |x-1| < \delta$.
Then $0 < |x-1| < \delta \leq \frac{1}{2}$, so $0 < |x-1| < \frac{1}{2}$.
Hence, $\frac{-1}{2} < x - 1 < \frac{1}{2}$, so $\frac{-1}{2} < x - 1$.
Thus, $\frac{1}{2} < x$, so $0 < \frac{3}{2} < x + 1$.
Therefore, $0 < \frac{1}{x+1} < \frac{2}{3}$.

Since $|x-1| < \frac{1}{2}$, then $|2x-1| = |2(x-1)+1| \leq 2|x-1| + 1 < 2 \cdot \frac{1}{2} + 1 = 2$, so $0 \leq |2x-1| < 2$.

Observe that

$$
\begin{aligned}
|f(x) - \frac{1}{2}| &= |\frac{x^2 - x + 1}{x + 1} - \frac{1}{2}| \\
&= |\frac{2x^2 - 3x + 1}{2(x + 1)}| \\
&= |\frac{(x - 1)(2x - 1)}{2(x + 1)}| \\
&= \frac{1}{2}|x - 1||2x - 1| \cdot \frac{1}{x + 1} \\
&< \frac{1}{2}\delta \cdot 2 \cdot \frac{2}{3} \\
&= \frac{2}{3} \cdot \delta \\
&\leq \frac{2}{3} \cdot \frac{3\epsilon}{2} \\
&= \epsilon.
\end{aligned}
$$

Therefore, $|f(x) - \frac{1}{2}| < \epsilon$, as desired. ∎

Exercise 16. Show that $\lim_{x\to 2} \frac{x^3-4}{x^2+1} = \frac{4}{5}$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \frac{x^3-4}{x^2+1}$ for all $x \in \mathbb{R}$.

Observe that 2 is an accumulation point of $\mathbb{R}$, the domain of f.

Let $\epsilon > 0$ be given.

Let $\delta = \min\{1, \frac{2\epsilon}{15}\}$.

Then $\delta \leq 1$ and $\delta \leq \frac{2\epsilon}{15}$ and $\delta > 0$.

Let $x \in \mathbb{R}$ such that $0 < |x - 2| < \delta$.

Since $|x - 2| < \delta$ and $\delta \leq 1$, then $|x - 2| < 1$, so $-1 < x - 2 < 1$.

Hence, $1 < x < 3$.

Since $1 < x$, then $1 < x^2$, so $0 < 2 < x^2 + 1$.

Thus, $0 < \frac{1}{x^2+1} < \frac{1}{2}$.

Since $|x| = |(x - 2) + 2| \leq |x - 2| + 2 < \delta + 2 \leq 3$, then $|x| < 3$.

Since

$$
\begin{aligned}
|5x^2 + 6x + 12| &\leq |5x^2 + 6x| + 12 \\
&\leq |5x^2| + |6x| + 12 \\
&= 5|x|^2 + 6|x| + 12 \\
&< 5(3)^2 + 6(3) + 12 \\
&= 75,
\end{aligned}
$$

then $|5x^2 + 6x + 12| < 75$.

Hence,

$$\begin{aligned}
|\frac{x^3-4}{x^2+1}-\frac{4}{5}| &= |\frac{5x^3-4x^2-24}{5(x^2+1)}| \\
&= |\frac{(x-2)(5x^2+6x+12)}{5(x^2+1)}| \\
&= \frac{|x-2|}{5}\cdot|5x^2+6x+12|\cdot\frac{1}{x^2+1} \\
&< \frac{\delta}{5}\cdot\frac{75}{2} \\
&= \delta\cdot\frac{15}{2} \\
&\leq \frac{2\epsilon}{15}\cdot\frac{15}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|\frac{x^3-4}{x^2+1}-\frac{4}{5}|<\epsilon$. □

Exercise 17. Show that $\lim_{x\to 0}\frac{x^2}{|x|}=0$.

Solution. Let $f:\mathbb{R}^*\to\mathbb{R}$ be a function defined by $f(x)=\frac{x^2}{|x|}$ for all $x\neq 0$.
Let $x\in\mathbb{R}^*$.
Then $x\neq 0$, so either $x>0$ or $x<0$.
If $x>0$, then $f(x)=\frac{x^2}{|x|}=\frac{x^2}{x}=x$.
If $x<0$, then $f(x)=\frac{x^2}{|x|}=\frac{x^2}{-x}=-x$.
Thus,

$$f(x)=\begin{cases} x & \text{if } x>0 \\ -x & \text{if } x<0 \end{cases}$$

Observe that 0 is an accumulation point of $\mathbb{R}^*=\{x\in\mathbb{R}:x\neq 0\}$, the domain of f.
We prove $\lim_{x\to 0}f(x)=0$.
Let $\epsilon>0$ be given.
Let $\delta=\epsilon$.
Then $\delta>0$.
Let $x\in\mathbb{R}$ such that $0<|x|<\delta$.
Then $0<|x|$ and $|x|<\delta$.
Thus, $|f(x)|=|\frac{x^2}{|x|}|=\frac{|x^2|}{||x||}=\frac{|x|^2}{|x|}=|x|<\delta=\epsilon$.
Therefore, $|f(x)|<\epsilon$, as desired. □

Exercise 18. Let $f:\mathbb{Q}\to\mathbb{R}$ be a function defined by $f(x)=x^2$ for all $x\in\mathbb{Q}$.
Then $\lim_{x\to\sqrt{2}}f(x)=2$.

Solution.

Since every real number is an accumulation point of $\mathbb{Q}$, then $\sqrt{2}$ is an accumulation point of $\mathbb{Q}$, the domain of f.

We prove $\lim_{x\to\sqrt{2}} f(x) = 2$.

Let $\epsilon > 0$ be given.

Let $\delta = \min\{1, \frac{\epsilon}{1+2\sqrt{2}}\}$.

Then $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{1+2\sqrt{2}}$ and $\delta > 0$.

Let $x \in \mathbb{Q}$ such that $0 < |x - \sqrt{2}| < \delta$.

Then

$$\begin{aligned} |x+\sqrt{2}| &= |(x-\sqrt{2}) + 2\sqrt{2}| \\ &\leq |x-\sqrt{2}| + 2\sqrt{2} \\ &< \delta + 2\sqrt{2} \\ &\leq 1 + 2\sqrt{2}. \end{aligned}$$

Thus, $0 \leq |x+\sqrt{2}| < 1 + 2\sqrt{2}$.

Hence,

$$\begin{aligned} |f(x) - 2| &= |x^2 - 2| \\ &= |(x-\sqrt{2})(x+\sqrt{2})| \\ &= |x-\sqrt{2}||x+\sqrt{2}| \\ &< \delta(1+2\sqrt{2}) \\ &\leq \frac{\epsilon}{1+2\sqrt{2}} \cdot (1+2\sqrt{2}) \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x) - 2| < \epsilon$, as desired. □

Exercise 19. Show that $\lim_{x\to a}(-12x+4) = -12a+4$ using the sequential characterization of a limit.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = -12x + 4$ for all $x \in \mathbb{R}$.

Observe that $a \in \mathbb{R}$ is an accumulation point of $\mathbb{R}$, the domain of f.

Let (x_n) be an arbitrary sequence of points in $\mathbb{R} - \{a\}$ such that $\lim_{n\to\infty} x_n = a$.

To prove $\lim_{x\to a}(-12x+4) = -12a+4$, we must prove $\lim_{n\to\infty} f(x_n) = -12a + 4$.

Observe that

$$\begin{aligned} -12a+4 &= -12(\lim_{n\to\infty} x_n)+4 \\ &= \lim_{n\to\infty}(-12x_n)+4 \\ &= \lim_{n\to\infty}(-12x_n)+\lim_{n\to\infty} 4 \\ &= \lim_{n\to\infty}(-12x_n+4) \\ &= \lim_{n\to\infty} f(x_n). \end{aligned}$$

Therefore, $\lim_{n\to\infty} f(x_n) = -12a+4$, as desired. ∎

Exercise 20. Let m and b be fixed real numbers.
Then for every real number a, $\lim_{x\to a}(mx+b) = ma+b$.

Proof. Let a be an arbitrary real number.
Let $f : \mathbb{R} \to \mathbb{R}$ be the linear function defined by $f(x) = mx+b$.
Observe that a is an accumulation point of $dom f = \mathbb{R}$.
We must prove $\lim_{x\to a}(mx+b) = ma+b$.
Either $m = 0$ or $m \neq 0$.
We consider these cases separately.
Case 1: Suppose $m = 0$.
Then $\lim_{x\to a}(mx+b) = \lim_{x\to a}(0x+b) = \lim_{x\to a} b = b = 0+b = 0a+b = ma+b$.
Case 2: Suppose $m \neq 0$.
Let $\epsilon > 0$ be given.
Let $\delta = \frac{\epsilon}{|m|}$.
Since $m \neq 0$, then $|m| > 0$, so $\delta > 0$.
Let $x \in \mathbb{R}$ such that $0 < |x-a| < \delta$.
Then

$$\begin{aligned} |f(x)-(ma+b)| &= |(mx+b)-(ma+b)| \\ &= |mx+b-ma-b| \\ &= |mx-ma| \\ &= |m(x-a)| \\ &= |m||x-a| \\ &< |m|\delta \\ &= |m|\cdot\frac{\epsilon}{|m|} \\ &= \epsilon. \end{aligned}$$

Therefore, $|(mx+b)-(ma+b)| < \epsilon$, so $\lim_{x\to a}(mx+b) = ma+b$. ∎

Proof. Let a be an arbitrary real number.
Let $f : \mathbb{R} \to \mathbb{R}$ be the linear function defined by $f(x) = mx+b$.
Observe that a is an accumulation point of $dom f = \mathbb{R}$.

To prove $\lim_{x\to a}(mx+b) = ma+b$ using the sequential characterization of a limit, let (x_n) be an arbitrary sequence of points in $\mathbb{R} - \{a\}$ such that $\lim_{n\to\infty} x_n = a$.

We must prove $\lim_{n\to\infty} f(x_n) = ma+b$.

Observe that

$$\begin{aligned} ma+b &= m \lim_{n\to\infty} x_n + b \\ &= m \lim_{n\to\infty} x_n + \lim_{n\to\infty} b \\ &= \lim_{n\to\infty} mx_n + \lim_{n\to\infty} b \\ &= \lim_{n\to\infty} (mx_n + b) \\ &= \lim_{n\to\infty} f(x_n). \end{aligned}$$

Therefore, $\lim_{n\to\infty} f(x_n) = ma+b$, as desired. □

Exercise 21. limit of the square function

Prove that for all $a \in \mathbb{R}$, $\lim_{x\to a} x^2 = a^2$ using using the sequential characterization of a limit.

Proof. Let a be an arbitrary real number.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$ for all $x \in \mathbb{R}$.

Since every real number is an accumulation point of $\mathbb{R}$, then a is an accumulation point of $\mathbb{R}$, the domain of f.

Since a is an accumulation point of $\mathbb{R}$, then there exists a sequence of points in $\mathbb{R} - \{a\}$ that converges to a.

To prove $\lim_{x\to a} x^2 = a^2$ using the sequential characterization of a limit, let (x_n) be an arbitrary sequence of points in $\mathbb{R} - \{a\}$ such that $\lim_{n\to\infty} x_n = a$.

We must prove $\lim_{n\to\infty} f(x_n) = a^2$.

Observe that

$$\begin{aligned} a^2 &= (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} x_n) \\ &= \lim_{n\to\infty} (x_n x_n) \\ &= \lim_{n\to\infty} (x_n)^2 \\ &= \lim_{n\to\infty} f(x_n). \end{aligned}$$

Therefore, $\lim_{n\to\infty} f(x_n) = a^2$, as desired. □

Exercise 22. Let $a > 0$.

Let $I = (0, a)$.

Then for any $x, c \in I$, $|x^2 - c^2| \leq 2a|x - c|$ and $\lim_{x\to c} x^2 = c^2$ for all $c \in I$.

Proof. Let $x, c \in I$.

Since $x \in I$, then $x \in (0, a)$, so $0 < x < a$.

Since $c \in I$, then $c \in (0, a)$, so $0 < c < a$.

Thus,

$$\begin{aligned} |x+c| &\leq |x|+|c| \\ &= x+c \\ &< a+a \\ &= 2a. \end{aligned}$$

Hence, $|x+c| < 2a$.

Since $|x-c| \geq 0$, then $|x+c||x-c| \leq 2a|x-c|$, so $|x^2-c^2| \leq 2a|x-c|$.

Let $f : I \to \mathbb{R}$ be the function defined by $f(x) = x^2$ for all $x \in I$.

We prove $\lim_{x\to c} x^2 = c^2$ for all $c \in I$.

Let $c \in I$ be given.

Then $c \in (0, a)$, so c is an accumulation point of I.

Let $\epsilon > 0$ be given.

Let $\delta = \frac{\epsilon}{2a}$.

Since $\epsilon > 0$ and $a > 0$, then $\delta > 0$.

Let $x \in I$ such that $0 < |x-c| < \delta$.

Then

$$\begin{aligned} |x^2-c^2| &\leq 2a|x-c| \\ &< 2a\delta \\ &= 2a \cdot \frac{\epsilon}{2a} \\ &= \epsilon. \end{aligned}$$

Therefore, $|x^2-c^2| < \epsilon$, as desired. □

Exercise 23. limit of a square root equals square root of a limit

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function such that $f(x) \geq 0$ for all $x \in E$.

If $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a} \sqrt{f(x)} = \sqrt{\lim_{x\to a} f(x)}$.

Proof. Suppose $\lim_{x\to a} f(x)$ exists.

Then there exists $L \in \mathbb{R}$ such that $\lim_{x\to a} f(x) = L$, so a is an accumulation point of E.

Let $g(x) = \sqrt{f(x)}$.

Then g is a function and $domg = \{x \in E : g(x) \in \mathbb{R}\} = \{x \in E : \sqrt{f(x)} \in \mathbb{R}\} = \{x \in E : f(x) \geq 0\}$.

We must prove $\lim_{x\to a} g(x) = \sqrt{L}$.

We first prove a is an accumulation point of $domg$.

Let $\epsilon > 0$ be given.

Since a is an accumulation point of E, then there exists $x \in E$ such that $x \in N'(a;\epsilon)$.

Since $x \in E$, then $f(x) \geq 0$.

Since $x \in E$ and $f(x) \geq 0$, then $x \in domg$.

Since there exists $x \in domg$ such that $x \in N'(a;\epsilon)$, then a is an accumulation point of $domg$.

We next prove $\lim_{x\to a} g(x) = \sqrt{L}$.

Since a is an accumulation point of $domg$, then there exists a sequence in $domg - \{a\}$ that converges to a.

Let (x_n) be an arbitrary sequence in $domg - \{a\}$ such that $\lim_{n\to\infty} x_n = a$.

Let $n \in \mathbb{N}$ be given.

Then $x_n \in domg - \{a\}$, so $x_n \in domg$ and $x_n \neq a$.

Since $x_n \in domg$ and $domg \subset E$, then $x_n \in E$, so $f(x_n) \geq 0$.

Since $x_n \in E$ and $x_n \neq a$, then $x_n \in E - \{a\}$.

Thus, $f(x_n) \geq 0$ and $x_n \in E - \{a\}$ for all $n \in \mathbb{N}$, so $f(x_n) \geq 0$ for all $n \in \mathbb{N}$ and $x_n \in E - \{a\}$ for all $n \in \mathbb{N}$.

Since $x_n \in E - \{a\}$ for all $n \in \mathbb{N}$, then (x_n) is a sequence of points in $E - \{a\}$.

Since a is an accumulation point of E and $\lim_{x\to a} f(x) = L$ and (x_n) is a sequence of points in $E - \{a\}$ and $\lim_{n\to\infty} x_n = a$, then by the sequential characterization of a function limit, we have $\lim_{n\to\infty} f(x_n) = L$, so $\lim_{n\to\infty} f(x_n)$ exists.

Since $f(x_n) \geq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f(x_n)$ exists, then by a previous proposition, $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} \sqrt{f(x_n)} = \sqrt{\lim_{n\to\infty} f(x_n)} = \sqrt{L}$.

Since (x_n) is an arbitrary sequence of points in $domg - \{a\}$ such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} g(x_n) = \sqrt{L}$, then by the sequential characterization of a function limit, we have $\lim_{x\to a} g(x) = \sqrt{L}$, as desired. ☐

Exercise 24. Let $I \subset \mathbb{R}$ be an interval with at least two elements.

Let $f : I \to \mathbb{R}$ be a function defined for all $x \in I$.

Let $a \in I$.

If there exist real numbers K, L such that $|f(x) - L| \leq K|x - a|$ for all $x \in I$, then $\lim_{x\to a} f(x) = L$.

Proof. Suppose there exist real numbers K, L such that $|f(x) - L| \leq K|x - a|$ for all $x \in I$.

Since I has at least two elements and $a \in I$, then a is an accumulation point of I, the domain of f.

Since $a \in I$ and I has at least two elements, then there exists at least one element of I that is distinct from a.

Hence, there exists $b \in I$ such that $b \neq a$.

Thus, $d(b, a) = |b - a| > 0$.

Since $b \in I$, then $0 \leq |f(b) - L| \leq K|b - a|$, so $0 \leq K|b - a|$.

Since $|b - a| > 0$, then $0 \leq K$, so $K \geq 0$.

Thus, either $K > 0$ or $K = 0$.

We consider these cases separately.

Case 1: Suppose $K = 0$.

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon$.

Then $\delta > 0$.

Since there is at least one element of I distinct from a, let $x \in I$ such that $0 < |x - a| < \delta$.

Since $x \in I$, then

$$\begin{aligned} |f(x) - L| &\leq K|x - a| \\ &= 0|x - a| \\ &= 0 \\ &< \epsilon. \end{aligned}$$

Therefore, $|f(x) - L| < \epsilon$, so $\lim_{x \to a} f(x) = L$.

Case 2: Suppose $K > 0$.

Let $\epsilon > 0$ be given.

Let $\delta = \frac{\epsilon}{K}$.

Since $\epsilon > 0$ and $K > 0$, then $\delta > 0$.

Since there is at least one element of I distinct from a, let $x \in I$ such that $0 < |x - a| < \delta$.

Then $|x - a| < \delta$.

Since $x \in I$, then

$$\begin{aligned} |f(x) - L| &\leq K|x - a| \\ &< K\delta \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x) - L| < \epsilon$, so $\lim_{x \to a} f(x) = L$.

Thus, in all cases, $\lim_{x \to a} f(x) = L$, as desired. $\square$

Exercise 25. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x^2}$ for all $x > 0$.

Then $\lim_{x \to 0} f(x)$ does not exist in $\mathbb{R}$.

Solution.

Observe that 0 is an accumulation point of $\mathbb{R}^+ = (0, \infty)$, the domain of f.

We prove there is no real L such that $\lim_{x \to 0} f(x) = L$.

Observe that

$$\begin{aligned} \neg(\exists L \in \mathbb{R})(\lim_{x \to 0} f(x) = L) &\Leftrightarrow \\ \neg(\exists L \in \mathbb{R})(\forall \epsilon > 0)(\exists \delta > 0)(\forall x > 0)(0 < |x| < \delta \to |f(x) - L| < \epsilon) &\Leftrightarrow \\ (\forall L \in \mathbb{R})(\exists \epsilon > 0)(\forall \delta > 0)(\exists x > 0)(0 < |x| < \delta \wedge |f(x) - L| \geq \epsilon). \end{aligned}$$

Thus, we prove $(\forall L \in \mathbb{R})(\exists \epsilon > 0)(\forall \delta > 0)(\exists x > 0)(0 < |x| < \delta \wedge |f(x) - L| \geq \epsilon)$.

Let L be an arbitrary real number.

Let $\epsilon = \max\{0, -L\} + 1$.

Then $\epsilon - 1 = \max\{0, -L\}$, so $\epsilon - 1 \geq 0$ and $\epsilon - 1 \geq -L$.

Since $\epsilon - 1 \geq 0$, then $\epsilon \geq 1 > 0$, so $\epsilon > 0$.
Since $\epsilon - 1 \geq -L$, then $L + \epsilon \geq 1 > 0$, so $L + \epsilon > 0$.
Hence, $\frac{1}{L+\epsilon} > 0$, so $\frac{1}{\sqrt{L+\epsilon}} > 0$.
Let $\delta > 0$ be given.
We must prove there exists $x > 0$ such that $0 < |x| < \delta$ and $|f(x) - L| \geq \epsilon$.

Let $x = \min\{\frac{\delta}{2}, \frac{1}{\sqrt{L+\epsilon}}\}$.
Then $x \leq \frac{\delta}{2}$ and $x \leq \frac{1}{\sqrt{L+\epsilon}}$ and $x > 0$.
Since $0 < |x| = x \leq \frac{\delta}{2} < \delta$, then $0 < |x| < \delta$.
Since $0 < x \leq \frac{1}{\sqrt{L+\epsilon}}$, then $0 < x^2 \leq \frac{1}{L+\epsilon}$, so $0 < L + \epsilon \leq \frac{1}{x^2}$.
Hence, $0 < \epsilon \leq \frac{1}{x^2} - L$, so $0 < \frac{1}{x^2} - L$ and $\epsilon \leq \frac{1}{x^2} - L$.
Thus, $|f(x) - L| = |\frac{1}{x^2} - L| = \frac{1}{x^2} - L \geq \epsilon$, so $|f(x) - L| \geq \epsilon$.
Therefore, there exists $x > 0$ such that $0 < |x| < \delta$ and $|f(x) - L| \geq \epsilon$, as desired. $\square$

Exercise 26. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{\sqrt{x}}$ for all $x > 0$.
Then $\lim_{x \to 0} f(x)$ does not exist in $\mathbb{R}$.

Solution.
Observe that 0 is an accumulation point of $\mathbb{R}^+ = (0, \infty)$, the domain of f.
We prove there is no real L such that $\lim_{x \to 0} f(x) = L$.
Observe that

$$\begin{aligned}
\neg(\exists L \in \mathbb{R})(\lim_{x \to 0} f(x) = L) \quad &\Leftrightarrow \\
\neg(\exists L \in \mathbb{R})(\forall \epsilon > 0)(\exists \delta > 0)(\forall x > 0)(0 < |x| < \delta \to |f(x) - L| < \epsilon) \quad &\Leftrightarrow \\
(\forall L \in \mathbb{R})(\exists \epsilon > 0)(\forall \delta > 0)(\exists x > 0)(0 < |x| < \delta \wedge |f(x) - L| \geq \epsilon).&
\end{aligned}$$

Thus, we prove $(\forall L \in \mathbb{R})(\exists \epsilon > 0)(\forall \delta > 0)(\exists x > 0)(0 < |x| < \delta \wedge |f(x) - L| \geq \epsilon)$.
Let L be an arbitrary real number.
Let $\epsilon = \max\{0, -L\} + 1$.
Then $\epsilon - 1 = \max\{0, -L\}$, so $\epsilon - 1 \geq 0$ and $\epsilon - 1 \geq -L$.
Since $\epsilon - 1 \geq 0$, then $\epsilon \geq 1 > 0$, so $\epsilon > 0$.
Since $\epsilon - 1 \geq -L$, then $L + \epsilon \geq 1 > 0$, so $L + \epsilon > 0$.
Let $\delta > 0$ be given.
We must prove there exists $x > 0$ such that $0 < |x| < \delta$ and $|f(x) - L| \geq \epsilon$.

Let $x = \min\{\frac{\delta}{2}, \frac{1}{(L+\epsilon)^2}\}$.
Then $x \leq \frac{\delta}{2}$ and $x \leq \frac{1}{(L+\epsilon)^2}$.
Since $\delta > 0$ and $L + \epsilon > 0$, then $x > 0$.
Since $0 < x = |x| \leq \frac{\delta}{2} < \delta$, then $0 < |x| < \delta$.
Since $0 < x \leq \frac{1}{(L+\epsilon)^2}$, then $0 < \sqrt{x} \leq \frac{1}{L+\epsilon}$, so $L + \epsilon \leq \frac{1}{\sqrt{x}}$.
Hence, $0 < \epsilon \leq \frac{1}{\sqrt{x}} - L$, so $0 < \frac{1}{\sqrt{x}} - L$ and $\epsilon \leq \frac{1}{\sqrt{x}} - L$.

Therefore, $|f(x) - L| = |\frac{1}{\sqrt{x}} - L| = \frac{1}{\sqrt{x}} - L \geq \epsilon$, so $|f(x) - L| \geq \epsilon$.

Thus, there exists $x > 0$ such that $0 < |x| < \delta$ and $|f(x) - L| \geq \epsilon$, as desired. ☐

Exercise 27. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined for all $x \in \mathbb{R}$.

Let $a, L \in \mathbb{R}$.

Then $\lim_{x \to a} f(x) = L$ iff $\lim_{x \to 0} f(x + a) = L$.

Proof. We first prove if $\lim_{x \to a} f(x) = L$, then $\lim_{x \to 0} f(x + a) = L$.

Suppose $\lim_{x \to a} f(x) = L$.

Let $\epsilon > 0$ be given.

Then there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Let $x \in \mathbb{R}$ such that $0 < |x| < \delta$.

Since $x \in \mathbb{R}$, then $x + a \in \mathbb{R}$, so if $0 < |(x+a) - a| < \delta$, then $|f(x+a) - L| < \epsilon$.

Hence, if $0 < |x| < \delta$, then $|f(x + a) - L| < \epsilon$.

Since $0 < |x| < \delta$, then we conclude $|f(x + a) - L| < \epsilon$.

Therefore, $\lim_{x \to 0} f(x + a) = L$.

Conversely, we prove if $\lim_{x \to 0} f(x + a) = L$, then $\lim_{x \to a} f(x) = L$.

Suppose $\lim_{x \to 0} f(x + a) = L$.

Let $\epsilon > 0$ be given.

Then there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x| < \delta$, then $|f(x + a) - L| < \epsilon$.

Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$.

Since $x \in \mathbb{R}$, then $x - a \in \mathbb{R}$, so if $0 < |x - a| < \delta$, then $|f((x-a)+a) - L| < \epsilon$.

Hence, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Since $0 < |x - a| < \delta$, then we conclude $|f(x) - L| < \epsilon$.

Therefore, $\lim_{x \to a} f(x) = L$. ☐

Exercise 28. Let $a, L \in \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x \to a}(f(x))^2 = L$.

If $L = 0$, then $\lim_{x \to a} f(x) = 0$.

Provide an example such that $L \neq 0$ and $\lim_{x \to a} f(x)$ does not exist.

Proof. We must prove if $L = 0$, then $\lim_{x \to a} f(x) = 0$.

Suppose $L = 0$.

Then $\lim_{x \to a}(f(x))^2 = 0$.

To prove $\lim_{x \to a} f(x) = 0$, let $\epsilon > 0$ be given.

Then $\epsilon^2 > 0$.

Since $\lim_{x \to a}(f(x))^2 = 0$, then there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|(f(x))^2| < \epsilon^2$.

Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$.

Then $|(f(x))^2| < \epsilon^2$, so $0 \leq |(f(x))^2| < \epsilon^2$.

Hence, $0 \leq |f(x)|^2 < \epsilon^2$, so $0 \leq |f(x)| < \epsilon$.

Therefore, $|f(x)| < \epsilon$, as desired. ☐

Solution. Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = 1$ if $x > 0$ and $f(x) = -1$ if $x < 0$.

We shall show that $\lim_{x\to 0}(f(x))^2 = 1 \neq 0$ and $\lim_{x\to 0} f(x)$ does not exist.

We first prove $\lim_{x\to 0}(f(x))^2 = 1$.

Observe that 0 is an accumulation point of $\mathbb{R}^*$, the domain of f.

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon$.

Then $\delta > 0$.

Let $x \in \mathbb{R}^*$.

Then either $x > 0$ or $x < 0$.

If $x > 0$, then $(f(x))^2 = 1^2 = 1$.

If $x < 0$, then $(f(x))^2 = (-1)^2 = 1$.

Thus, in all cases, $(f(x))^2 = 1$.

Since $|(f(x))^2 - 1| = |1 - 1| = 0 < \epsilon$, then $|(f(x))^2 - 1| < \epsilon$.

Hence, the implication if $0 < |x| < \delta$, then $|(f(x))^2 - 1| < \epsilon$ is trivially true.

Therefore, $\lim_{x\to 0}(f(x))^2 = 1$.

We next prove $\lim_{x\to 0} f(x)$ does not exist.

We already proved by contradiction that the function f fails to have a limit at 0 in the list of examples.

It has a jump discontinuity at zero.

Therefore, $\lim_{x\to 0} f(x)$ does not exist. □

Exercise 29. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x$ if x is rational and $f(x) = 0$ if x is irrational.

Show that $\lim_{x\to 0} f(x) = 0$.

Show that if $a \neq 0$, then the limit of f at a does not exist.

Proof. Observe that 0 is an accumulation point of $\mathbb{R}$, the domain of f.

We prove $\lim_{x\to 0} f(x) = 0$.

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon$.

Then $\delta > 0$.

Let $x \in \mathbb{R}$ such that $0 < |x| < \delta$.

Since $x \in \mathbb{R}$, then either x is rational or x is not rational.

If x is rational, then $|f(x)| = |x| < \delta = \epsilon$, so $|f(x)| < \epsilon$.

If x is not rational, then x is irrational, so $|f(x)| = 0 < \epsilon$.

Therefore, in all cases, $|f(x)| < \epsilon$, so $\lim_{x\to 0} f(x) = 0$, as desired. □

Proof. Let $a \neq 0$ be given.

Observe that a is an accumulation point of $\mathbb{R}$, the domain of f.

To prove the limit of f at a does not exist, we must prove there is no real L such that $\lim_{x\to a} f(x) = L$.

Observe that

$$\neg(\exists L \in \mathbb{R})(\lim_{x\to a} f(x) = L) \quad \Leftrightarrow$$
$$\neg(\exists L \in \mathbb{R})(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(0 < |x-a| < \delta \rightarrow |f(x) - L| < \epsilon) \quad \Leftrightarrow$$
$$(\forall L \in \mathbb{R})(\exists \epsilon > 0)(\forall \delta > 0)(\exists x)(0 < |x-a| < \delta \wedge |f(x) - L| \geq \epsilon).$$

Thus, we prove $(\forall L \in \mathbb{R})(\exists \epsilon > 0)(\forall \delta > 0)(\exists x)(0 < |x-a| < \delta \wedge |f(x) - L| \geq \epsilon)$.

Let L be an arbitrary real number.

Either $L = 0$ or $L \neq 0$.

We consider these cases separately.

Case 1: Suppose $L \neq 0$.

Let $\epsilon = |L|$.

Since $L \neq 0$, then $|L| > 0$, so $\epsilon > 0$.

Let $\delta > 0$ be given.

Since a and $a + \delta$ are real numbers and $a < a + \delta$, then there exists an irrational number x such that $a < x < a + \delta$.

Thus, $a < x$ and $0 < x - a < \delta$.

Since $0 < x - a < \delta$, then $|x - a| = x - a < \delta$, so $|x - a| < \delta$.

Since $x > a$, then $x - a > 0$, so $0 < |x - a| < \delta$.

Since x is irrational, then $|f(x) - L| = |0 - L| = |-L| = |L| = \epsilon$, so $|f(x) - L| = \epsilon$.

Thus, $|f(x) - L| \geq \epsilon$.

Case 2: Suppose $L = 0$.

Let $\epsilon = |a|$.

Since $a \neq 0$, then $|a| > 0$, so $\epsilon > 0$.

Let $\delta > 0$ be given.

Since $a \neq 0$, then either $a > 0$ or $a < 0$.

Case 2a: Suppose $a > 0$.

Since a and $a+\delta$ are real numbers and $a < a+\delta$, then there exists a rational number x such that $a < x < a + \delta$.

Thus, $a < x$ and $0 < x - a < \delta$.

Since $0 < x - a < \delta$, then $|x - a| = x - a < \delta$, so $|x - a| < \delta$.

Since $x > a$, then $x - a > 0$, so $0 < |x - a| < \delta$.

Since $0 < a < x$, then $|x| = x > a = |a|$, so $|x| > |a|$.

Since x is rational, then $|f(x) - L| = |x - 0| = |x| > |a| = \epsilon$, so $|f(x) - L| > \epsilon$.

Case 2b: Suppose $a < 0$.

Since a and $a+\delta$ are real numbers and $a-\delta < a$, then there exists a rational number x such that $a - \delta < x < a$.

Thus, $x < a$ and $-\delta < x - a < 0$.

Since $\delta > x - a > 0$, then $|x - a| = x - a < \delta$, so $|x - a| < \delta$.

Since $x < a$, then $x - a < 0$, so $x - a \neq 0$.

Hence, $|x - a| > 0$, so $0 < |x - a| < \delta$.

Since $x < a < 0$, then $|x| = -x > -a = |a|$, so $|x| > |a|$.

Since x is rational, then $|f(x) - L| = |x - 0| = |x| > |a| = \epsilon$, so $|f(x) - L| > \epsilon$.

Therefore, in either case, $|f(x) - L| > \epsilon$, so $|f(x) - L| \geq \epsilon$.

Thus, there is no real L such that $\lim_{x \to a} f(x) = L$, so if $a \neq 0$, then the limit of f at a does not exist. $\square$

Exercise 30. Given that the definition of limit requires that a be an accumulation point of $domf$, what real values of a would be excluded from consideration in the limit $\lim_{x \to a} \sqrt{x^2 - 2}$?

Solution. Let $f : [\sqrt{2}, \infty) \to \mathbb{R}$ be the function given by $f(x) = \sqrt{x^2 - 2}$.

The domain of f is the interval $[\sqrt{2}, \infty)$, a closed, unbounded set.

Thus, $domf = [\sqrt{2}, \infty)$.

Values of a that should be included in the consideration of a limit must be accumulation points of $domf$.

Hence, values of a that should be excluded from consideration of a limit must not be accumulation points of $domf$.

Let S be the set of all real values of a that are excluded from consideration of a limit of f at a.

Then S is the set of all real values of a that are not accumulation points of $domf$.

Thus, $S = \{x \in \mathbb{R} : x \text{ is not an accumulation point of } domf\}$.

We prove $S = (-\infty, \sqrt{2})$.

We first prove $(-\infty, \sqrt{2}) \subset S$.

Let $t \in (-\infty, \sqrt{2})$.

Since $domf$ is a closed set, then if x is an accumulation point of $domf$, then $x \in domf$.

Hence, if $x \notin domf$, then x is not an accumulation point of $domf$.

Since $t \in (-\infty, \sqrt{2})$ and the interval $(-\infty, \sqrt{2})$ is the complement of $[\sqrt{2}, \infty) = domf$, then $t \notin domf$.

Thus, t is not an accumulation point of $domf$, so $t \in S$.

Hence, $(-\infty, \sqrt{2}) \subset S$.

We now prove $S \subset (-\infty, \sqrt{2})$.

Let $s \in S$.

Then $s \in \mathbb{R}$ and s is not an accumulation point of $domf$.

Suppose for the sake of contradiction $s \in domf$.

Then $s \in [\sqrt{2}, \infty)$, so s is an accumulation point of $[\sqrt{2}, \infty)$.

Hence, s is an accumulation point of $domf$, contradicting the fact that s is not an accumulation point of $domf$.

Therefore, $s \notin domf$, so s is in the complement of $domf$.

Thus, $s \in (-\infty, \sqrt{2})$, the complement of $domf$.

Hence, $S \subset (-\infty, \sqrt{2})$.

Since $S \subset (-\infty, \sqrt{2})$ and $(-\infty, \sqrt{2}) \subset S$, then $S = (-\infty, \sqrt{2})$. $\square$

Exercise 31. Let $f : (0, 1) \to \mathbb{R}$ be a function defined by $f(x) = \sin(\frac{1}{x})$ for all $x \in (0, 1)$.

Then $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist in $\mathbb{R}$.

Solution.

Observe that 0 is an accumulation point of $(0,1)$, the domain of f.

We prove there is no real L such that $\lim_{x\to 0} f(x) = L$.

Observe that

$$\begin{aligned} \neg(\exists L\in\mathbb{R})(\lim_{x\to 0} f(x) = L) &\Leftrightarrow \\ \neg(\exists L\in\mathbb{R})(\forall \epsilon>0)(\exists\delta>0)(\forall x\in(0,1))(0<|x|<\delta \to |f(x)-L|<\epsilon) &\Leftrightarrow \\ (\forall L\in\mathbb{R})(\exists\epsilon>0)(\forall\delta>0)(\exists x\in(0,1))(0<|x|<\delta \wedge |f(x)-L|\geq\epsilon). \end{aligned}$$

Thus, we prove $(\forall L\in\mathbb{R})(\exists\epsilon>0)(\forall\delta>0)(\exists x\in(0,1))(0<|x|<\delta \wedge |f(x)-L|\geq\epsilon)$.

Let L be an arbitrary real number.

Let $\epsilon = \frac{1}{3}$.

Let $\delta > 0$ be given.

We must prove there exists $x \in (0,1)$ such that $0<|x|<\delta$ and $|f(x)-L|\geq\epsilon$.

We first prove either $0 \notin N(L;\epsilon)$ or $1 \notin N(L;\epsilon)$ by contradiction.

Suppose $0 \in N(L;\epsilon)$ and $1 \in N(L;\epsilon)$.

Then $d(0,L)<\epsilon$ and $d(1,L)<\epsilon$, so $|0-L|<\epsilon$ and $|1-L|<\epsilon$.

Hence, $|L|<\epsilon$ and $|1-L|<\epsilon$.

Observe that

$$\begin{aligned} 1 &= |(1-L)+L| \\ &\leq |1-L|+|L| \\ &< \epsilon+\epsilon \\ &= \frac{1}{3}+\frac{1}{3} \\ &= \frac{2}{3}. \end{aligned}$$

Therefore, $1 < \frac{2}{3}$, a contradiction.

Hence, either $0 \notin N(L;\epsilon)$ or $1 \notin N(L;\epsilon)$.

We consider these cases separately.

Case 1: Suppose $0 \notin N(L;\epsilon)$.

Then $d(0,L)\geq\epsilon$, so $|0-L|\geq\epsilon$.

Since $\delta>0$, then $\delta\pi>0$, so by the Archimedean property of $\mathbb{R}$, there exists $n\in\mathbb{N}$ such that $\frac{1}{n}<\delta\pi$.

Hence, $\frac{1}{n\pi}<\delta$.

Let $x = \frac{1}{n\pi}$.

Since $0<\frac{1}{\pi}<n$ for any $n\in\mathbb{N}$, then $0<1<n\pi$, so $0<\frac{1}{n\pi}<1$.

Thus, $\frac{1}{n\pi}\in(0,1)$.

Since $0<\frac{1}{n\pi}=|\frac{1}{n\pi}|<\delta$, then $0<|\frac{1}{n\pi}|<\delta$.

Since $|f(\frac{1}{n\pi}) - L| = |\sin(n\pi) - L| = |0 - L| \geq \epsilon$, then $|f(\frac{1}{n\pi}) - L| \geq \epsilon$.

Case 2: Suppose $1 \notin N(L; \epsilon)$.

Then $d(1, L) \geq \epsilon$, so $|1 - L| \geq \epsilon$.

Since $\delta > 0$, then $\frac{\delta\pi}{2} > 0$, so by the Archimedean property of $\mathbb{R}$, there exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{\delta\pi}{2}$.

Thus, $\frac{2}{k\pi} < \delta$.

Let $m = 4k + 1$.

Then $m - 1 = 4k$, so $4|(m - 1)$.

Hence, $m \equiv 1 \pmod 4$.

Since $m - k = (4k + 1) - k = 3k + 1 > 0$, then $m - k > 0$, so $m > k$.

Since $m > k > 0$, then $0 < \frac{1}{m} < \frac{1}{k}$, so $0 < \frac{2}{m\pi} < \frac{2}{k\pi}$.

Thus, $0 < \frac{2}{m\pi} < \frac{2}{k\pi} < \delta$, so $0 < \frac{2}{m\pi} < \delta$.

Let $x = \frac{2}{m\pi}$.

Since $0 < \frac{2}{\pi} < m$ for any $m \in \mathbb{N}$, then $0 < 2 < m\pi$, so $0 < \frac{2}{m\pi} < 1$.

Thus, $\frac{2}{m\pi} \in (0, 1)$.

Since $0 < \frac{2}{m\pi} = |\frac{2}{m\pi}| < \delta$, then $0 < |\frac{2}{m\pi}| < \delta$.

Since $m \equiv 1 \pmod 4$, then $|f(\frac{2}{m\pi}) - L| = |\sin(\frac{m\pi}{2}) - L| = |1 - L| \geq \epsilon$, so $|f(\frac{2}{m\pi}) - L| \geq \epsilon$.

Therefore, in all cases, there exists $x \in (0, 1)$ such that $0 < |x| < \delta$ and $|f(x) - L| \geq \epsilon$, as desired. $\square$

Exercise 32. Show that $\lim_{x\to 0} \cos(\frac{1}{x})$ does not exist in $\mathbb{R}$.

Solution. Let $f : \mathbb{R} - \{0\} \to \mathbb{R}$ be a function defined by $f(x) = \cos(\frac{1}{x})$.

Since 0 is an accumulation point of $\mathbb{R}$, then 0 is an accumulation point of $\mathbb{R} - \{0\}$, the domain of f.

Suppose $\lim_{x\to 0} f(x)$ does exist in $\mathbb{R}$.

Then there is a real number L such that $\lim_{x\to 0} \cos(\frac{1}{x}) = L$.

Thus, for every $\epsilon > 0$, there is $\delta > 0$ such that for every $x \neq 0$, if $0 < |x| < \delta$, then $|\cos(\frac{1}{x}) - L| < \epsilon$.

Let $\epsilon = \frac{1}{2}$.

Then there is $\delta > 0$ such that for every $x \in \mathbb{R} - \{0\}$, if $0 < |x| < \delta$, then $|\cos(\frac{1}{x}) - L| < \frac{1}{2}$.

Let $M = \frac{1}{2\pi\delta}$.

Then $M \in \mathbb{R}$, so by the Archimedean property of $\mathbb{R}$, there exists $n_1 \in \mathbb{N}$ such that $n_1 > M$.

Let $x_1 = \frac{1}{2\pi n_1}$.

Since $n_1 \in \mathbb{N}$, then $n_1 > 0$, so $x_1 > 0$.

Hence, $x_1 \neq 0$.

Since $n_1 > M$ and $M = \frac{1}{2\pi\delta}$, then $n_1 > \frac{1}{2\pi\delta}$.

Since $\delta > 0$ and $n_1 > 0$, then $\delta > \frac{1}{2\pi n_1}$, so $\delta > x_1$.

Thus, $\delta > x_1 = |x_1| > 0$.

Since $x_1 \neq 0$ and $0 < |x_1| < \delta$, then $|\cos(\frac{1}{x_1}) - L| < \frac{1}{2}$.

Let $N = \frac{\frac{1}{\pi\delta}-1}{2}$.

Then $N \in \mathbb{R}$, so by the Archimedean property of $\mathbb{R}$, there exists $n_2 \in \mathbb{N}$ such that $n_2 > N$.

Let $x_2 = \frac{1}{(2n_2+1)\pi}$.

Since $n_2 \in \mathbb{N}$, then $n_2 > 0$, so $2n_2 + 1 > 0$.

Hence, $x_2 > 0$, so $x_2 \neq 0$.

Since $n_2 > N$ and $N = \frac{\frac{1}{\pi\delta}-1}{2}$, then $n_2 > \frac{\frac{1}{\pi\delta}-1}{2}$.

Thus, $2n_2 > \frac{1}{\pi\delta} - 1$, so $2n_2 + 1 > \frac{1}{\pi\delta}$.

Since $\delta > 0$ and $n_2 > 0$, then $\delta > \frac{1}{(2n_2+1)\pi}$, so $\delta > x_2$.

Thus, $\delta > x_2 = |x_2| > 0$.

Since $x_2 \neq 0$ and $0 < |x_2| < \delta$, then $|\cos(\frac{1}{x_2}) - L| < \frac{1}{2}$.

Observe that

$$\begin{aligned}
2 &= |1-(-1)| \\
&= |\cos(2\pi n_1) - \cos((2n_2+1)\pi)| \\
&= |\cos(\frac{1}{\frac{1}{2\pi n_1}}) - \cos(\frac{1}{\frac{1}{(2n_2+1)\pi}})| \\
&= |\cos(\frac{1}{x_1}) - \cos(\frac{1}{x_2})| \\
&= |\cos(\frac{1}{x_1}) - L + L - \cos(\frac{1}{x_2})| \\
&\leq |\cos(\frac{1}{x_1}) - L| + |L - \cos(\frac{1}{x_2})| \\
&= |\cos(\frac{1}{x_1}) - L| + |\cos(\frac{1}{x_2}) - L| \\
&< \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}$$

Thus, we have $2 < 1$, a contradiction.

Therefore, $\lim_{x\to 0} \cos(\frac{1}{x})$ does not exist in $\mathbb{R}$. ∎

Exercise 33. Show that $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$.

Proof. Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = x\sin(\frac{1}{x})$ for all $x \neq 0$.

Observe that 0 is an accumulation point of the set $\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\}$, the domain of f.

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon$.

Then $\delta > 0$.

Let $x \in \mathbb{R}^*$ such that $0 < |x| < \delta$.

Since $x \in \mathbb{R}^*$, then $x \in \mathbb{R}$ and $x \neq 0$, so $\frac{1}{x} \in \mathbb{R}$.

Since $|\sin\theta| \leq 1$ for all $\theta \in \mathbb{R}$, then in particular, $|\sin(\frac{1}{x})| \leq 1$, so $0 \leq |\sin(\frac{1}{x})| \leq 1$.

Therefore, $|x\sin(\frac{1}{x})| = |x| \cdot |\sin(\frac{1}{x})| \leq |x| < \delta = \epsilon$.

Thus, $|x\sin(\frac{1}{x})| < \epsilon$, as desired. $\square$

Proof. Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = x\sin(\frac{1}{x})$ for all $x \neq 0$.

Observe that 0 is an accumulation point of the set $\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\}$, the domain of f.

Let $x \in \mathbb{R}^*$.

Then $x \in \mathbb{R}$ and $x \neq 0$.

Hence, $|x| > 0$ and $-|x| < 0$ and $\frac{1}{x} \in \mathbb{R}$.

Since $|\sin\theta| \leq 1$ for all $\theta \in \mathbb{R}$, then in particular, $|\sin(\frac{1}{x})| \leq 1$.

Since $x \neq 0$, then $x\sin(\frac{1}{x}) \in \mathbb{R}$, so $-|x\sin(\frac{1}{x})| \leq x\sin(\frac{1}{x}) \leq |x\sin(\frac{1}{x})|$.

Thus, $-|x\sin(\frac{1}{x})| \leq x\sin(\frac{1}{x})$ and $x\sin(\frac{1}{x}) \leq |x\sin(\frac{1}{x})|$.

Since $x\sin(\frac{1}{x}) \leq |x\sin(\frac{1}{x})| = |x||\sin(\frac{1}{x})| \leq |x|$, then $x\sin(\frac{1}{x}) \leq |x|$.

Since $-|x\sin(\frac{1}{x})| \leq x\sin(\frac{1}{x})$, then $x\sin(\frac{1}{x}) \geq -|x\sin(\frac{1}{x})| = -|x||\sin(\frac{1}{x})| \geq -|x|$, so $x\sin(\frac{1}{x}) \geq -|x|$.

Thus, $-|x| \leq x\sin(\frac{1}{x}) \leq |x|$, so $-|x| \leq x\sin(\frac{1}{x}) \leq |x|$ for all $x \in \mathbb{R}^*$.

Since $-|x| \leq x\sin(\frac{1}{x}) \leq |x|$ for all $x \in \mathbb{R}^*$ and $\lim_{x\to 0}|x| = |0| = 0 = -|0| = -\lim_{x\to 0}|x| = \lim_{x\to 0} -|x|$, then by the squeeze rule, $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$. $\square$

Proof. Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = x\sin(\frac{1}{x})$ for all $x \neq 0$.

Observe that 0 is an accumulation point of the set $\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\}$, the domain of f.

We prove by using the sequential criterion for function limits.

Since 0 is an accumulation point of $\mathbb{R}^*$, then there exists a sequence (x_n) of points in $\mathbb{R}^*$ such that $\lim_{n\to\infty} x_n = 0$.

Let (x_n) be an arbitrary sequence of points in $\mathbb{R}^*$ such that $\lim_{n\to\infty} x_n = 0$.

Since (x_n) is a sequence of points in $\mathbb{R}^* = \mathbb{R} - \{0\}$, then $x_n \neq 0$ for all $n \in \mathbb{N}$.

We first prove $\lim_{n\to\infty} f(x_n) = 0$.

Let $\epsilon > 0$ be given.

Since $\lim_{n\to\infty} x_n = 0$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|x_n| < \epsilon$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|x_n| < \epsilon$, so $0 < |x_n| < \epsilon$.

Since $x_n \neq 0$, then $\frac{1}{x_n} \in \mathbb{R}$.

Since $|\sin\theta| \leq 1$ for all $\theta \in \mathbb{R}$, then in particular, $|\sin(\frac{1}{x_n})| \leq 1$, so $0 \leq |\sin(\frac{1}{x_n})| \leq 1$.

Hence, $|f(x_n)| = |x_n\sin(\frac{1}{x_n})| = |x_n| \cdot |\sin(\frac{1}{x_n})| \leq |x_n| < \epsilon$, so $\lim_{n\to\infty} f(x_n) = 0$.

Since (x_n) is an arbitrary sequence of points in $\mathbb{R}^* = domf = domf - \{0\}$ such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} f(x_n) = 0$, then by the sequential criterion for function limits, $\lim_{x\to 0} f(x) = 0$.

Therefore, $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$. $\square$

Algebraic properties of function limits

Exercise 34. Let $f : (0, 1) \to \mathbb{R}$ be a function defined by $f(x) = \frac{x^3+6x^2+x}{x^2-6x}$.

Then $\lim_{x\to 0} f(x) = \frac{-1}{6}$.

Proof. Observe that 0 is an accumulation point of the interval $(0, 1)$, the domain of f.

Let $x \in (0, 1)$.

Then $0 < x < 1$, so $0 < x$.

Since $x > 0$, then $x \neq 0$.

Thus, $x \neq 0$ for all $x \in (0, 1)$, so

$$\begin{aligned} \lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{x^3 + 6x^2 + x}{x^2 - 6x} \\ &= \lim_{x\to 0} \frac{x(x^2 + 6x + 1)}{x(x - 6)} \\ &= \lim_{x\to 0} \frac{x^2 + 6x + 1}{x - 6}. \end{aligned}$$

Since $\lim_{x\to 0}(x^2 + 6x + 1) = 1$ and $\lim_{x\to 0}(x - 6) = -6 \neq 0$, then by the quotient rule for limits we have

$$\begin{aligned} \lim_{x\to 0} \frac{x^2 + 6x + 1}{x - 6} &= \frac{\lim_{x\to 0}(x^2 + 6x + 1)}{\lim_{x\to 0}(x - 6)} \\ &= \frac{1}{-6}. \end{aligned}$$

Therefore, $\lim_{x\to 0} f(x) = \frac{-1}{6}$. ∎

Exercise 35. Show that $\lim_{x\to -1} \frac{x+1}{x^3+1} = \frac{1}{3}$.

Proof. Let $f : \mathbb{R} - \{-1\} \to \mathbb{R}$ be a function defined by $f(x) = \frac{x+1}{x^3+1}$.

Since -1 is an accumulation point of $\mathbb{R}$, then -1 is an accumulation point of $\mathbb{R} - \{-1\}$, the domain of f.

Since $x^3 + 1 = (x + 1)(x^2 - x + 1)$, then $\frac{x^3+1}{x+1} = x^2 - x + 1$ if $x \neq -1$.

Let $x \in \mathbb{R} - \{-1\}$.

Then $x \in \mathbb{R}$ and $x \neq -1$, so $\frac{x^3+1}{x+1} = x^2 - x + 1$.

Since $\lim_{x\to -1} 1 = 1$ and $\lim_{x\to -1}(x^2 - x + 1) = 3 \neq 0$, then

$$\begin{aligned} \frac{1}{3} &= \frac{\lim_{x\to -1} 1}{\lim_{x\to -1}(x^2 - x + 1)} \\ &= \lim_{x\to -1} \frac{1}{x^2 - x + 1} \\ &= \lim_{x\to -1} \frac{1}{\frac{x^3+1}{x+1}} \\ &= \lim_{x\to -1} \frac{x + 1}{x^3 + 1}. \end{aligned}$$

Therefore, $\lim_{x\to -1} \frac{x+1}{x^3+1} = \frac{1}{3}$. ☐

Exercise 36. Show that $\lim_{h\to 0} \frac{\sin(\frac{\pi}{6}+h)-\frac{1}{2}}{h} = \frac{\sqrt{3}}{2}$.

Solution. We observe that this limit is simply the definition of the derivative of sin at $\frac{\pi}{6}$ which is $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$. ☐

Proof. Since $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$, then

$$\begin{aligned}
\frac{\sqrt{3}}{2} &= \frac{1}{2}\cdot\sqrt{3}\cdot 1 \\
&= \frac{1}{2}\cdot\sqrt{3}\cdot\lim_{h\to 0}\frac{\sin h}{h} \\
&= \frac{1}{2}\cdot\lim_{h\to 0}\frac{\sqrt{3}\sin h}{h} \\
&= \frac{1}{2}(0+\lim_{h\to 0}\frac{\sqrt{3}\sin h}{h}) \\
&= \frac{1}{2}(\lim_{h\to 0}\frac{\cos h-1}{h}+\lim_{h\to 0}\frac{\sqrt{3}\sin h}{h}) \\
&= \frac{1}{2}\cdot\lim_{h\to 0}(\frac{\cos h-1}{h}+\frac{\sqrt{3}\sin h}{h}) \\
&= \frac{1}{2}\cdot\lim_{h\to 0}\frac{\cos h-1+\sqrt{3}\sin h}{h} \\
&= \lim_{h\to 0}\frac{1}{2}\cdot\frac{\cos h-1+\sqrt{3}\sin h}{h} \\
&= \lim_{h\to 0}\frac{\frac{\cos h}{2}-\frac{1}{2}+\frac{\sqrt{3}}{2}\sin h}{h} \\
&= \lim_{h\to 0}\frac{\frac{\cos h}{2}+\frac{\sqrt{3}}{2}\sin h-\frac{1}{2}}{h} \\
&= \lim_{h\to 0}\frac{\sin(\frac{\pi}{6})\cos(h)+\cos(\frac{\pi}{6})\sin h-\frac{1}{2}}{h} \\
&= \lim_{h\to 0}\frac{\sin(\frac{\pi}{6}+h)-\frac{1}{2}}{h}.
\end{aligned}$$

Therefore, $\lim_{h\to 0} \frac{\sin(\frac{\pi}{6}+h)-\frac{1}{2}}{h} = \frac{\sqrt{3}}{2}$. ☐

Exercise 37. Let $f : (0, 1) \to \mathbb{R}$ be a function defined by $f(x) = \frac{\sqrt{1+x}-1}{x}$.
Then $\lim_{x\to 0} f(x) = \frac{1}{2}$.

Proof. Observe that 0 is an accumulation point of the interval $(0, 1)$, the domain of f.

Let $x \in (0, 1)$.

Then $0 < x < 1$, so $0 < x$.

Since $x > 0$, then $x \neq 0$.
Since $-1 < 0$ and $0 < x$, then $-1 < x$, so $0 < 1 + x$.
Hence, $1 + x > 0$, so $\sqrt{1+x} > 0$.
Thus, $\sqrt{1+x} + 1 > 1 > 0$, so $\sqrt{1+x} + 1 > 0$.
Therefore, $\sqrt{1+x} + 1 \neq 0$.
Since $x \neq 0$ and $\sqrt{1+x} + 1 \neq 0$, then

$$\begin{aligned}
\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} \\
&= \lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} \cdot \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} \\
&= \lim_{x\to 0} \frac{(1+x) - 1}{x(\sqrt{1+x} + 1)} \\
&= \lim_{x\to 0} \frac{x}{x(\sqrt{1+x} + 1)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{1+x} + 1}.
\end{aligned}$$

Since $\lim_{x\to 0}(1+x) = 1 > 0$, then $\lim_{x\to 0} \sqrt{1+x} = \sqrt{\lim_{x\to 0}(1+x)}$.
Since $\lim_{x\to 0} 1 = 1$, then

$$\begin{aligned}
2 &= \sqrt{1} + 1 \\
&= \sqrt{\lim_{x\to 0}(1+x)} + \lim_{x\to 0} 1 \\
&= \lim_{x\to 0} \sqrt{1+x} + \lim_{x\to 0} 1 \\
&= \lim_{x\to 0}(\sqrt{1+x} + 1).
\end{aligned}$$

Since $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0}(\sqrt{1+x} + 1) = 2 \neq 0$, then by the quotient rule

$$\begin{aligned}
\lim_{x\to 0} \frac{1}{\sqrt{1+x} + 1} &= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0}(\sqrt{1+x} + 1)} \\
&= \frac{1}{2}.
\end{aligned}$$

Therefore, $\lim_{x\to 0} f(x) = \frac{1}{2}$. $\square$

Exercise 38. Let $f : (0,1) \to \mathbb{R}$ be a function defined by $f(x) = \frac{\sqrt{9-x}-3}{x}$.
Then $\lim_{x\to 0} f(x) = \frac{-1}{6}$.

Proof. Observe that 0 is an accumulation point of the interval $(0,1)$, the domain of f.
Let $x \in (0,1)$.

Then $0 < x < 1$, so $0 < x$ and $x < 1$.
Since $x > 0$, then $x \neq 0$.
Since $x < 1$ and $1 < 9$, then $x < 9$, so $0 < 9 - x$.
Hence, $9 - x > 0$, so $\sqrt{9-x} > 0$.
Thus, $\sqrt{9-x} + 3 > 3 > 0$, so $\sqrt{9-x} + 3 > 0$.
Therefore, $\sqrt{9-x} + 3 \neq 0$.
Since $x \neq 0$ and $\sqrt{9-x} + 3 \neq 0$, then

$$\begin{aligned}
\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{\sqrt{9-x}-3}{x} \\
&= \lim_{x\to 0} \frac{\sqrt{9-x}-3}{x} \cdot \frac{\sqrt{9-x}+3}{\sqrt{9-x}+3} \\
&= \lim_{x\to 0} \frac{(9-x)-9}{x(\sqrt{9-x}+3)} \\
&= \lim_{x\to 0} \frac{-x}{x(\sqrt{9-x}+3)} \\
&= \lim_{x\to 0} \frac{-1}{\sqrt{9-x}+3}.
\end{aligned}$$

Since $\lim_{x\to 0}(9-x) = 9 > 0$, then $\lim_{x\to 0} \sqrt{9-x} = \sqrt{\lim_{x\to 0}(9-x)}$.
Since $\lim_{x\to 0} 3 = 3$, then

$$\begin{aligned}
6 &= \sqrt{9} + 3 \\
&= \sqrt{\lim_{x\to 0}(9-x)} + \lim_{x\to 0} 3 \\
&= \lim_{x\to 0} \sqrt{9-x} + \lim_{x\to 0} 3 \\
&= \lim_{x\to 0}(\sqrt{9-x} + 3).
\end{aligned}$$

Since $\lim_{x\to 0} -1 = -1$ and $\lim_{x\to 0}(\sqrt{9-x}+3) = 6 \neq 0$, then by the quotient rule

$$\begin{aligned}
\lim_{x\to 0} \frac{-1}{\sqrt{9-x}+3} &= \frac{\lim_{x\to 0} -1}{\lim_{x\to 0}(\sqrt{9-x}+3)} \\
&= \frac{-1}{6}.
\end{aligned}$$

Therefore, $\lim_{x\to 0} f(x) = \frac{-1}{6}$. ☐

Exercise 39. Let f and g be real valued functions defined on $E \subset \mathbb{R}$.
Let a be an accumulation point of E.
a. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a}(f+g)(x)$ exist, then $\lim_{x\to a} g(x)$ exists.
b. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a}(fg)(x)$ exist, then it does not necessarily follow that $\lim_{x\to a} g(x)$ exists.

Proof. We prove a.

Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a}(f+g)(x)$ exist.

Then there exist real numbers L and M such that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a}(f+g)(x) = M$.

Thus,

$$\begin{aligned} M - L &= \lim_{x\to a}(f+g)(x) - \lim_{x\to a} f(x) \\ &= \lim_{x\to a}[f(x)+g(x)] - \lim_{x\to a} f(x) \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) - \lim_{x\to a} f(x) \\ &= \lim_{x\to a} g(x). \end{aligned}$$

Therefore, $\lim_{x\to a} g(x) = M - L$, so the limit of g at a exists. $\square$

Solution. We provide a counterexample to the assertion that $\lim_{x\to a} f(x)$ and $\lim_{x\to a}(fg)(x)$ exist implies $\lim_{x\to a} g(x)$ exists.

Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = x$.

Let $g : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $g(x) = \sin(\frac{1}{x})$.

Then $\lim_{x\to 0} f(x) = \lim_{x\to 0} x = 0$ and $\lim_{x\to 0}(fg)(x) = \lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$, but $\lim_{x\to 0}\sin(\frac{1}{x})$ does not exist. $\square$

Exercise 40. Show that $\lim_{x\to 0} x^2\cos(\frac{1}{x}) = 0$.

Proof. Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = x^2\cos(\frac{1}{x})$ for all $x \neq 0$.

Observe that 0 is an accumulation point of the set $\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\}$, the domain of f.

Let $\epsilon > 0$ be given.

Then $\sqrt{\epsilon} > 0$.

Let $\delta = \sqrt{\epsilon}$.

Then $\delta > 0$.

Let $x \in \mathbb{R}^*$ such that $0 < |x| < \delta$.

Since $0 < |x| < \delta$, then $0 < |x|$ and $|x| < \delta$.

Since $x \in \mathbb{R}^*$, then $x \in \mathbb{R}$ and $x \neq 0$, so $\frac{1}{x} \in \mathbb{R}$.

Since $|\cos\theta| \leq 1$ for all $\theta \in \mathbb{R}$, then in particular, $|\cos(\frac{1}{x})| \leq 1$, so $0 \leq |\cos(\frac{1}{x})| \leq 1$.

Therefore, $|x^2\cos(\frac{1}{x})| = |x^2| \cdot |\cos(\frac{1}{x})| = |x|^2 \cdot |\cos(\frac{1}{x})| \leq |x|^2 < \delta^2 = \epsilon$, so $|x^2\cos(\frac{1}{x})| < \epsilon$, as desired. $\square$

Exercise 41. Show that $\lim_{x\to c}(x^2+x+1) = c^2+c+1$ for any $c \in \mathbb{R}$.

Solution. Let $c \in \mathbb{R}$ be given.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2+x+1$.

Since f is a polynomial function and $c \in \mathbb{R}$, then we conclude that $\lim_{x\to c}(x^2+x+1) = \lim_{x\to c} f(x) = f(c) = c^2+c+1$. $\square$

Exercise 42. Let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions.

Let a be an accumulation point of E.

If $\lim_{x\to a} f(x) = 0$ and $g(x)$ is bounded locally near a, then $\lim_{x\to a} f(x)g(x) = 0$.

Proof. Suppose $\lim_{x\to a} f(x) = 0$ and $g(x)$ is bounded near a.

Let $\epsilon > 0$ be given.

Since $g(x)$ is bounded locally near a, then there exist δ_1 and $M > 0$ such that $|g(x)| \leq M$ for all $x \in N(a;\delta_1) \cap E$.

Since $\epsilon > 0$ and $M > 0$, then $\frac{\epsilon}{M} > 0$.

Since $\lim_{x\to a} f(x) = 0$, then there exists $\delta_2 > 0$ such that $|f(x)| < \frac{\epsilon}{M}$ for all $x \in \cap N'(a;\delta_2) \cap E$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.

Let $x \in E \cap N'(a;\delta)$.

Then $x \in E$ and $x \in N'(a;\delta)$.

Since $x \in N'(a;\delta)$, then $x \in N(a;\delta)$.

Since $0 < \delta \leq \delta_1$, then $N(a;\delta) \subset N(a;\delta_1)$.

Since $x \in N(a;\delta)$ and $N(a;\delta) \subset N(a;\delta_1)$, then $x \in N(a;\delta_1)$.

Since $x \in N(a;\delta_1)$ and $x \in E$, then $x \in N(a;\delta_1) \cap E$, so $|g(x)| \leq M$.

Since $0 < \delta \leq \delta_2$, then $N(a;\delta) \subset N(a;\delta_2)$.

Since $x \in N(a;\delta)$ and $N(a;\delta) \subset N(a;\delta_2)$, then $x \in N(a;\delta_2)$.

Since $x \in N(a;\delta_2)$ and $x \in E$, then $x \in N(a;\delta_2) \cap E$, so $|f(x)| < \frac{\epsilon}{M}$.

Since $0 \leq |f(x)| < \frac{\epsilon}{M}$ and $0 \leq |g(x)| < M$, then

$$\begin{aligned} |f(x)g(x)| &= |f(x)||g(x)| \\ &< \frac{\epsilon}{M} \cdot M \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x)g(x)| < \epsilon$, so $\lim_{x\to a} f(x)g(x) = 0$. □

Exercise 43. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions.

Let a be an accumulation point of A and B.

Assume $f(x) = g(x)$ for all $x \in A \cap B$.

a. What conditions on A and B ensure that if $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a} g(x)$ exists?

b. What conditions on A and B ensure that if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then they must be equal?

Proof. b. We assume that $B \subset A$.

Then if $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a} g(x)$ exists.

Suppose $\lim_{x\to a} f(x)$ exists and $B \subset A$.

Since $\lim_{x\to a} f(x)$ exists, then there is a real number L such that $\lim_{x\to a} f(x) = L$.

Let $\epsilon > 0$ be given.

Since $\lim_{x\to a} f(x) = L$, then there exists $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Since a is an accumulation point of B and $\delta > 0$, let $x \in B$ such that $0 < |x - a| < \delta$.

Since $x \in B$ and $B \subset A$, then $x \in A$.

Since $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Since $x \in A$ and $x \in B$, then $x \in A \cap B$, so $f(x) = g(x)$.
Hence, $|g(x) - L| = |f(x) - L| < \epsilon$, so $\lim_{x \to a} g(x) = L$.
Therefore, the limit of g at a exists. □

Proof. b. We assume that a is an accumulation point of $A \cap B$.

If this assumption holds, then if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

Suppose a is an accumulation point of $A \cap B$ and $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

Since $\lim_{x \to a} f(x)$ exists, then there is a real number L such that $\lim_{x \to a} f(x) = L$.

To prove $\lim_{x \to a} g(x) = L$, we must prove for every $\epsilon > 0$, there exists $\delta > 0$ such that $|g(x) - L| < \epsilon$ for all $x \in N'(a; \delta) \cap B$.

Let $\epsilon > 0$ be given.

Since $\lim_{x \to a} f(x) = L$, then there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in N'(a; \delta) \cap A$.

Since a is an accumulation point of $A \cap B$ and $\delta > 0$, then there exists $x \in A \cap B$ such that $x \in N'(a; \delta)$.

Let x be an arbitrary element of $A \cap B$ such that $x \in N'(a; \delta)$.
Since $x \in A \cap B$, then $x \in A$ and $x \in B$.
Since $x \in N'(a; \delta)$ and $x \in A$, then $x \in N'(a; \delta) \cap A$, so $|f(x) - L| < \epsilon$.
Since $x \in A \cap B$, then $f(x) = g(x)$.
Since $x \in N'(a; \delta)$ and $x \in B$, then $x \in N'(a; \delta) \cap B$.
Observe that $|g(x) - L| = |f(x) - L| < \epsilon$.
Therefore, $\lim_{x \to a} g(x) = L$. □

Proposition 44. *Let $a, b \in \mathbb{R}$ with $a < b$.*
Let $f : [a, b] \to \mathbb{R}$ be a monotonic increasing function.
Then $\lim_{x \to b} f(x) = \sup\{f(x) : x \in [a, b], x < b\}$.

Proof. Let $S = \{f(x) : x \in [a, b], x < b\}$.
We first prove $\sup S$ exists.
If $x \in [a, b]$, then $f(x) \in \mathbb{R}$ since f is a real valued function.
Hence, $S \subset \mathbb{R}$.
Since $a \in [a, b]$ and $a < b$, then $f(a) \in S$, so $S \neq \emptyset$.
Let $y \in S$ be arbitrary.
Then there exists $x \in [a, b]$ with $x < b$ such that $f(x) = y$.

Since f is monotonic increasing on $[a, b]$ and $x \in [a, b]$ and $b \in [a, b]$ and $x < b$, then $f(x) \leq f(b)$.

Hence, $y \leq f(b)$, so $y \leq f(b)$ for all $y \in S$.
Therefore, $f(b)$ is an upper bound of S, so S is bounded above in $\mathbb{R}$.

Since $S \subset \mathbb{R}$ and $S \neq \emptyset$ and S is bounded above in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\sup S$ exists.

We next prove $\lim_{x \to b} f(x) = \sup S$.

Let $\epsilon > 0$ be given.

Since $\sup S - \epsilon < \sup S$, then $\sup S - \epsilon$ is not an upper bound of S, so there exists $s \in S$ such that $s > \sup S - \epsilon$.

Since $s \in S$, then there exists $c \in [a, b]$ with $c < b$ such that $f(c) = s$.

Let $\delta = b - c$.

Since $b > c$, then $\delta = b - c > 0$, so $\delta > 0$.

Let $x \in [a, b]$ such that $0 < |x - b| < \delta$.

Since $0 < |x - b| < \delta$, then $0 < |x - b|$ and $|x - b| < \delta$.

Since $|x - b| > 0$, then $x - b \neq 0$, so $x \neq b$.

Since $x \in [a, b]$, then $a \leq x \leq b$, so $x \leq b$.

Since $x \leq b$ and $x \neq b$, then $x < b$.

Since $x \in [a, b]$ and $x < b$, then $f(x) \in S$.

Since $\sup S$ is an upper bound of S, then $f(x) \leq \sup S$.

Since $f(x) \leq \sup S$ and $\sup S < \sup S + \epsilon$, then $f(x) < \sup S + \epsilon$, so $f(x) - \sup S < \epsilon$.

Since $\delta = b - c$, then $c = b - \delta$.

Since $x < b$, then $0 < b - x = |b - x| = |x - b| < \delta$, so $b - x < \delta$.

Thus, $b - \delta < x$, so $c < x$.

Since f is monotonic increasing on $[a, b]$ and $c \in [a, b]$ and $x \in [a, b]$ and $c < x$, then $s = f(c) \leq f(x)$.

Since $\sup S - \epsilon < s$ and $s \leq f(x)$, then $\sup S - \epsilon < f(x)$, so $-\epsilon < f(x) - \sup S$.

Since $-\epsilon < f(x) - \sup S$ and $f(x) - \sup S < \epsilon$, then $-\epsilon < f(x) - \sup S < \epsilon$, so $|f(x) - \sup S| < \epsilon$, as desired. ☐

Proposition 45. *Let $I \subset \mathbb{R}$ be an interval.*

Let $f : I \to \mathbb{R}$ be a function such that f is (monotonic) increasing on I.

Let a be an interior point of I.

Then $\lim_{x \to a^+} f(x) = \inf\{f(x) : x \in I, x > a\}$.

Proof. Let $S = \{f(x) : x \in I, x > a\}$.

We first prove $S \neq \emptyset$.

Since a is an interior point of I, then I is not empty, so there exists $\delta > 0$ such that $N(a; \delta) \subset I$.

Since $|(a + \frac{\delta}{2}) - a| = \frac{\delta}{2} < \delta$, then $a + \frac{\delta}{2} \in N(a; \delta)$.

Since $N(a; \delta) \subset I$, then $a + \frac{\delta}{2} \in I$, so $f(a + \frac{\delta}{2})$ exists.

Since $a + \frac{\delta}{2} > a$, then $f(a + \frac{\delta}{2}) \in S$, so $S \neq \emptyset$.

We next prove S is bounded below in $\mathbb{R}$.

Since $S \neq \emptyset$, let $x \in I$ such that $x > a$.

Then $f(x) \in S$.

Since $a < x$ and f is increasing, then $f(a) \leq f(x)$.

Hence, $f(a) \leq f(x)$ for every $f(x) \in S$, so $f(a)$ is a lower bound of S.

Thus, S is bounded below in $\mathbb{R}$.

Since $S \neq \emptyset$ and S is bounded below in $\mathbb{R}$, then $\inf S$ exists.

Hence, there exists $L \in \mathbb{R}$ such that $L = \inf S$.

To prove $\lim_{x \to a^+} f(x) = L$, let $\epsilon > 0$ be given.

Since L is a lower bound of S, then $L + \epsilon$ is not a lower bound of S, so there exists $f(b) \in S$ such that $f(b) < L + \epsilon$.

Thus, $f(b) - L < \epsilon$.

Since $f(b) \in S$, then $b \in I$ and $b > a$.

Let $\delta = b - a$.

Since $b - a > 0$, then $\delta > 0$.

Let $x \in I$ such that $0 < x - a < \delta$.

Then $0 < x - a < b - a$, so $0 < x - a$ and $x - a < b - a$.

Since $0 < x - a$, then $a < x$.

Since $x - a < b - a$, then $x < b$.

Since f is increasing, then $f(x) \leq f(b)$, so $f(x) - L \leq f(b) - L < \epsilon$.

Hence, $f(x) - L < \epsilon$.

Since $x \in I$ and $x > a$, then $f(x) \in S$.

Since L is a lower bound of S, then $L \leq f(x)$, so $f(x) - L \geq 0$.

Therefore, $0 \leq f(x) - L < \epsilon$, so $|f(x) - L| < \epsilon$, as desired. $\square$