Real Number System Examples

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Boundedness of sets in an ordered field

Example 1. $\sup(0, 1) = 1$.

Proof. Let $S = (0, 1)$.

Since $0 < \frac{1}{2} < 1$, then $\frac{1}{2} \in (0, 1) = S$, so $S \neq \emptyset$.

Let $x \in S$.

Then $0 < x < 1$, so $x < 1$.

Hence, $x < 1$ for all $x \in S$, so 1 is an upper bound of S.

Let $r \in \mathbb{R}$ such that $r < 1$.

To prove r is not an upper bound of S, we must prove there exists $s \in S$ such that $s > r$, so we must prove there exists s such that $0 < s < 1$ and $s > r$.

Let $T = \{0, r\}$.

Let $s = \frac{\max T+1}{2}$.

Since $\max T \geq 0$, then $\max T + 1 \geq 1$, so $\frac{\max T+1}{2} \geq \frac{1}{2}$.

Thus, $s \geq \frac{1}{2} > 0$, so $s > 0$.

Since $0 < 1$ and $r < 1$ and either $\max T = 0$ or $\max T = r$, then $\max T < 1$, so $\max T + 1 < 2$.

Hence, $\frac{\max T+1}{2} < 1$, so $s < 1$.

Since $0 < s$ and $s < 1$, then $0 < s < 1$.

Since $r \leq \max T$ and $r < 1$, then $2r < \max T + 1$.

Hence, $r < \frac{\max T+1}{2}$, so $r < s$.

Since there exists s such that $0 < s < 1$ and $r < s$, then there exists $s \in S$ such that $s > r$, so r is not an upper bound of S.

Thus, every real number $r < 1$ is not an upper bound of S.

Since 1 is an upper bound of S and every real number $r < 1$ is not an upper bound of S, then 1 is the least upper bound of S.

Therefore, $1 = \sup S$, so $1 = \sup(0, 1)$. □

Example 2. $\inf(0, 1) = 0$.

Proof. Let $S = (0, 1)$.

Since $0 < \frac{1}{2} < 1$, then $\frac{1}{2} \in (0, 1) = S$, so $S \neq \emptyset$.

Let $x \in S$.

Then $0 < x < 1$, so $0 < x$.
Hence, $0 < x$ for all $x \in S$, so 0 is a lower bound of S.

Let $r \in \mathbb{R}$ such that $r > 0$.
To prove r is not a lower bound of S, we must prove there exists $s \in S$ such that $s < r$, so we must prove there exists s such that $0 < s < 1$ and $s < r$.
Let $T = \{1, r\}$.
Let $s = \frac{\min T}{2}$.
Since $1 > 0$ and $r > 0$ and either $\min T = 1$ or $\min T = r$, then $\min T > 0$.
Thus, $\frac{\min T}{2} > 0$, so $s > 0$.
Since $0 < 1$ and $\min T \leq 1$, then $\min T < 2$.
Hence, $\frac{\min T}{2} < 1$, so $s < 1$.
Since $0 < s$ and $s < 1$, then $0 < s < 1$.
Since $\min T \leq r$ and $0 < r$, then $\min T < 2r$.
Hence, $\frac{\min T}{2} < r$, so $s < r$.
Since there exists s such that $0 < s < 1$ and $s < r$, then there exists $s \in S$ such that $s < r$, so r is not a lower bound of S.
Thus, every real number $r > 0$ is not a lower bound of S.
Since 0 is a lower bound of S and every real number $r > 0$ is not a lower bound of S, then 0 is the greatest lower bound of S.
Therefore, $0 = \inf S$, so $0 = \inf(0, 1)$. □

Complete ordered fields

Example 3. $\mathbb{Q}$ is not a complete ordered field.
The set $\{q \in \mathbb{Q} : q^2 < 2\}$ is bounded above in $\mathbb{Q}$, but does not have a least upper bound in $\mathbb{Q}$.

Proof. Let $S = \{q \in \mathbb{Q} : q^2 < 2\}$.
We prove 2 is an upper bound of S in $\mathbb{Q}$.
Observe that $2 = \frac{2}{1} \in \mathbb{Q}$.
Since $1 \in \mathbb{Q}$ and $1^2 = 1 < 2$, then $1 \in S$.
Hence, S is not empty.
Let $x \in \mathbb{Q}$ such that $x > 2$.
Since $x > 2 > 0$, then $x > 0$.
Thus, $x^2 > 2x > 2 \cdot 2 > 2 \cdot 1 = 2$, so $x^2 > 2$.
Hence, $x \notin S$.
Thus, if $x \in S$, then $x \leq 2$.
Therefore, $x \leq 2$ for every $x \in S$, so 2 is an upper bound of S in $\mathbb{Q}$.
Consequently, S is bounded above in $\mathbb{Q}$.
We prove S does not have a least upper bound in $\mathbb{Q}$.
For the sake of contradiction suppose that S has a least upper bound in $\mathbb{Q}$.
Let $M = \sup(S)$ in $\mathbb{Q}$.
Then $M \in \mathbb{Q}$.
Either $M^2 < 2$ or $M^2 = 2$ or $M^2 > 2$.

We consider these cases separately.

Case 1: Suppose $M^2 = 2$.

Since $M \in \mathbb{Q}$ and $M^2 = 2$, then there is a rational number whose square is 2.

This contradicts the fact that there is no rational number whose square is 2.

Therefore, $M^2 \neq 2$.

Case 2: Suppose $M^2 < 2$.

Let $q = M + \frac{2-M^2}{4}$.

Since $M \in \mathbb{Q}$ and $\mathbb{Q}$ is a field, then $q \in \mathbb{Q}$.

Since $M^2 < 2$, then $0 < 2 - M^2$, so $2 - M^2 > 0$.

Hence, $q - M = \frac{2-M^2}{4} > 0$.

Therefore, $q - M > 0$, so $q > M$.

Observe that

$$\begin{aligned}
q^2 - 2 &= (M + \frac{2-M^2}{4})^2 - 2 \\
&= M^2 + \frac{2M}{4}(2 - M^2) + (\frac{2-M^2}{4})^2 - 2 \\
&= (M^2 - 2) - \frac{2M}{4}(M^2 - 2) + (\frac{M^2-2}{4})^2 \\
&= 4^2\frac{M^2-2}{4^2} - (2 \cdot 4M)\frac{M^2-2}{4^2} + \frac{(M^2-2)^2}{4^2} \\
&= \frac{M^2-2}{4^2}(4^2 - 2 \cdot 4M + M^2 - 2) \\
&= \frac{M^2-2}{16}(M^2 - 4M - 4M + 10 + 4) \\
&= \frac{M^2-2}{16}[(M-2)^2 + (10 - 4M)].
\end{aligned}$$

Since $M^2 < 2$, then $M^2 - 2 < 0$, so $\frac{M^2-2}{16} < 0$.

Observe that $(M-2)^2 > 0$ iff $M - 2 \neq 0$.

If $M - 2 = 0$, then $M = 2$, so $M^2 = 4 > 2$.

Thus, $M^2 > 2$, contradicting the assumption $M^2 < 2$.

Hence, $M - 2 \neq 0$, so $(M-2)^2 > 0$.

Observe that $10 - 4M > 0$ iff $M < \frac{5}{2}$.

If $M \geq \frac{5}{2}$, then $M^2 \geq \frac{25}{4} > 2$.

Thus, $M^2 > 2$, contradicting the assumption $M^2 < 2$.

Hence, $M < \frac{5}{2}$, so $10 - 4M > 0$.

Since $(M-2)^2 > 0$ and $10 - 4M > 0$, then $(M-2)^2 + (10 - 4M) > 0$.

Since $\frac{M^2-2}{16} < 0$ and $(M-2)^2 + (10-4M) > 0$, then $q^2 - 2 < 0$, so $q^2 < 2$.

Since $q \in \mathbb{Q}$ and $q^2 < 2$, then $q \in S$.

Therefore, there exists $q \in S$ such that $q > M$.

This contradicts the fact that M is an upper bound of S.

Hence, M^2 cannot be less than 2.

Case 3: Suppose $M^2 > 2$.

Suppose y is a positive rational such that $y^2 > 2$.
Then $y \in \mathbb{Q}$ and $y > 0$ and $y^2 > 2$.
Since S is not empty, let $s \in S$.
Then $s \in \mathbb{Q}$ and $s^2 < 2$.
Suppose that $s > y$.
Then $0 < y < s$.
Hence, $2 < y^2 < s^2$, so $2 < s^2$.
Thus, $s^2 > 2$ and $s^2 < 2$, a contradiction.
Therefore, $s \leq y$, so y is an upper bound of S.
Hence, if y is any positive rational such that $y^2 > 2$, then y is an upper bound of S.
Let $U = \frac{M^2+2}{2M}$.
Since $M \in \mathbb{Q}$ and $\mathbb{Q}$ is a field, then $U \in \mathbb{Q}$.
Since $1 \in S$ and M is an upper bound of S, then $1 \leq M$, so $M > 0$.
Thus, $2M > 0$ and $M^2 + 2 > 0$, so $U > 0$.
Since $2 < M^2$, then $M^2 + 2 < 2M^2$.
We divide by positive $2M$ to get $U = \frac{M^2+2}{2M} < M$.
Therefore, $U < M$.
Observe that

$$\begin{aligned}
U^2 - 2 &= (\frac{M^2+2}{2M})^2 - 2 \\
&= -2 + \frac{(M^2+2)^2}{4M^2} \\
&= \frac{-8M^2 + M^4 + 4M^2 + 4}{4M^2} \\
&= \frac{M^4 - 4M^2 + 4}{4M^2} \\
&= \frac{(M^2-2)^2}{(2M)^2} \\
&= (\frac{M^2-2}{2M})^2.
\end{aligned}$$

Since $M \in \mathbb{Q}$ and $2M \neq 0$, then $\frac{M^2-2}{2M} \in \mathbb{Q}$.
Since $M^2 - 2 > 0$, then $M^2 - 2 \neq 0$, so $\frac{M^2-2}{2M} \neq 0$.
Hence, $U^2 - 2 = (\frac{M^2-2}{2M})^2 > 0$.
Therefore, $U^2 - 2 > 0$, so $U^2 > 2$.
Thus, U is a positive rational such that $U^2 > 2$.
Hence, U is an upper bound of S in $\mathbb{Q}$.
Therefore, there exists an upper bound U of S such that $U < M$.
This contradicts the fact that M is the least upper bound of S.
Thus, M^2 cannot be greater than 2.
In all cases a contradiction is reached.
Therefore, S does not have a least upper bound in $\mathbb{Q}$. $\square$

Archimedean ordered fields

Example 4. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.
Then $\max S = \sup S = 1$ and $\min S$ does not exist and $\inf S = 0$.

Proof. Since $1 \in \mathbb{N}$ and $\frac{1}{1} = 1$, then $1 \in S$.
Hence, S is not empty.
Let $x \in S$.
Then there exists $n \in \mathbb{N}$ such that $x = \frac{1}{n}$.
Since $n \in \mathbb{N}$, then $n \geq 1 > 0$, so $n > 0$.
Since $1 \leq n$ and $n > 0$, then we divide by n to obtain $\frac{1}{n} \leq 1$.
Thus, $x \leq 1$, so $x \leq 1$ for all $x \in S$.
Therefore, 1 is an upper bound of S.
Since $1 \in S$ and 1 is an upper bound of S, then $1 = \max S$.
Since $S \subset \mathbb{R}$ and $\mathbb{R}$ is an ordered field and $\max S$ exists, then $\sup S = \max S = 1$.

Since we proved 1 is an upper bound of S, we may equivalently prove $\sup S = 1$ by showing for every $\epsilon > 0$, there exists $x \in S$ such that $x > 1 - \epsilon$.
Let $\epsilon > 0$ be given.
Let $x = 1$.
Since $1 = \frac{1}{1}$ and $1 \in \mathbb{N}$, then $x \in S$.
Since $\epsilon > 0 = 1 - 1$, then $1 + \epsilon > 1$, so $1 > 1 - \epsilon$.
Since $x = 1$, then $x > 1 - \epsilon$, as desired.

□

Proof. Suppose for the sake of contradiction that $\min S$ exists.
Since $\min S \in S$, then there exists $n \in \mathbb{N}$ such that $\min S = \frac{1}{n}$.
Since $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$, so $\frac{1}{n+1} \in S$.
Since $0 < n < n + 1$, then $\frac{1}{n+1} < \frac{1}{n} = \min S$.
Thus, there exists $\frac{1}{n+1} \in S$ such that $\frac{1}{n+1} < \min S$.
But, this contradicts the fact that $\min S$ is a lower bound of S.
Therefore, $\min S$ does not exist. □

Proof. We prove $0 = \inf S$.
Let $x \in S$.
Then there exists $n \in \mathbb{N}$ such that $x = \frac{1}{n}$.
Since $n \in \mathbb{N}$, then $n > 0$.
Hence, $\frac{1}{n} > 0$, so $0 < \frac{1}{n}$.
Thus, $0 < x$, so $0 < x$ for all $x \in S$.
Therefore, 0 is a lower bound of S.

Let $\epsilon > 0$.

To prove 0 is the greatest lower bound of S, we must prove there exists $s \in S$ such that $s < \epsilon$.

By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Let $s = \frac{1}{n}$.

Then $s \in S$ and $s < \epsilon$.

Therefore, $0 = \inf S$. $\square$

Existence of square roots in $\mathbb{R}$

Example 5. If $S = \{q \in \mathbb{Q} : q^2 < 2\}$, then $\sup S = \sqrt{2}$.

Proof. We prove $\sqrt{2}$ is an upper bound of S in $\mathbb{R}$.

Since $0 \in \mathbb{Q}$ and $0^2 = 0 < 2$, then $0 \in S$, so $S \neq \emptyset$.

Let $x \in S$.

Then $x \in \mathbb{Q}$ and $x^2 < 2$.

Since $\mathbb{Q}$ is an ordered field and $x \in \mathbb{Q}$, then $x^2 \geq 0$.

Thus, $0 \leq x^2 < 2$, so $0 \leq \sqrt{x^2} < \sqrt{2}$.

Hence, $0 \leq |x| < \sqrt{2}$, so $|x| < \sqrt{2}$.

Consequently, $-\sqrt{2} < x < \sqrt{2}$, so $x < \sqrt{2}$.

Therefore, $\sqrt{2}$ is an upper bound of S in $\mathbb{R}$.

Let $r \in \mathbb{R}$ such that $r < \sqrt{2}$.

To prove r is not an upper bound of S, we must prove there exists $q \in S$ such that $q > r$.

Since $r \in \mathbb{R}$, then $|r| \in \mathbb{R}$, so either $|r| < \sqrt{2}$ or $|r| \geq \sqrt{2}$.

We consider these cases separately.

Case 1: Suppose $|r| < \sqrt{2}$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $q \in \mathbb{Q}$ such that $|r| < q < \sqrt{2}$.

Since $0 \leq |r| < q < \sqrt{2}$, then $|r| < q$ and $0 < q < \sqrt{2}$.

Since $0 < q < \sqrt{2}$, then $q^2 < 2$.

Since $q \in \mathbb{Q}$ and $q^2 < 2$, then $q \in S$.

Since $|r| < q$, then $-q < r < q$, so $r < q$.

Thus, there exists $q \in S$ such that $q > r$.

Case 2: Suppose $|r| \geq \sqrt{2}$.

Then either $r \geq \sqrt{2}$ or $r \leq -\sqrt{2}$.

Since $r < \sqrt{2}$, then r cannot be greater than or equal to $\sqrt{2}$.

Hence, $r \leq -\sqrt{2}$.

Let $q = 0$.

Since $0 \in S$, then $q \in S$.

Since $q = 0 > -\sqrt{2} \geq r$, then $q > r$.

Thus, there exists $q \in S$ such that $q > r$.

In all cases, there exists $q \in S$ such that $q > r$, so r is not an upper bound of S.

Hence, every real number $r < \sqrt{2}$ is not an upper bound of S.

Since $\sqrt{2}$ is an upper bound of S and every real number $r < \sqrt{2}$ is not an upper bound of S, then $\sqrt{2}$ is the least upper bound of S, so $\sqrt{2} = \sup S$. □

Proof. We prove $\sqrt{2}$ is an upper bound of S in $\mathbb{R}$.
Clearly, $\sqrt{2} \in \mathbb{R}$.
Let $x \in S$.
Then $x \in \mathbb{Q}$ and $x^2 < 2$.
Since $\mathbb{Q}$ is an ordered field and $x \in \mathbb{Q}$, then $x^2 \geq 0$.
Thus, $0 \leq x^2 < 2$, so $0 \leq \sqrt{x^2} < \sqrt{2}$.
Hence, $0 \leq |x| < \sqrt{2}$, so $0 \leq x < \sqrt{2}$.
Therefore, $x < \sqrt{2}$, so $\sqrt{2}$ is an upper bound of S in $\mathbb{R}$.
To prove $\sqrt{2}$ is the least upper bound of S, let $\epsilon > 0$.
We must prove there exists $s \in S$ such that $s > \sqrt{2} - \epsilon$.
Either $\epsilon < \sqrt{2}$ or $\epsilon = \sqrt{2}$ or $\epsilon > \sqrt{2}$.
We consider these cases separately.
Case 1: Suppose $\epsilon = \sqrt{2}$.
Since $1 \in \mathbb{Q}$ and $1^2 = 1 < 2$, then $1 \in S$.
Observe that

$$\begin{aligned} 1 &> 0 \\ &= \sqrt{2} - \sqrt{2} \\ &= \sqrt{2} - \epsilon. \end{aligned}$$

Thus, there exists $1 \in S$ such that $1 > \sqrt{2} - \epsilon$.
Case 2: Suppose $\epsilon > \sqrt{2}$.
Then $\epsilon - \sqrt{2} > 0$, so $\sqrt{2} - \epsilon < 0$.
Since $0 \in \mathbb{Q}$ and $0^2 = 0 < 2$, then $0 \in S$.
Thus, there exists $0 \in S$ such that $0 > \sqrt{2} - \epsilon$.
Case 3: Suppose $\epsilon < \sqrt{2}$.
Then $\sqrt{2} - \epsilon > 0$.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\sqrt{2} - \epsilon \in \mathbb{R}$ and $\sqrt{2} \in \mathbb{R}$ and $\sqrt{2} - \epsilon < \sqrt{2}$, then there exists a rational number q between $\sqrt{2} - \epsilon$ and $\sqrt{2}$.
Therefore, there exists $q \in \mathbb{Q}$ such that $\sqrt{2} - \epsilon < q < \sqrt{2}$.
Thus, $0 < \sqrt{2} - \epsilon < q < \sqrt{2}$, so $0 < q < \sqrt{2}$ and $\sqrt{2} - \epsilon < q$.
Since $0 < q < \sqrt{2}$, then $0 < q^2 < 2$, so $q^2 < 2$.
Since $q \in \mathbb{Q}$ and $q^2 < 2$, then $q \in S$.
Thus, there exists $q \in S$ such that $q > \sqrt{2} - \epsilon$.
Therefore, in all cases, there is an element of S greater than $\sqrt{2} - \epsilon$.
Hence, $\sqrt{2}$ is the least upper bound of S in $\mathbb{R}$, as desired. □