Limits of real valued functions Notes

Jason Sass

May 25, 2023

Sets of Numbers

$\mathbb{R} =$ set of all real numbers
$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} = (0, \infty) =$ set of all positive real numbers
$\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty) =$ set of all nonzero real numbers

Limit of a real valued function

The limit concept expresses the idea that for a given function f, $f(x)$ is arbitrarily close to some number L if x is sufficiently close to some number a.

Definition 1. ϵ, δ **definition of a limit of a real valued function**
Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Let a be an accumulation point of E.
A real number L **is a limit of** f **at** a, denoted $\lim_{x \to a} f(x) = L$, iff for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.
Therefore, $\lim_{x \to a} f(x) = L$ iff
$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(0 < |x - a| < \delta \to |f(x) - L| < \epsilon)$.

Observe that

$$\begin{aligned}
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(0 < |x - a| < \delta \to |f(x) - L| < \epsilon) &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(|x - a| > 0 \wedge |x - a| < \delta \to |f(x) - L| < \epsilon) &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(x - a \neq 0 \wedge |x - a| < \delta \to |f(x) - L| < \epsilon) &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(x \neq a \wedge |x - a| < \delta \to |f(x) - L| < \epsilon) &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(x \neq a \wedge x \in N(a; \delta) \to f(x) \in N(L; \epsilon)) &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(x \in N'(a; \delta) \to f(x) \in N(L; \epsilon)) &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E \cap N'(a; \delta))(f(x) \in N(L; \epsilon)). &
\end{aligned}$$

Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Let a be an accumulation point of E.

Suppose the limit of f at a exists.
Then there exists $L \in \mathbb{R}$ such that $\lim_{x \to a} f(x) = L$.
Therefore, $\lim_{x \to a} f(x) = L$ iff
$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(0 < |x-a| < \delta \to |f(x) - L| < \epsilon)$ iff
$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E \cap N'(a;\delta))(f(x) \in N(L;\epsilon))$.

Suppose a real number L is not the limit of f at a.
Then $\lim_{x \to a} f(x) \neq L$.
Therefore, $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(0 < |x-a| < \delta \to |f(x) - L| < \epsilon)$ is false.
Observe that

$$\begin{aligned}
\neg(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(0 < |x-a| < \delta \to |f(x) - L| < \epsilon) &\Leftrightarrow \\
(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in E)(0 < |x-a| < \delta \wedge \neg(|f(x) - L| < \epsilon)) &\Leftrightarrow \\
(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in E)(0 < |x-a| \wedge |x-a| < \delta \wedge \neg(f(x) \in N(L;\epsilon))) &\Leftrightarrow \\
(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in E)(0 < |x-a| \wedge x \in N(a;\delta) \wedge f(x) \notin N(L;\epsilon)) &\Leftrightarrow \\
(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in E)(x - a \neq 0 \wedge x \in N(a;\delta) \wedge f(x) \notin N(L;\epsilon)) &\Leftrightarrow \\
(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in E)(x \neq a \wedge x \in N(a;\delta) \wedge f(x) \notin N(L;\epsilon)) &\Leftrightarrow \\
(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in E)(x \in N'(a;\delta) \wedge f(x) \notin N(L;\epsilon)) &\Leftrightarrow \\
(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in E \cap N'(a;\delta))(f(x) \notin N(L;\epsilon)).&
\end{aligned}$$

Therefore, $\lim_{x \to a} f(x) \neq L$ iff $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in E \cap N'(a;\delta))(f(x) \notin N(L;\epsilon))$.

Suppose the limit of f at a does not exist.
Then there is no $L \in \mathbb{R}$ such that $\lim_{x \to a} f(x) = L$.
Thus, $\neg(\exists L \in \mathbb{R})(\lim_{x \to a} f(x) = L)$.

Example 2. limit of a constant function
For every $k \in \mathbb{R}$, $\lim_{x \to a} k = k$. (limit of a constant k is k)

Example 3. limit of the identity function
For all $a \in \mathbb{R}$, $\lim_{x \to a} x = a$.

Example 4. limit of the square function
For all $a \in \mathbb{R}$, $\lim_{x \to a} x^2 = a^2$.

Example 5. limit of the cube function
For all $a \in \mathbb{R}$, $\lim_{x \to a} x^3 = a^3$.

Example 6. limit of the reciprocal function
For all positive real a, $\lim_{x \to a} \frac{1}{x} = \frac{1}{a}$.

Example 7. limit of the absolute value function
For all $a \in \mathbb{R}$, $\lim_{x \to a} |x| = |a|$.

Example 8. limit of the square root function

For all $a \geq 0$, $\lim_{x\to a} \sqrt{x} = \sqrt{a}$.

Proposition 9. *Let f be a real valued function.*

Let a be an accumulation point of $\text{dom} f$.

If L is a real number, then $\lim_{x\to a} f(x) = L$ iff $\lim_{x\to a} |f(x) - L| = 0$.

Proposition 10. *Let $E \subset \mathbb{R}$.*

If a is an accumulation point of E, then a is an accumulation point of $E - \{a\}$.

Proposition 11. *Let $E \subset \mathbb{R}$.*

A point a is an accumulation point of E iff there is a sequence (x_n) of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$.

Theorem 12. ***sequential characterization of a function limit***

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

Let a be an accumulation point of E.

Let $L \in \mathbb{R}$.

Then $\lim_{x\to a} f(x) = L$ iff for every sequence (x_n) of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} f(x_n) = L$.

Therefore, $\lim_{x\to a} f(x) \neq L$ iff there exists a sequence (x_n) of points in $E - \{a\}$ such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} f(x_n) \neq L$.

Proposition 13. ***limit of an absolute value equals absolute value of a limit***

Let f be a real valued function.

Let a be an accumulation point of $\text{dom} f$.

If the limit of f at a exists, then $\lim_{x\to a} |f(x)| = |\lim_{x\to a} f(x)|$.

Lemma 14. *Let $E \subset \mathbb{R}$.*

Let $f : E \to \mathbb{R}$ be a function.

Let a be a point.

If the limit of f at a exists and is positive, then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in N'(a;\delta) \cap E$.

Proposition 15. ***limit of a square root equals square root of a limit***

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

If $\lim_{x\to a} f(x)$ exists and is positive, then $\lim_{x\to a} \sqrt{f(x)} = \sqrt{\lim_{x\to a} f(x)}$.

Example 16. limit of f at a need not equal $f(a)$, function with a removable discontinuity

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ 5 & \text{if } x = 1 \end{cases}$$

Then $\lim_{x\to 1} f(x) = 2$ and $\lim_{x\to 1} f(x) \neq f(1)$.

Example 17. function with a jump discontinuity
Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then $\lim_{x \to 0} f(x)$ does not exist in $\mathbb{R}$.

Example 18. unbounded function, infinite discontinuity
Show that $\lim_{x \to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$.

Example 19. oscillating function
Show that $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist in $\mathbb{R}$.

Algebraic properties of function limits

Theorem 20. ***scalar multiple rule for limits***
Let f be a real valued function.
Let a be a point.
If the limit of f at a exists and is a real number, then for every $\lambda \in \mathbb{R}$, the limit of λf exists and $\lim_{x \to a} \lambda f(x) = \lambda \lim_{x \to a} f(x)$.

Therefore, $\lim_{x \to a} \lambda f = \lambda \lim_{x \to a} f$.

Theorem 21. ***limit of a sum equals sum of limits***
Let f and g be real valued functions.
Let a be an accumulation point of $dom f \cap dom g$.
If the limit of f at a exists and is a real number and the limit of g at a exists and is a real number, then the limit of $f + g$ exists and
$\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

Therefore, $\lim_{x \to a}(f + g) = \lim_{x \to a} f + \lim_{x \to a} g$.

Corollary 22. ***limit of a difference equals difference of limits***
Let f and g be real valued functions.
Let a be an accumulation point of $dom f \cap dom g$.
If the limit of f at a exists and is a real number and the limit of g at a exists and is a real number, then the limit of $f - g$ exists and
$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$.

Therefore, $\lim_{x \to a}(f - g) = \lim_{x \to a} f - \lim_{x \to a} g$.

Corollary 23. ***limit of a finite sum equals finite sum of limits***
Let $n \in \mathbb{N}$ with $n \geq 2$.
Let a be an accumulation point of $\bigcap_{i=1}^{n} dom f_i$.
Let $f_1, f_2, ..., f_n$ be real valued functions.
Then $\lim_{x \to a}[f_1(x) + f_2(x) + ... + f_n(x)] = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) + ... + \lim_{x \to a} f_n(x)$.

Therefore, $\lim_{x\to a}(f_1+f_2+...+f_n) = \lim_{x\to a} f_1+\lim_{x\to a} f_2+...+\lim_{x\to a} f_n$.

Lemma 24. ***local boundedness of a function limit***

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

Let a be a point.

If the limit of f at a exists, then there exist $\delta > 0$ and $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in N(a;\delta) \cap E$.

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

Let a be a point.

Suppose the limit of f at a exists.

Then a is an accumulation point of E and there exist $\delta > 0$ and $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in N(a;\delta) \cap E$.

Thus, f is bounded in some δ neighborhood of a, so f is bounded locally near a.

Since $M \in \mathbb{R}$ and $|f(x)| \leq M$ for all $x \in N(a;\delta) \cap E$, then there exists $b > 0$ such that $|f(x)| < b$ for all $x \in N(a;\delta) \cap E$.

Therefore, if the limit of f at a exists, then f is bounded locally near a.

Therefore, if f is not locally bounded near a, then the limit of f at a does not exist.

Theorem 25. ***limit of a product equals product of limits***

Let f and g be real valued functions.

Let a be an accumulation point of $\text{dom} f \cap \text{dom} g$.

If the limit of f at a exists and is a real number and the limit of g at a exists and is a real number, then the limit of fg exists and

$\lim_{x\to a}(f(x)g(x)) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$.

Therefore, $\lim_{x\to a}(fg) = (\lim_{x\to a} f)(\lim_{x\to a} g)$.

Lemma 26. ***boundedness away from zero***

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

Let a be a point.

If there is a real number L such that $\lim_{x\to a} f(x) = L$ and $L \neq 0$, then there exists $\delta > 0$ such that $|f(x)| > \frac{|L|}{2}$ for all $x \in N'(a;\delta) \cap E$.

Lemma 27. *Let f be a real valued function.*

Let a be a point.

If the limit of f at a exists and is a nonzero real number, then the limit of $\frac{1}{f}$ exists and $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{\lim_{x\to a} f(x)}$.

Theorem 28. ***limit of a quotient equals quotient of limits***

Let f and g be real valued functions.

Let a be an accumulation point of $\text{dom} f \cap \text{dom} g$.

If the limit of f at a exists and is a real number and the limit of g at a exists and is a nonzero real number, then the limit of $\frac{f}{g}$ exists and

$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

Therefore, $\lim_{x\to a}(\frac{f}{g}) = \frac{\lim_{x\to a} f}{\lim_{x\to a} g}$.

If $f(x) \to L$ and $g(x) \to M$, then

1. Scalar Multiple Rule
$\lambda f(x) \to \lambda L$ for every $\lambda \in \mathbb{R}$.
2. Sum Rule
$f(x) + g(x) \to L + M$.
3. Difference Rule
$f(x) - g(x) \to L - M$.
4. Product Rule
$f(x)g(x) \to LM$.
5. Quotient Rule
If $M \neq 0$, then $\frac{f(x)}{g(x)} \to \frac{L}{M}$.

Lemma 29. *For all $n \in \mathbb{N}$ and all $a \in \mathbb{R}$, $\lim_{x\to a} x^n = a^n$.*

Theorem 30. ***limit of a polynomial function***

If p is a polynomial function and $c \in \mathbb{R}$, then $\lim_{x\to c} p(x) = p(c)$.

Theorem 31. ***limit of a rational function***

Let r be a rational function defined by $r(x) = \frac{p(x)}{q(x)}$ such that p and q are polynomial functions.

If $c \in \mathbb{R}$ such that $q(c) \neq 0$, then $\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}$.

Theorem 32. ***a limit preserves a non strict inequality***

Let f and g be real valued functions such that the limit of f at a exists and the limit of g at a exists and a is an accumulation point of $domf \cap domg$.

If $f(x) \leq g(x)$ for all $x \in domf \cap domg$, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

Corollary 33. *Let f be a real valued function such that $\lim_{x\to a} f(x)$ exists.*

1. If $M \in \mathbb{R}$ is an upper bound of $rngf$, then $\lim_{x\to a} f(x) \leq M$.
2. If $m \in \mathbb{R}$ is a lower bound of $rngf$, then $m \leq \lim_{x\to a} f(x)$.

Corollary 34. ***limit of a function is between any upper and lower bound of the range of a function***

Let f be a real valued function.

If $\lim_{x\to a} f(x)$ exists and there exist real numbers m and M such that $m \leq f(x) \leq M$ for all $x \in domf$, then $m \leq \lim_{x\to a} f(x) \leq M$.

Example 35. If $\lim_{n\to\infty} a_n = L$ and $a_n \geq 0$ for all $n \in \mathbb{N}$, then $L \geq 0$.

However, $a_n > 0$ for all $n \in \mathbb{N}$ does not imply $L > 0$.

Proof. Suppose $\lim_{n\to\infty} a_n = L$ and $a_n \geq 0$ for all $n \in \mathbb{N}$.

Since $a_n \geq 0$ for all $n \in \mathbb{N}$, then 0 is a lower bound of (a_n).

Since $\lim_{n\to\infty} a_n = L$ and 0 is a lower bound of (a_n), then $0 \leq L$, so $L \geq 0$, as desired.

Here is a counterexample:

Let $a_n = \frac{1}{n}$.

Then $\lim_{n\to\infty} a_n = 0$ and $a_n = \frac{1}{n} > 0$ for all $n \in \mathbb{N}$, but $0 \not> 0$. □

Theorem 36. ***squeeze rule for function limits***

Let f, g, h be real valued functions with common domain E.

Let a be an accumulation point of E.

If $f(x) \leq h(x) \leq g(x)$ for all $x \in E$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, then $\lim_{x\to a} h(x) = \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

Example 37. If $x > 0$, then $\lim_{x\to 0} x^{\frac{3}{2}} = 0$.

Solution. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function given by $f(x) = x^{\frac{3}{2}}$.

Then $dom f = (0, \infty)$.

Suppose $x > 0$.

If $0 < x < 1$, then $x < \sqrt{x} < 1$, so $x^2 < x^{\frac{3}{2}} < x$ for all $x \in (0, 1)$.

Since 0 is an accumulation point of the open interval $(0, 1)$ and $x^2 < x^{\frac{3}{2}} < x$ for all $x \in (0, 1)$ and $\lim_{x\to 0} x^2 = 0 = \lim_{x\to 0} x$, then by the squeeze theorem, $\lim_{x\to 0} x^{\frac{3}{2}} = 0$. □