Real Number System Exercises

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Exercise 1. For every $x \in \mathbb{R}$, if $x > 0$, then $x^2 > 0$.

Proof. Let $x \in \mathbb{R}$ such that $x > 0$.
Since $x > 0$, then $x > 0$ and $x > 0$.
Since $\mathbb{R}$ is an ordered field and $x > 0$ and $x > 0$, then $x^2 = xx > 0$, so $x^2 > 0$. □

Exercise 2. Show that the statement $(\forall x, y \in \mathbb{R})[(x > 1 \to y > 2) \to x - y \notin (0, 3)]$ is false.

Proof. Let $p(x, y)$ be the predicate defined over $\mathbb{R} \times \mathbb{R}$ such that $p(x, y) : (x > 1 \to y > 2) \to x - y \notin (0, 3)$.
Observe that

$$(\forall x, y \in \mathbb{R})(x > 1 \to y > 2) \to x - y \notin (0,3) \quad \Leftrightarrow \quad (\forall x, y \in \mathbb{R})(x \leq 1 \vee y > 2) \to x - y \notin (0,3).$$

Thus, the negation of the statement is: there exists $x, y \in \mathbb{R}$ such that either $x \leq 1$ or $y > 2$ and $x - y \in (0, 3)$.
So, to prove the statement is false, we must prove there exists $x, y \in \mathbb{R}$ such that either $x \leq 1$ or $y > 2$ and the difference $x - y$ is in the open interval $(0, 3)$.
Let $x = 4$ and $y = 2.5$.
Since $2.5 > 2$, then $4 \leq 1$ or $2.5 > 2$ is true.
Observe that

$$\begin{aligned} 0 < 1.5 < 3 &\Leftrightarrow 0 < 4 - 2.5 < 3 \\ &\Leftrightarrow 0 < x - y < 3 \\ &\Leftrightarrow x - y \in (0, 3). \end{aligned}$$

Therefore, the statement $(\forall x, y \in \mathbb{R})[(x > 1 \to y > 2) \to x - y \notin (0, 3)]$ is false. □

Exercise 3. Let $x, y \in \mathbb{R}$.
If $xy = 0$, then $x = 0$ or $y = 0$.

Solution. The hypothesis is: $xy = 0$.
The conclusion is: $x = 0 \vee y = 0$.
Thus, we have a statement of the form $P \to Q \vee R$.

We know that $P \to Q \vee R \Leftrightarrow (P \wedge \neg Q) \to R$.

Thus, we assume $x \neq 0$, in addition to $xy = 0$.

We must prove $y = 0$.

Since $x \neq 0$, then x has a multiplicative inverse, $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$.

Thus, we can multiply x^{-1} with xy to get $x^{-1}(xy) = x^{-1}0$, so $(x^{-1}x)y = x^{-1}0 = 0$.

Thus, $1y = 0$, so $y = 0$.

To write up a correct proof, we write up in a transitive format, so we need to reverse the steps above in the proof. $\square$

Proof. Suppose $xy = 0$ and $x \neq 0$.

We must prove $y = 0$.

Since x is nonzero, then the multiplicative inverse x^{-1} exists, so $x^{-1}x = 1$.

Observe that

$$\begin{aligned} y &= 1 \cdot y \\ &= (x^{-1}x)y \\ &= x^{-1}(xy) \\ &= x^{-1}(0) \\ &= 0, \text{ as desired.} \end{aligned}$$

$\square$

Exercise 4. Let $a \in \mathbb{R}$.

If $a \cdot a = a$, then either $a = 0$ or $a = 1$.

Proof. We prove by contrapositive.

Suppose $a \cdot a = a$ and $a \neq 0$.

We must prove $a = 1$.

Since $a^2 = a \cdot a = a$, then $a^2 - a = 0$, so $a(a-1) = 0$.

Hence, either $a = 0$ or $a - 1 = 0$.

Since $a \neq 0$, then $a - 1 = 0$, so $a = 1$, as desired. $\square$

Exercise 5. reciprocal of a product equals product of the reciprocals

Let $a, b \in \mathbb{R}$.

If $a \neq 0$ and $b \neq 0$, then $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$.

Proof. Suppose $a \neq 0$ and $b \neq 0$.

Then $ab \in \mathbb{R}$ and $ab \neq 0$.

Hence, there exists a unique real number $\frac{1}{ab}$ such that $(ab)(\frac{1}{ab}) = (\frac{1}{ab})(ab) = 1$.

Since $a \neq 0$, then there is a unique real number $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.

Since $b \neq 0$, then there is a unique real number $\frac{1}{b}$ such that $b \cdot \frac{1}{b} = \frac{1}{b} \cdot b = 1$.

Hence, $\frac{1}{a} \cdot \frac{1}{b} \in \mathbb{R}$ and

$$
\begin{aligned}
(ab)(\frac{1}{a}\cdot\frac{1}{b}) &= (\frac{1}{a}\cdot\frac{1}{b})(ab) \\
&= (\frac{1}{a}\cdot\frac{1}{b})(ba) \\
&= \frac{1}{a}\cdot(\frac{1}{b}\cdot b)\cdot a \\
&= \frac{1}{a}\cdot 1\cdot a \\
&= \frac{1}{a}\cdot a \\
&= 1
\end{aligned}
$$

Since $\frac{1}{ab}$ is unique, then this implies $\frac{1}{ab} = \frac{1}{a}\cdot\frac{1}{b}$, as desired. □

Ordered Fields

Exercise 6. Let $x, y \in \mathbb{R}$ such that $xy = 10$ and $x > 5$.
Then $y < 2$.

Proof. Since $x > 5$, then $2x > 2\cdot 5 = 10 = xy$, so $2x > xy$.
Since $x > 5 > 0$, then $x > 0$.
Since $2x > xy$ and $x > 0$, then $\frac{2x}{x} > \frac{xy}{x}$, so $2 > y$.
Therefore, $y < 2$, as desired. □

Exercise 7. Let $x, y \in \mathbb{R}$.
If $xy = 6$ and $x > 2$, then $y < 3$.

Proof. Suppose $xy = 6$ and $x > 2$.
Since $x > 2 > 0$, then $x > 0$.
Since $xy = 6 > 0$, then $xy > 0$.
Since $xy > 0$ and $x > 0$, then $\frac{xy}{x} > \frac{0}{x}$, so $y > 0$.
Observe that

$$
\begin{aligned}
x > 2 &\Rightarrow xy > 2y \\
&\Leftrightarrow 6 > 2y \\
&\Leftrightarrow 2\cdot 3 > 2y \\
&\Leftrightarrow 3 > y
\end{aligned}
$$

Since $x > 2$ and $x > 2$ implies $3 > y$, then $3 > y$, so $y < 3$, as desired. □

Lemma 8. *Let $a, b, c \in \mathbb{R}$ and $c > 0$.*
If $a > b + c$, then $a > b$.

Proof. Suppose $a > b + c$.
Since $c > 0$, then $b + c > b$.
Since $a > b + c$ and $b + c > b$, then $a > b$. □

Proposition 9. *Let $a, b, c, d \in \mathbb{R}$.*
If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.

Proof. Suppose $a \leq b$ and $c \leq d$.
Then either $a < b$ or $a = b$ and either $c < d$ or $c = d$.
Thus, either $a < b$ and $c < d$ or $a < b$ and $c = d$ or $a = b$ and $c < d$ or $a = b$ and $c = d$.
We consider these cases separately.
Case 1: Suppose $a < b$ and $c < d$.
Then $a + c < b + d$.
Case 2: Suppose $a < b$ and $c = d$.
Then $a + c < b + c = b + d$.
Case 3: Suppose $a = b$ and $c < d$.
Then $a + c = b + c < b + d$.
Case 4: Suppose $a = b$ and $c = d$.
Then $a + c = b + c = b + d$.
Thus, in all cases, $a + c \leq b + d$. □

Corollary 10. *Let a_k and b_k be real numbers such that $a_k \leq b_k$ for every $k \in \mathbb{Z}^+$.*
Then $\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k$ for all positive integers n.

Proof. We prove by induction on n.
Let $S = \{n \in \mathbb{Z}^+ : \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k\}$.
Then $\sum_{k=1}^{1} a_k = a_1 \leq b_1 = \sum_{k=1}^{1} b_k$.
Since $1 \in \mathbb{Z}^+$ and $\sum_{k=1}^{1} a_k \leq \sum_{k=1}^{1} b_k$, then $1 \in S$.
Let $m \in S$.
Then $m \in \mathbb{Z}^+$ and $\sum_{k=1}^{m} a_k \leq \sum_{k=1}^{m} b_k$.
Since $m \in \mathbb{Z}^+$, then $m + 1 \in \mathbb{Z}^+$, so $a_{m+1} \leq b_{m+1}$.
Observe that

$$\begin{aligned}
\sum_{k=1}^{m+1} a_k &= \sum_{k=1}^{m} a_k + a_{m+1} \\
&\leq \sum_{k=1}^{m} b_k + a_{m+1} \\
&\leq \sum_{k=1}^{m} b_k + b_{m+1} \\
&= \sum_{k=1}^{m+1} b_k.
\end{aligned}$$

Since $m + 1 \in \mathbb{Z}^+$ and $\sum_{k=1}^{m+1} a_k \leq \sum_{k=1}^{m+1} b_k$, then $m + 1 \in S$.
Since $m \in S$ implies $m + 1 \in S$, then by PMI, $S = \mathbb{Z}^+$.
Therefore, $\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k$ for every $n \in \mathbb{Z}^+$. □

Exercise 11. Let $a, b, c, d \in \mathbb{R}$.
If $a < b$ and $c \leq d$, then $a + c < b + d$.

Proof. Suppose $a < b$ and $c \leq d$.
Since $c \leq d$, then either $c < d$ or $c = d$.
We consider these cases separately.
Case 1: Suppose $c < d$.
Since $a < b$ and $c < d$, then $a + c < b + d$.
Case 2: Suppose $c = d$.
Since $a < b$, then $a + c < b + c$, so $a + c < b + d$. □

Exercise 12. Show that the statement $(\forall x \in \mathbb{R})(x \leq 0) \vee (x^2 > 2) \vee (x^3 > 3)$ is false.

Proof. To prove the statement is false, we must prove $(\exists x \in \mathbb{R})(x > 0) \wedge (x^2 \leq 2) \wedge (x^3 \leq 3)$.
Let $x = 1$.
Observe that $1 > 0$ and $1^2 = 1 < 2 \leq 2$ and $1^3 = 1 < 3 \leq 3$. □

Exercise 13. Show that the statement $(\exists m, n \in \mathbb{Z})(n \geq 5 \rightarrow m \leq 4) \wedge (m + n \leq 9)$ is true.

Proof. The statement is equivalent to $(\exists m, n \in \mathbb{Z})(n < 5 \vee m \leq 4) \wedge (m + n \leq 9)$.
Let $m = 1$ and $n = 2$.
Then m and n are integers and $2 < 5$ and $1 < 4 \leq 4$ and $m + n = 1 + 2 = 3 < 9 \leq 9$. □

Exercise 14. Let a, b, c, d be elements of an ordered field F.
If $a < b$ and $c < d$, then $ad + bc < ac + bd$.

Proof. Suppose $a < b$ and $c < d$.
Since $c < d$, then $d > c$, so $d - c > 0$.
Since $a < b$ and $d - c > 0$, then $a(d - c) < b(d - c)$.
Hence, $ad - ac < bd - bc$.
Observe that

$$\begin{array}{rcl} ad - ac + ac & < & bd - bc + ac \\ ad + 0 & < & bd - bc + ac \\ ad & < & bd + ac - bc \\ ad + bc & < & bd + ac - bc + bc \\ ad + bc & < & bd + ac + 0 \\ ad + bc & < & bd + ac \\ ad + bc & < & ac + bd. \end{array}$$

Therefore, $ad + bc < ac + bd$. □

Exercise 15. Let $x \in \mathbb{R}$.
If $0 \leq x \leq 2$, then $-x^3 + 4x + 1 > 0$.

Proof. Suppose $0 \leq x \leq 2$.
Then $0 \leq x$ and $x \leq 2$.
Observe that $-x^3+4x+1 = x(-x^2+4)+1 = x(4-x^2)+1 = x(2-x)(2+x)+1$.
Since $x \leq 2$, then $2 - x \geq 0$.
Since $x \geq 0$, then $2 + x \geq 2 > 0$, so $2 + x > 0$.
Since $x \geq 0$ and $2 - x \geq 0$ and $2 + x > 0$, then $x(2 - x)(2 + x) \geq 0$.
Thus, $x(2 - x)(2 + x) + 1 \geq 1 > 0$, so $x(2 - x)(2 + x) + 1 > 0$.
Therefore, $-x^3 + 4x + 1 > 0$. ☐

Exercise 16. Let a, b, x, y be positive elements of an ordered field F.
If $\frac{x}{y} < \frac{a}{b}$, then $\frac{x}{y} < \frac{x+a}{y+b} < \frac{a}{b}$.

Proof. Suppose $\frac{x}{y} < \frac{a}{b}$.
Since $y > 0$ and $b > 0$, then $y + b > 0$.
Hence, $\frac{x}{y} < \frac{a}{b}$ implies $xb < ya$, so $xy + xb < xy + ya$ and $xb + ab < ya + ab$.
Thus, $x(y + b) < y(x + a)$ and $(x + a)b < (y + b)a$, so $\frac{x}{y} < \frac{x+a}{y+b}$ and $\frac{x+a}{y+b} < \frac{a}{b}$.
Therefore, $\frac{x}{y} < \frac{x+a}{y+b} < \frac{a}{b}$. ☐

Exercise 17. Let $a, b, c, d \in \mathbb{R}$.
If $0 < a < b$ and $0 \leq c \leq d$, then $0 \leq ac \leq bd$.

Proof. Suppose $0 < a < b$ and $0 \leq c \leq d$.
Since $0 < a < b$, then $0 < a$ and $a < b$, so $0 < b$.
Since $0 \leq c \leq d$, then $0 \leq c$ and $c \leq d$.
Since $0 \leq c$ and $a > 0$, then $0 = a0 \leq ac$, so $0 \leq ac$.
Since $0 \leq c$ and $c \leq d$, then either $0 < c$ and $c < d$ or $0 < c$ and $c = d$ or $0 = c$ and $c < d$ or $0 = c$ and $c = d$.
We consider these cases separately.
Case 1: Suppose $0 < c$ and $c < d$.
Then $0 < c < d$.
Since $0 < a < b$ and $0 < c < d$, then $0 < ac < bd$, so $ac < bd$.
Case 2: Suppose $0 < c$ and $c = d$.
Since $a < b$ and $c > 0$, then $ac < bc = bd$, so $ac < bd$.
Case 3: Suppose $0 = c$ and $c < d$.
Then $0 < d$.
Since $b > 0$ and $d > 0$, then $bd > 0$.
Since $ac = a0 = 0 < bd$, then $ac < bd$.
Case 4: Suppose $0 = c$ and $c = d$.
Then $ac = a0 = 0 = b0 = bc = bd$, so $ac = bd$.
Thus, in all cases, either $ac < bd$ or $ac = bd$, so $ac \leq bd$.
Therefore, $0 \leq ac$ and $ac \leq bd$, so $0 \leq ac \leq bd$, as desired. ☐

Exercise 18. Let $a, b \in \mathbb{R}$.
If $0 \leq a < b$, then $a^2 \leq ab < b^2$.

Proof. Suppose $0 \leq a < b$.

Then $0 \leq a$ and $a < b$, so $0 < b$.

Since $a < b$ and $b > 0$, then $ab < b^2$.

Since $a \geq 0$, then either $a > 0$ or $a = 0$.

We consider these cases separately.

Case 1: Suppose $a > 0$.

Since $a < b$ and $a > 0$, then $a^2 < ab$.

Case 2: Suppose $a = 0$.

Then $a^2 = 0^2 = 0 = 0b = ab$.

Thus, in either case, either $a^2 < ab$ or $a^2 = ab$, so $a^2 \leq ab$.

Since $a^2 \leq ab$ and $ab < b^2$, then $a^2 \leq ab < b^2$, as desired. □

Exercise 19. Prove $x^2 + x + 1 > 0$ for all $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$.

Then $x + \frac{1}{2} \in \mathbb{R}$, so $(x + \frac{1}{2})^2 \geq 0$.

Hence, $0 \leq (x + \frac{1}{2})^2 = x^2 + x + \frac{1}{4}$, so $0 \leq x^2 + x + \frac{1}{4}$.

Therefore $0 < \frac{3}{4} \leq x^2 + x + 1$, so $0 < x^2 + x + 1$. □

Exercise 20. Let $r \in \mathbb{R}$ with $0 < r < 1$.

Then $0 < r^n < 1$ for all $n \in \mathbb{N}$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : 0 < r^n < 1\}$.

Since $1 \in \mathbb{N}$ and $0 < r^1 = r < 1$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $0 < r^k < 1$.

Since $0 < r < 1$ and $0 < r^k < 1$, then $0 < rr^k < 1 \cdot 1$, so $0 < r^{k+1} < 1$.

Since $k + 1 \in \mathbb{N}$ and $0 < r^{k+1} < 1$, then $k + 1 \in S$.

Thus, $k \in S$ implies $k + 1 \in S$.

By the principle of mathematical induction, $0 < r^n < 1$ for all $n \in \mathbb{N}$, as desired. □

Proposition 21. *Let $a, b \in \mathbb{R}$ with $a > 0$.*

Then $a^n < b^n$ for all $n \in \mathbb{N}$ iff $a < b$.

Proof. We first prove if $a^n < b^n$ for all $n \in \mathbb{N}$, then $a < b$ by contrapositive.

Suppose $a \geq b$.

Then $a^1 = a \geq b = b^1$, so $a^1 \geq b^1$.

Since $1 \in \mathbb{N}$ and $a^1 \geq b^1$, then there is a natural number n such that $a^n \geq b^n$, as desired. □

Proof. We next prove if $a < b$, then $a^n < b^n$ for all $n \in \mathbb{N}$.

Suppose $a < b$.

Since $b > a$ and $a > 0$, then $b > 0$.

We must prove $a^n < b^n$ for all $n \in \mathbb{N}$.

Let $p(n) : a^n < b^n$ be a predicate defined over $\mathbb{N}$.

Basis:

Since $a < b$, then $a^1 < b^1$, so the statement $p(1)$ is true.
Induction:
Let $k \in \mathbb{N}$ such that $p(k)$ is true.
Then $a^k < b^k$.
Since $k \in \mathbb{N}$, then $k > 0$.
Since $b > 0$ and $k > 0$, then $b^k > 0$.
Since $a^k < b^k$ and $a > 0$ and $a < b$, then

$$\begin{aligned} a^{k+1} &= a^k a \\ &< b^k a \\ &< b^k b \\ &= b^{k+1}. \end{aligned}$$

Therefore, $a^{k+1} < b^{k+1}$, so $p(k+1)$ is true.
Hence, $p(k)$ implies $p(k+1)$ for all $k \in \mathbb{N}$.
Since $p(1)$ is true and $p(k)$ implies $p(k+1)$ for all $k \in \mathbb{N}$, then by induction, $p(n)$ is true for all $n \in \mathbb{N}$.
Therefore, $a^n < b^n$ for all $n \in \mathbb{N}$, as desired. □

Proposition 22. *Let $c \in \mathbb{R}$.*
If $c > 1$, then $c^n > c$ for all natural numbers $n > 1$.

Proof. Suppose $c > 1$.
To prove $c^n > c$ for all natural numbers $n > 1$, let $S = \{n \in \mathbb{N} : c^n > c \wedge n > 1\}$.
We prove by induction on n.

Since $c > 1$ and $1 > 0$, then $c > 0$, so $c^2 > c$.
Since $2 \in \mathbb{N}$ and $c^2 > c$ and $2 > 1$, then $2 \in S$.

Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $c^k > c$ and $k > 1$.
Since $c^k > c$ and $c > 1$, then $c^k > 1$.
Since $c > 1$ and $1 > 0$, then $c > 0$.
Thus, $c^k \cdot c > 1 \cdot c$, so $c^{k+1} > c$.
Since $k \in \mathbb{N}$, then $k+1 \in \mathbb{N}$.
Since $k > 1$ and $1 > 0$, then $k > 0$, so $k+1 > 1$.
Since $k+1 \in \mathbb{N}$ and $c^{k+1} > c$ and $k+1 > 1$, then $k+1 \in S$.
Hence, $k \in S$ implies $k+1 \in S$, so $k \in S$ implies $k+1 \in S$ for all $k \in S$.
Since $2 \in S$ and $k \in S$ implies $k+1 \in S$ for all $k \in S$, then by PMI, $c^n > c$ for all natural numbers $n > 1$. □

Proposition 23. *Let $c \in \mathbb{R}$.*
If $0 < c < 1$, then $c^n < c$ for all natural numbers $n > 1$.

Proof. Suppose $0 < c < 1$.
Then $0 < c$ and $c < 1$.
To prove $c^n < c$ for all natural numbers $n > 1$, let $S = \{n \in \mathbb{N} : c^n < c \wedge n > 1\}$.
We prove by induction on n.

Since $c < 1$ and $c > 0$, then $c \cdot c < 1 \cdot c$, so $c^2 < c$.
Since $2 \in \mathbb{N}$ and $c^2 < c$ and $2 > 1$, then $2 \in S$.

Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $c^k < c$ and $k > 1$.
Since $c^k < c$ and $c < 1$, then $c^k < 1$.
Since $c > 0$, then $c^k \cdot c < 1 \cdot c$, so $c^{k+1} < c$.
Since $k \in \mathbb{N}$, then $k + 1 \in \mathbb{N}$.
Since $k > 1$ and $1 > 0$, then $k > 0$, so $k + 1 > 1$.
Since $k + 1 \in \mathbb{N}$ and $c^{k+1} < c$ and $k + 1 > 1$, then $k + 1 \in S$.
Hence, $k \in S$ implies $k + 1 \in S$, so $k \in S$ implies $k + 1 \in S$ for all $k \in S$.
Since $2 \in S$ and $k \in S$ implies $k + 1 \in S$ for all $k \in S$, then by PMI, $c^n < c$ for all natural numbers $n > 1$. ☐

Lemma 24. *Let $c \in \mathbb{R}$ with $c > 1$.*
Then $c^n \geq c$ for all $n \in \mathbb{N}$.

Proof. We prove $c^n \geq c$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : c^n \geq c\}$.
Since $1 \in \mathbb{N}$ and $c^1 = c$, then $1 \in S$.

Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $c^k \geq c$.
Since $c > 1 > 0$, then $c > 0$.
Since $k \in \mathbb{N}$, then $k + 1 \in \mathbb{N}$.
Since $c^k \geq c > 1$, then $c^k > 1$.
Thus, $c^{k+1} = c^k \cdot c > 1 \cdot c = c$.
Since $k + 1 \in \mathbb{N}$ and $c^{k+1} > c$, then $k + 1 \in S$.
Therefore, by PMI, $c^n \geq c$ for all $n \in \mathbb{N}$. ☐

Proposition 25. *Let $c \in \mathbb{R}$ with $c > 1$ and $m, n \in \mathbb{N}$.*
Then $c^m > c^n$ iff $m > n$.

Proof. We prove if $m > n$, then $c^m > c^n$.
Suppose $m > n$.
Then $m - n > 0$.
Since $m, n \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{Z}$, then $m, n \in \mathbb{Z}$, so $m - n \in \mathbb{Z}$.
Since $m - n \in \mathbb{Z}$ and $m - n > 0$, then $m - n \in \mathbb{Z}^+$, so $m - n \in \mathbb{N}$.
Since $c > 1$ and $n \in \mathbb{N}$, then by the previous lemma, $c^n \geq c$.
Since $c^n \geq c > 1 > 0$, then $c^n > 0$.
Since $c > 1$ and $m - n \in \mathbb{N}$, then by the previous lemma, $c^{m-n} \geq c$.

Since $c^{m-n} \geq c > 1$, then $c^{m-n} > 1$, so $\frac{c^m}{c^n} > 1$.
Since $c^n > 0$, then $c^m > c^n$.
Therefore, if $m > n$, then $c^m > c^n$.

Conversely, we prove if $c^m > c^n$, then $m > n$.
Suppose $c^m > c^n$.
Suppose $m \leq n$.
Then $n \geq m$, so either $n > m$ or $n = m$.
If $n = m$, then $c^n = c^m$.
If $n > m$, then $c^n > c^m$.
Hence, either $c^n > c^m$ or $c^n = c^m$, so $c^n \geq c^m$.
Thus, we have $c^m > c^n$ and $c^m \leq c^n$, a violation of trichotomy.
Therefore, $m < n$, as desired.

□

Proposition 26. ***Bernoulli's inequality***
Let $x \in \mathbb{R}$ *with* $x > -1$.
Then $(1+x)^n \geq 1 + nx$ *for all* $n \in \mathbb{N}$.

Proof. Let $S = \{n \in \mathbb{N} : (1+x)^n \geq 1 + nx\}$.
We prove using mathematical induction(weak).
Basis:
Let $n = 1$.
Then $(1+x)^1 = 1 + x = 1 + 1x$.
Since $1 \in \mathbb{N}$ and $(1+x)^1 = 1 + 1x$, then $1 \in S$.
Induction:
Let $k \in S$.
Then $k \in \mathbb{N}$ and $(1+x)^k \geq 1 + kx$.
Since $k \in \mathbb{N}$, then $k + 1 \in \mathbb{N}$ and $k > 0$.
Since $x > -1$, then $1 + x > 0$.
Observe that

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k(1+x) \\
&\geq (1+kx)(1+x) \\
&= 1 + x + kx + kx^2 \\
&= 1 + kx + x + kx^2 \\
&= 1 + (k+1)x + kx^2.
\end{aligned}$$

Thus, $(1+x)^{k+1} \geq 1 + (k+1)x + kx^2$.
Since $x^2 \geq 0$ and $k > 0$, then $kx^2 \geq 0$.
Thus, $1 + (k+1)x + kx^2 \geq 1 + (k+1)x + 0 = 1 + (k+1)x$.
Since $(1+x)^{k+1} \geq 1 + (k+1)x + kx^2$ and $1 + (k+1)x + kx^2 \geq 1 + (k+1)x$, then $(1+x)^{k+1} \geq 1 + (k+1)x$.
Since $k + 1 \in \mathbb{N}$ and $(1+x)^{k+1} \geq 1 + (k+1)x$, then $k + 1 \in S$, so $k \in S$ implies $k + 1 \in S$.
Hence, $k \in S$ implies $k + 1 \in S$ for all $k \in \mathbb{N}$.

Since $1 \in S$ and $k \in S$ implies $k+1 \in S$ for all $k \in \mathbb{N}$, then by PMI, $S = \mathbb{N}$, so $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$. $\square$

Exercise 27. Let $n \in \mathbb{N}$.
Then $2^0 + 2^1 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 1$.

Proof. Let $S = 2^0 + 2^1 + 2^2 + 2^3 + ... + 2^n$.
We must prove $S = 2^{n+1} - 1$.

Observe that $2S = 2^1+2^2+2^3+...+2^n+2^{n+1}$ and $S-1 = 2^1+2^2+2^3+...+2^n$.
Thus, $2S = (2^1 + 2^2 + 2^3 + ... + 2^n) + 2^{n+1} = (S-1) + 2^{n+1}$.
Hence, $S = 2S - S = (S-1) + 2^{n+1} - S = -1 + 2^{n+1} = 2^{n+1} - 1$.
Therefore, $S = 2^{n+1} - 1$, as desired. $\square$

Proposition 28. *Let $x \in \mathbb{R}$.*
Then $x^n - 1 = (x-1)\sum_{k=0}^{n-1} x^k$ for all $n \in \mathbb{Z}^+$.

Proof. We prove $x^n - 1 = (x-1)\sum_{k=0}^{n-1} x^k$ by induction on n.
Let $S = \{n \in \mathbb{Z}^+ : x^n - 1 = (x-1)(\sum_{k=0}^{n-1} x^k)\}$.
Basis:
Observe that $x^1 - 1 = x - 1 = (x-1)(1) = (x-1)(x^0) = (x-1)(\sum_{k=0}^{1-1} x^k)$.
Hence, $1 \in S$.
Induction:
Suppose $m \in S$.
Then $m \in \mathbb{Z}^+$ and $x^m - 1 = (x-1)(\sum_{k=0}^{m-1} x^k)$.
Observe that

$$\begin{aligned}
(x-1)(\sum_{k=0}^{m} x^k) &= (x-1)[\sum_{k=0}^{m-1} x^k + x^m] \\
&= (x-1)\sum_{k=0}^{m-1} x^k + (x-1)x^m \\
&= (x^m - 1) + (x^{m+1} - x^m) \\
&= x^{m+1} - 1.
\end{aligned}$$

Since $m+1 \in \mathbb{Z}^+$ and $x^{m+1} - 1 = (x-1)(\sum_{k=0}^{m} x^k)$, then $m+1 \in S$.
Hence, $m \in S$ implies $m+1 \in S$, so by PMI, $S = \mathbb{Z}^+$.
Therefore, $x^n - 1 = (x-1)\sum_{k=0}^{n-1} x^k$ for all $n \in \mathbb{Z}^+$. $\square$

Corollary 29. *Let $x \in \mathbb{R}$ with $x \neq 1$.*
Then $\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$ for all $n \in \mathbb{Z}^+$.

Proof. Let $n \in \mathbb{Z}^+$ be given.
Then $x^n - 1 = (x-1)\sum_{k=0}^{n-1} x^k$.
Since $x \neq 1$, then $x - 1 \neq 0$, so we divide to obtain $\frac{x^n-1}{x-1} = \sum_{k=0}^{n-1} x^k$.

Observe that

$$\begin{aligned}
\sum_{k=0}^{n} x^k &= \sum_{k=0}^{n-1} x^k + x^n \\
&= \frac{x^n - 1}{x - 1} + x^n \\
&= \frac{x^n - 1 + x^n(x-1)}{x-1} \\
&= \frac{x^n - 1 + x^{n+1} - x^n}{x-1} \\
&= \frac{x^{n+1} - 1}{x-1}.
\end{aligned}$$

Therefore, $\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$. □

Proposition 30. ***Difference of powers***

Let $a, b \in \mathbb{R}^*$.

Then $a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$ *for all* $n \in \mathbb{N}$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}\}$.

Basis:

Since $1 \in \mathbb{N}$ and $a^1 - b^1 = a - b = (a-b)(1) = (a-b)(ab)^0 = (a-b)a^0b^0 = (a-b)\sum_{k=0}^{1-1} a^k b^{1-1-k}$, then $1 \in S$.

Induction:

Suppose $m \in S$.

Then $m \in \mathbb{N}$ and $a^m - b^m = (a-b)\sum_{k=0}^{m-1} a^k b^{m-1-k}$.

Since $m \in \mathbb{N}$, then $m + 1 \in \mathbb{N}$.

Observe that

$$\begin{aligned}
a^{m+1} - b^{m+1} &= a^{m+1} - ab^m + ab^m - b^{m+1} \\
&= a(a^m - b^m) + b^m(a-b) \\
&= a(a-b)\sum_{k=0}^{m-1} a^k b^{m-1-k} + b^m(a-b) \\
&= (a-b)[a\sum_{k=0}^{m-1} a^k b^{m-1-k} + b^m] \\
&= (a-b)[\sum_{k=0}^{m-1} a^{k+1} b^{m-1-k} + b^m] \\
&= (a-b)[(a^1b^{m-1} + a^2b^{m-2} + a^3b^{m-3} + ... + a^mb^0) + a^0b^m] \\
&= (a-b)\sum_{k=0}^{m} a^k b^{m-k}.
\end{aligned}$$

Since $m+1 \in \mathbb{N}$ and $a^{m+1} - b^{m+1} = (a-b)\sum_{k=0}^{m} a^k b^{m-k}$, then $m+1 \in S$.
Thus, $m \in S$ implies $m+1 \in S$.
Therefore, by the principle of mathematical induction, $a^n - b^n = (a - b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$ for all $n \in \mathbb{N}$, as desired. □

Exercise 31. Let $x, y \in \mathbb{R}$.
If $xy = 10$ and $|x| > 2$, then $|y| \leq 5$.

Proof. We prove by contradiction.
Suppose $xy = 10$ and $|x| > 2$ and $|y| > 5$.
Since $|x| > 2$ and $|y| > 5$, then $|x| \cdot |y| > 2 \cdot 5$, so $|xy| > 10$.
Thus, $|10| > 10$, so $10 > 10$, a contradiction.
Therefore, if $xy = 10$ and $|x| > 2$, then $|y| \leq 5$. □

Exercise 32. Let $x, y \in \mathbb{R}$.
If $xy \leq 9$ and $x > 3$, then $y < 3$.

Proof. Suppose $xy \leq 9$ and $x > 3$.
Since $x > 3 > 0$, then $x > 0$.
Thus, $3x > 3 \cdot 3 = 9 \geq xy$, so $3x > xy$.
Hence, $3 > y$, so $y < 3$. □

Proof. We prove by contradiction.
Suppose $xy \leq 9$ and $x > 3$ and $y \geq 3$.
Since $x > 3$ and $y \geq 3$, then $xy > 9$.
But, this contradicts the fact that $xy \leq 9$.
Therefore, if $xy \leq 9$ and $x > 3$, then $y < 3$. □

Exercise 33. Let $S = \{x \in \mathbb{R} : x^2 - 4x + 5 \leq 10\}$.
Then $S = [-1, 5]$.

Proof. Since $1^2 - 4(1) + 5 = 1 - 4 + 5 = 2 < 10$, then $1 \in S$, so $S \neq \emptyset$.

We first prove $S \subset [-1, 5]$.
Let $x \in S$.
Then $x \in \mathbb{R}$ and $x^2 - 4x + 5 \leq 10$, so $x^2 - 4x - 5 \leq 0$.
Hence, $(x-5)(x+1) \leq 0$, so either $(x-5)(x+1) < 0$ or $(x-5)(x+1) = 0$.
We consider these cases separately.
Case 1: Suppose $(x-5)(x+1) < 0$.
Then either $x - 5 > 0$ and $x + 1 < 0$ or $x - 5 < 0$ and $x + 1 > 0$, so either $x > 5$ and $x < -1$, or $x < 5$ and $x > -1$.
Since x cannot be both less than -1 and greater than 5, then $x < 5$ and $x > -1$, so $-1 < x < 5$.
Hence, $x \in (-1, 5)$.
Case 2: Suppose $(x-5)(x+1) = 0$.
Then either $x - 5 = 0$ or $x + 1 = 0$, so either $x = 5$ or $x = -1$.
Hence, $x \in \{-1, 5\}$.

Both cases show that either $x \in (-1,5)$ or $x \in \{-1,5\}$, so $x \in (-1,5) \cup \{-1,5\}$.

Therefore, $x \in [-1,5]$.

Since $x \in S$ implies $x \in [-1,5]$, then $S \subset [-1,5]$.

We next prove $[-1,5] \subset S$.

Let $y \in [-1,5]$.

Then $-1 \leq y \leq 5$, so $-1 \leq y$ and $y \leq 5$.

Hence, $0 \leq y+1$ and $y-5 \leq 0$.

Since $y+1 \geq 0$ and $y-5 \leq 0$, then $(y+1)(y-5) \leq 0$, so $y^2 - 4y - 5 \leq 0$.

Thus, $y^2 - 4y + 5 \leq 10$, so $y \in S$.

Therefore, if $y \in [-1,5]$, then $y \in S$, so $[-1,5] \subset S$.

Since $S \subset [-1,5]$ and $[-1,5] \subset S$, then $S = [-1,5]$, as desired. □

Exercise 34. Let $S = \{x \in \mathbb{R} : |\frac{x}{x-2}| < 4\}$.

Then $S = (-\infty, \frac{8}{5}) \cup (\frac{8}{3}, \infty)$.

Proof. Let $T = (-\infty, \frac{8}{5}) \cup (\frac{8}{3}, \infty)$.

We must prove $S = T$.

Since $|\frac{3}{3-2}| = |\frac{3}{1}| = |3| = 3 < 4$, then $3 \in S$, so $S \neq \emptyset$.

We first prove $S \subset T$.

Let $x \in S$.

Then $x \in \mathbb{R}$ and $|\frac{x}{x-2}| < 4$, so $-4 < \frac{x}{x-2} < 4$.

Since $\frac{x}{x-2} \in \mathbb{R}$ and division by zero is undefined in $\mathbb{R}$, then $x - 2 \neq 0$, so $x \neq 2$.

Thus, either $x > 2$ or $x < 2$.

We consider these cases separately.

Case 1: Suppose $x > 2$.

Then $x - 2 > 0$.

Observe that

$$\begin{aligned}
-4 < \frac{x}{x-2} < 4 \quad &\Rightarrow \quad -4(x-2) < x < 4(x-2) \\
&\Leftrightarrow \quad -4x + 8 < x < 4x - 8 \\
&\Leftrightarrow \quad -5x + 8 < 0 < 3x - 8 \\
&\Leftrightarrow \quad -5x + 8 < 0 \text{ and } 0 < 3x - 8 \\
&\Leftrightarrow \quad 8 < 5x \text{ and } 8 < 3x \\
&\Leftrightarrow \quad \frac{8}{5} < x \text{ and } \frac{8}{3} < x
\end{aligned}$$

Thus, $-4 < \frac{x}{x-2} < 4$ implies $\frac{8}{5} < x$ and $\frac{8}{3} < x$.

Since $-4 < \frac{x}{x-2} < 4$, then we conclude $\frac{8}{5} < x$ and $\frac{8}{3} < x$, so $x > \frac{8}{5}$ and $x > \frac{8}{3}$.

Therefore, $x \in (\frac{8}{5}, \infty)$ and $x \in (\frac{8}{3}, \infty)$.

Hence, $x \in (\frac{8}{5}, \infty) \cap (\frac{8}{3}, \infty) = (\frac{8}{3}, \infty)$.

Case 2: Suppose $x < 2$.

Then $x-2<0$.
Observe that

$$
\begin{aligned}
-4 < \frac{x}{x-2} < 4 &\Rightarrow -4(x-2) > x > 4(x-2) \\
&\Leftrightarrow -4x+8 > x > 4x-8 \\
&\Leftrightarrow -5x+8 > 0 > 3x-8 \\
&\Leftrightarrow -5x+8 > 0 \text{ and } 0 > 3x-8 \\
&\Leftrightarrow 8 > 5x \text{ and } 8 > 3x \\
&\Leftrightarrow \frac{8}{5} > x \text{ and } \frac{8}{3} > x
\end{aligned}
$$

Thus, $-4 < \frac{x}{x-2} < 4$ implies $\frac{8}{5} > x$ and $\frac{8}{3} > x$.
Since $-4 < \frac{x}{x-2} < 4$, then we conclude $\frac{8}{5} > x$ and $\frac{8}{3} > x$, so $x < \frac{8}{5}$ and $x < \frac{8}{3}$.
Therefore, $x \in (-\infty, \frac{8}{5})$ and $x \in (-\infty, \frac{8}{3})$, so $x \in (-\infty, \frac{8}{5}) \cap (-\infty, \frac{8}{3}) = (-\infty, \frac{8}{5})$.

Both cases show that either $x \in (-\infty, \frac{8}{5})$ or $x \in (\frac{8}{3}, \infty)$, so $x \in (-\infty, \frac{8}{5}) \cup (\frac{8}{3}, \infty) = T$.
Therefore, $x \in S$ implies $x \in T$, so $S \subset T$.

We next prove $T \subset S$.
Let $y \in T$.
Then either $y \in (-\infty, \frac{8}{5})$ or $y \in (\frac{8}{3}, \infty)$.
We consider these cases separately.
Case 1: Suppose $y \in (\frac{8}{3}, \infty)$.
Then $y > \frac{8}{3}$.
Since $y > \frac{8}{3} > 0$, then $y > 0$.
Since $y > \frac{8}{3} > 2$, then $y > 2$, so $y-2 > 0$.
Observe that

$$
\begin{aligned}
\frac{8}{3} < y &\Leftrightarrow 8 < 3y \\
&\Leftrightarrow y+8 < 4y \\
&\Leftrightarrow y < 4y-8 \\
&\Leftrightarrow y < 4(y-2) \\
&\Rightarrow \frac{y}{y-2} < 4 \\
&\Rightarrow \frac{|y|}{|y-2|} < 4 \\
&\Leftrightarrow |\frac{y}{y-2}| < 4
\end{aligned}
$$

Since $\frac{8}{3} < y$ and $\frac{8}{3} < y$ implies $|\frac{y}{y-2}| < 4$, then we conclude $|\frac{y}{y-2}| < 4$.

Since $y \in \mathbb{R}$ and $|\frac{y}{y-2}| < 4$, then $y \in S$.
Case 2: Suppose $y \in (-\infty, \frac{8}{5})$.
Then $y < \frac{8}{5}$.
Since $y < \frac{8}{5}$ and $\frac{8}{5} < \frac{8}{3}$, then $y < \frac{8}{3}$.
Since $y < \frac{8}{5}$ and $\frac{8}{5} < 2$, then $y < 2$, so $y - 2 < 0$.
Observe that

$$\begin{aligned}
\frac{8}{5} > y \text{ and } \frac{8}{3} > y &\Leftrightarrow 8 > 5y \text{ and } 8 > 3y \\
&\Leftrightarrow 8 - 5y > 0 \text{ and } 0 > 3y - 8 \\
&\Leftrightarrow 8 - 5y > 0 > 3y - 8 \\
&\Leftrightarrow 8 - 4y > y > 4y - 8 \\
&\Leftrightarrow -4(y-2) > y > 4(y-2) \\
&\Rightarrow -4 < \frac{y}{y-2} < 4 \\
&\Leftrightarrow |\frac{y}{y-2}| < 4
\end{aligned}$$

Thus, $\frac{8}{5} > y$ and $\frac{8}{3} > y$ implies $|\frac{y}{y-2}| < 4$.
Since $\frac{8}{5} > y$ and $\frac{8}{3} > y$, then we conclude $|\frac{y}{y-2}| < 4$.
Since $y \in \mathbb{R}$ and $|\frac{y}{y-2}| < 4$, then $y \in S$.

In all cases, $y \in S$.
Thus, if $y \in T$, then $y \in S$, so $T \subset S$.
Since $S \subset T$ and $T \subset S$, then $S = T$, as desired. □

Exercise 35. Let $S = \{x \in \mathbb{R} : \frac{7}{x-3} > x + 3 > 0\}$.
Then $S = (3, 4)$.

Solution. Let $x \in S$.
Then $x \in \mathbb{R}$ and $\frac{7}{x-3} > x + 3 > 0$.
Since $x - 3 \neq 0$, then either $x - 3 > 0$ or $x - 3 < 0$.
Assume $x - 3 > 0$.
Then $7 > (x + 3)(x - 3) > 0$, so $7 > x^2 - 9 > 0$.
Hence, $16 > x^2 > 9$, so $9 < x^2 < 16$.
Thus, $3 < |x| < 4$.
Since $x - 3 > 0$, then $x > 3 > 0$, so $x > 0$.
Thus, $|x| = x$, so $3 < x < 4$.
Hence, $x \in (3, 4)$.

Assume $x - 3 < 0$.
Then $7 < (x + 3)(x - 3) < 0$, so $7 < x^2 - 9 < 0$.
Hence, $16 < x^2 < 9$, so $16 < 9$, a contradiction.
Therefore, $x - 3$ cannot be negative.
We conjecture that $S = (3, 4)$. □

Proof. To prove $S = (3, 4)$, we prove $S \subset (3, 4)$ and $(3, 4) \subset S$.

We first prove $S \subset (3,4)$.

Since $\frac{7}{3.5-3} = \frac{7}{.5} = 14 > 6.5$ and $6.5 > 0$, then $\frac{7}{3.5-3} > 6.5 > 0$, so $\frac{7}{3.5-3} > 3.5+3 > 0$.

Hence, $3.5 \in S$, so $S \neq \emptyset$.

Let $x \in S$.

Then $x \in \mathbb{R}$ and $\frac{7}{x-3} > x+3 > 0$.

Suppose $x-3=0$.

Then $\frac{7}{0} > x+3 > 0$, so $\frac{7}{0} \in \mathbb{R}$.

But, division by zero is not defined, so $x-3 \neq 0$.

Thus, either $x-3>0$ or $x-3<0$.

Suppose $x-3<0$.

Since $\frac{7}{x-3} > x+3 > 0$ and $x-3<0$, then $7 < (x+3)(x-3) < 0$, so $7 < x^2-9 < 0$.

Hence, $16 < x^2 < 9$, so $16 < 9$, a contradiction.

Therefore, $x-3$ cannot be negative.

Thus, $x-3>0$.

Since $\frac{7}{x-3} > x+3 > 0$ and $x-3>0$, then $7 > (x+3)(x-3) > 0$, so $7 > x^2-9 > 0$.

Hence, $16 > x^2 > 9$, so $9 < x^2 < 16$.

Thus, $3 < |x| < 4$.

Since $x-3>0$, then $x>3>0$, so $x>0$.

Thus, $|x| = x$, so $3 < x < 4$.

Hence, $x \in (3,4)$.

Therefore, if $x \in S$, then $x \in (3,4)$, so $S \subset (3,4)$. □

Proof. We prove $(3,4) \subset S$.

Let $y \in (3,4)$.

Then $y \in \mathbb{R}$ and $3 < y < 4$, so $3 < y$ and $y < 4$.

Since $3 < y$, then $y > 3$, so $y-3 > 0$.

Since $3 < y < 4$, then $9 < y^2 < 16$, so $0 < y^2 - 9 < 7$.

Hence, $0 < (y+3)(y-3) < 7$.

Since $y-3>0$, then $0 < y+3 < \frac{7}{y-3}$.

Since $y \in \mathbb{R}$ and $\frac{7}{y-3} > y+3 > 0$, then $y \in S$.

Therefore, $(3,4) \subset S$.

Since $S \subset (3,4)$ and $(3,4) \subset S$, then $S = (3,4)$. □

Exercise 36. Let $S = \{x \in \mathbb{R}^+ : |\frac{x-4}{x}| \leq 2\}$.

Then $S = [\frac{4}{3}, \infty)$.

Proof. Since $|\frac{2-4}{2}| = |\frac{-2}{2}| = |-1| = 1 < 2 \leq 2$, then $2 \in S$, so $S \neq \emptyset$.

We first prove $S \subset [\frac{4}{3}, \infty)$.

Let $x \in S$.

Then $x \in \mathbb{R}^+$ and $|\frac{x-4}{x}| \leq 2$, so $-2 \leq \frac{x-4}{x} \leq 2$.

Since $x \in \mathbb{R}^+$, then $x > 0$, so $-2x \leq x - 4 \leq 2x$.

Hence, $-3x \leq -4 \leq x$, so $-3x \leq -4$.

Thus, $x \geq \frac{4}{3}$, so $x \in [\frac{4}{3}, \infty)$.

Therefore, $S \subset [\frac{4}{3}, \infty)$.

We next prove $[\frac{4}{3}, \infty) \subset S$.

Let $y \in [\frac{4}{3}, \infty)$.

Then $y \geq \frac{4}{3}$, so $3y \geq 4$.

Since $y \geq \frac{4}{3} > 0$, then $y > 0$.

Hence, $3 \geq \frac{4}{y}$.

Since $y > 0$, then $\frac{4}{y} > 0$, so $3 \geq \frac{4}{y} > 0$.

Thus, $0 < \frac{4}{y} \leq 3$.

Observe that

$$\begin{aligned}
0 < \frac{4}{y} \leq 3 \quad &\Leftrightarrow \quad 0 - 1 < \frac{4}{y} - 1 \leq 3 - 1 \\
&\Leftrightarrow \quad -1 < \frac{4}{y} - 1 \leq 2 \\
&\Leftrightarrow \quad -2 < -1 < \frac{4}{y} - 1 \leq 2 \\
&\Rightarrow \quad -2 < \frac{4}{y} - 1 \leq 2 \\
&\Rightarrow \quad -2 \leq \frac{4}{y} - 1 \leq 2 \\
&\Leftrightarrow \quad |\frac{4}{y} - 1| \leq 2 \\
&\Leftrightarrow \quad |\frac{4 - y}{y}| \leq 2 \\
&\Leftrightarrow \quad |\frac{y - 4}{y}| \leq 2.
\end{aligned}$$

Since $0 < \frac{4}{y} \leq 3$ and $0 < \frac{4}{y} \leq 3$ implies $|\frac{y-4}{y}| \leq 2$, then $|\frac{y-4}{y}| \leq 2$.

Since $y > 0$ and $|\frac{y-4}{y}| \leq 2$, then $y \in S$, so $[\frac{4}{3}, \infty) \subset S$.

Since $S \subset [\frac{4}{3}, \infty)$ and $[\frac{4}{3}, \infty) \subset S$, then $S = [\frac{4}{3}, \infty)$, as desired. ☐

Exercise 37. Let $S = \{x \in \mathbb{R} : |\frac{x+4}{x}| < 1\}$.

Then $S = (-\infty, -2)$.

Proof. We prove $S \subset (-\infty, -2)$.

Since $|\frac{-3+4}{-3}| = |\frac{1}{-3}| = \frac{1}{3} < 1$, then $3 \in S$, so $s \neq \emptyset$.

Let $x \in S$.

Then $x \in \mathbb{R}$ and $|\frac{x+4}{x}| < 1$.
Since $\frac{x+4}{x} \in \mathbb{R}$ and division by zero is undefined in $\mathbb{R}$, then x cannot be zero.
Observe that

$$\begin{aligned} |\frac{x+4}{x}| < 1 &\Leftrightarrow |1 + \frac{4}{x}| < 1 \\ &\Leftrightarrow -1 < 1 + \frac{4}{x} < 1 \\ &\Leftrightarrow -2 < \frac{4}{x} < 0 \\ &\Leftrightarrow \frac{-1}{2} < \frac{1}{x} < 0. \end{aligned}$$

Since $|\frac{x+4}{x}| < 1$ and $|\frac{x+4}{x}| < 1$ if and only if $\frac{-1}{2} < \frac{1}{x} < 0$, then $\frac{-1}{2} < \frac{1}{x} < 0$, so $\frac{-1}{2} < \frac{1}{x}$ and $\frac{1}{x} < 0$.
Since $\frac{1}{x} < 0$, then $x < 0$.
Since $\frac{-1}{2} < \frac{1}{x}$ and $x < 0$, then $\frac{-x}{2} > 1$, so $x < -2$.
Hence, $x \in (-\infty, -2)$, so $S \subset (-\infty, -2)$.

We next prove $(-\infty, -2) \subset S$.
Let $y \in (-\infty, -2)$.
Then $y \in \mathbb{R}$ and $y < -2$.
Since $y < -2 < 0$, then $y < 0$, so $\frac{1}{y} < 0$.
Since $y < -2$ and $y < 0$, then $1 > \frac{-2}{y}$, so $\frac{-1}{2} < \frac{1}{y}$.
Thus, $\frac{-1}{2} < \frac{1}{y}$ and $\frac{1}{y} < 0$, so $\frac{-1}{2} < \frac{1}{y} < 0$.
Observe that

$$\begin{aligned} \frac{-1}{2} < \frac{1}{y} < 0 &\Leftrightarrow -2 < \frac{4}{y} < 0 \\ &\Leftrightarrow -1 < 1 + \frac{4}{y} < 1 \\ &\Leftrightarrow |1 + \frac{4}{y}| < 1 \\ &\Leftrightarrow |\frac{y+4}{y}| < 1. \end{aligned}$$

Since $\frac{-1}{2} < \frac{1}{y} < 0$ and $\frac{-1}{2} < \frac{1}{y} < 0$ if and only if $|\frac{y+4}{y}| < 1$, then $|\frac{y+4}{y}| < 1$.
Since $y \in \mathbb{R}$ and $|\frac{y+4}{y}| < 1$, then $y \in S$.
Hence, $y \in (-\infty, -2)$ implies $y \in S$, so $(-\infty, -2) \subset S$.
Since $S \subset (-\infty, -2)$ and $(-\infty, -2) \subset S$, then $S = (-\infty, -2)$, as desired. □

Exercise 38. Let $S = \{x \in \mathbb{R} : \frac{2x+1}{x+2} < 1\}$.
Then $S = (-2, 1)$.

Proof. Let $x \in (-2, 1)$.
Then $x \in \mathbb{R}$ and $-2 < x < 1$, so $-2 < x$ and $x < 1$.
Since $x > -2$, then $x + 2 > 0$.

Since $x < 1$ and $x < 1$ iff $x + 1 < 2$ iff $2x + 1 < x + 2$, then $2x + 1 < x + 2$.
Since $2x + 1 < x + 2$ and $x + 2 > 0$, then $\frac{2x+1}{x+2} < 1$.
Since $x \in \mathbb{R}$ and $\frac{2x+1}{x+2} < 1$, then $x \in S$.
Thus, $(-2, 1) \subset S$.

Let $y \in S$.
Then $y \in \mathbb{R}$ and $\frac{2y+1}{y+2} < 1$, so $\frac{2y+1}{y+2} - 1 < 0$.
Thus, $\frac{2y+1-(y+2)}{y+2} < 0$, so $\frac{y-1}{y+2} < 0$.
Hence, either $y - 1 > 0$ and $y + 2 < 0$ or $y - 1 < 0$ and $y + 2 > 0$.
Suppose $y - 1 > 0$ and $y + 2 < 0$.
Then $y > 1$ and $y < -2$.
Since $y < -2 < 1$, then $y < 1$.
Thus, $y > 1$ and $y < 1$, a contradiction.
Therefore, $y - 1 < 0$ and $y + 2 > 0$.
Hence, $y < 1$ and $y > -2$, so $-2 < y < 1$.
Thus, $y \in (-2, 1)$, so $S \subset (-2, 1)$.

Since $S \subset (-2, 1)$ and $(-2, 1) \subset S$, then $S = (-2, 1)$. □

Exercise 39. Let $S = \{x \in \mathbb{R} : \frac{x-1}{x-2} < \frac{x+1}{x+2}\}$.
Then $S = (-\infty, -2) \cup (0, 2)$.

Proof. To prove $S = (-\infty, -2) \cup (0, 2)$, we prove $S \subset (-\infty, -2) \cup (0, 2)$ and $(-\infty, -2) \cup (0, 2) \subset S$.

We first prove $S \subset (-\infty, -2) \cup (0, 2)$.
Since $\frac{1-1}{1-2} = 0 < \frac{2}{3} = \frac{1+1}{1+2}$, then $1 \in S$, so $S \neq \emptyset$.
Let $x \in S$.
Then $x \in \mathbb{R}$ and $\frac{x-1}{x-2} < \frac{x+1}{x+2}$.

Suppose $x = 2$.
Then $\frac{2-1}{2-2} < \frac{2+1}{2+2}$, so $\frac{1}{0} < \frac{3}{4}$.
Since division by 0 is undefined, then $x \neq 2$.
Suppose $x = -2$.
Then $\frac{-2-1}{-2-2} < \frac{-2+1}{-2+2}$, so $\frac{3}{4} < \frac{-1}{0}$.
Since division by 0 is undefined, then $x \neq -2$.
Since $x \neq 2$ and $x \neq -2$, then either $x > 2$ or $-2 < x < 2$ or $x < -2$.

Suppose $x > 2$.
Then $x - 2 > 0$.
Since $\frac{x-1}{x-2} < \frac{x+1}{x+2}$ and $x - 2 > 0$, then $x - 1 < \frac{x+1}{x+2} \cdot (x - 2)$.
Since $x > 2$, then $x + 2 > 4 > 0$, so $x + 2 > 0$.
Thus, $(x - 1)(x + 2) < (x + 1)(x - 2)$.
Hence, $x^2 + x - 2 < x^2 - x - 2$, so $x < -x$.
Therefore, $2x < 0$, so $x < 0$.

Thus, we have $x < 0$ and $x > 2$, a contradiction.
Hence, x cannot be greater than 2.
Thus, either $-2 < x < 2$ or $x < -2$.

Suppose $-2 < x < 2$.
Then $-2 < x$ and $x < 2$, so $0 < x + 2$ and $x - 2 < 0$.
Since $\frac{x-1}{x-2} < \frac{x+1}{x+2}$ and $x + 2 > 0$, then $\frac{(x-1)(x+2)}{x-2} < x + 1$.
Since $x - 2 < 0$, then $(x - 1)(x + 2) > (x + 1)(x - 2)$.
Hence, $x^2 + x - 2 > x^2 - x - 2$, so $x > -x$.
Therefore, $2x > 0$, so $x > 0$.
Since $0 < x$ and $x < 2$, then $0 < x < 2$, so $x \in (0, 2)$.
Since either $-2 < x < 2$ or $x < -2$ and if $-2 < x < 2$, then $x \in (0, 2)$, then either $x \in (0, 2)$ or $x < -2$.
Hence, either $x \in (0, 2)$ or $x \in (-\infty, -2)$, so $x \in (0, 2) \cup (-\infty, -2)$.
Thus, if $x \in S$, then $x \in (0, 2) \cup (-\infty, -2)$, so $S \subset (0, 2) \cup (-\infty, -2)$.
Therefore, $S \subset (-\infty, -2) \cup (0, 2)$. □

Proof. We prove $(-\infty, -2) \cup (0, 2) \subset S$.
Let $y \in (-\infty, -2) \cup (0, 2)$.
Then $y \in \mathbb{R}$ and either $y < -2$ or $0 < y < 2$.
We consider these cases separately.
Case 1: Suppose $y < -2$.
Then $y + 2 < 0$.
Since $y < -2 < 0$, then $y < 0$, so $-y > 0$.
Since $y < 0$ and $0 < -y$, then $y < -y$.
Thus, $y + (y^2 - 2) < -y + (y^2 - 2)$, so $(y - 1)(y + 2) < (y - 2)(y + 1)$.
Since $y < -2$ and $-2 < 2$, then $y < 2$, so $y - 2 < 0$.
We divide by negative $y - 2$ to get $\frac{(y-1)(y+2)}{y-2} > y + 1$.
Since $y + 2 < 0$, then $\frac{y-1}{y-2} < \frac{y+1}{y+2}$.
Since $y \in \mathbb{R}$ and $\frac{y-1}{y-2} < \frac{y+1}{y+2}$, then $y \in S$.
Case 2: Suppose $0 < y < 2$.
Then $0 < y$ and $y < 2$.
Since $y > 0$, then $-y < 0$.
Since $-y < 0$ and $0 < y$, then $-y < y$.
Thus, $-y + (y^2 - 2) < y + (y^2 - 2)$, so $(y - 2)(y + 1) < (y - 1)(y + 2)$.
Since $y < 2$, then $y - 2 < 0$, so we divide by negative $y - 2$ to get $y + 1 > \frac{(y-1)(y+2)}{y-2}$.
Since $y > 0$ and $0 > -2$, then $y > -2$, so $y + 2 > 0$.
We divide by positive $y + 2$ to get $\frac{y+1}{y+2} > \frac{y-1}{y-2}$.
Hence, $\frac{y-1}{y-2} < \frac{y+1}{y+2}$.
Since $y \in \mathbb{R}$ and $\frac{y-1}{y-2} < \frac{y+1}{y+2}$, then $y \in S$.
Therefore, in all cases, $y \in S$.
Hence, $(-\infty, -2) \cup (0, 2) \subset S$.

Since $S \subset (-\infty, -2) \cup (0, 2)$ and $(-\infty, -2) \cup (0, 2) \subset S$, then $S = (-\infty, -2) \cup (0, 2)$. ☐

Lemma 40. *Let $a, b \in \mathbb{R}$.*
If $a \leq t$ for every real number $t > b$, then $a \leq b$.

Proof. We prove by contrapositive.
Suppose $a > b$.
Let $t = \frac{b+a}{2}$.
Since $b < \frac{b+a}{2} < a$, then $b < t < a$, so $b < t$ and $t < a$.
Thus, there exists $t > b$ such that $a > t$, as desired. ☐

Exercise 41. Let a and b be real numbers.
If $a \leq b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$.

Solution. The hypothesis is $(\forall \epsilon > 0)(a \leq b + \epsilon)$ and the conclusion is $a \leq b$.
Since the conclusion is simple and hypothesis is complex, we try proof by contrapositive.
Thus, we assume $a > b$ and must find $\epsilon > 0$ such that $a > b + \epsilon$.
To find ϵ, let's try working backwards.
Suppose $a > b + \epsilon$.
Then $a - b > \epsilon$, so $\epsilon < a - b$.
Thus, we want ϵ such that $0 < \epsilon < a - b$.
We see that any real number between 0 and $a - b$ will work, so let's conveniently choose $\epsilon = \frac{a-b}{2}$. ☐

Proof. Suppose $a \leq b + \epsilon$ for every $\epsilon > 0$.
Let $\epsilon > 0$ be given.
Let $t = b + \epsilon$.
Then $\epsilon = t - b$, so $t - b > 0$.
Thus, $a \leq b + (t - b) = (t - b) + b = t$, so $a \leq t$.
Hence, $a \leq t$ for every $t - b > 0$, so $a \leq t$ for every $t > b$.
Therefore, by the previous lemma, we conclude $a \leq b$, as desired. ☐

Proof. We prove by contrapositive.
Suppose $a > b$.
Then $a - b > 0$, so $\frac{a-b}{2} > 0$.
Let $\epsilon = \frac{a-b}{2}$.
Then $\epsilon > 0$.
Since $1 > \frac{1}{2}$ and $a - b > 0$, then $a - b > \frac{a-b}{2} = \epsilon$, so $a > b + \epsilon$.
Therefore, there is some $\epsilon > 0$ such that $a > b + \epsilon$, as desired. ☐

Proof. We prove by contrapositive.
Suppose $a > b$.
Then $a - b > 0$, so $\frac{a-b}{2} > 0$.
Let $\epsilon = \frac{a-b}{2}$.
Then $\epsilon > 0$.

Since $a > b$ and

$$\begin{aligned} a > b &\Leftrightarrow 2a > b + a \\ &\Leftrightarrow 2a > 2b + (a - b) \\ &\Leftrightarrow a > b + \frac{a-b}{2} \\ &\Leftrightarrow a > b + \epsilon, \end{aligned}$$

then $a > b + \epsilon$.

Therefore, there exists $\epsilon > 0$ such that $a > b + \epsilon$, as desired. $\square$

Exercise 42. Let $a \in \mathbb{R}$.

If $0 \leq a < \epsilon$ for every real $\epsilon > 0$, then $a = 0$.

Proof. We prove by contradiction.

Suppose $0 \leq a < \epsilon$ for every real $\epsilon > 0$ and $a \neq 0$.

Since $1 > 0$, then $0 \leq a < 1$, so $0 \leq a$.

Since $a \geq 0$ and $a \neq 0$, then $a > 0$.

Hence, $0 \leq a < a$, so $a < a$, a contradiction.

Thus, either $0 \leq a < \epsilon$ for every real $\epsilon > 0$ is false or $a = 0$.

Therefore, $0 \leq a < \epsilon$ for every real $\epsilon > 0$ implies $a = 0$, as desired. $\square$

Exercise 43. Let $a, b \in \mathbb{R}$.

Then $(\frac{a+b}{2})^2 \leq \frac{a^2+b^2}{2}$.

Proof. Since $a, b \in \mathbb{R}$, then $a - b \in \mathbb{R}$, so $(a - b)^2 \in \mathbb{R}$.

Thus, $(a - b)^2 \geq 0$.

Observe that

$$\begin{aligned} (a-b)^2 \geq 0 &\Leftrightarrow a^2 - 2ab + b^2 \geq 0 \\ &\Leftrightarrow a^2 + b^2 \geq 2ab \\ &\Leftrightarrow 2a^2 + 2b^2 \geq a^2 + 2ab + b^2 \\ &\Leftrightarrow 2(a^2 + b^2) \geq (a + b)^2 \\ &\Leftrightarrow \frac{2(a^2 + b^2)}{4} \geq \frac{(a + b)^2}{4} \\ &\Leftrightarrow \frac{a^2 + b^2}{2} \geq (\frac{a + b}{2})^2. \end{aligned}$$

Hence, $\frac{a^2+b^2}{2} \geq (\frac{a+b}{2})^2$, so $(\frac{a+b}{2})^2 \leq \frac{a^2+b^2}{2}$.

$\square$

Exercise 44. At least one of the real numbers $a_1, a_2, ..., a_n$ is greater than or equal to the average of these numbers..

Solution. The statement to prove is shown below.

Let $n \in \mathbb{Z}^+$ and $n \geq 2$.

Let $a_1, a_2, ..., a_n \in \mathbb{R}$.

Then there exists $k \in \{1, 2, ..., n\}$ such that $a_k \geq \frac{\sum_{i=1}^n a_i}{n}$. $\square$

Proof. Let $n \in \mathbb{Z}^+$ and $n \geq 2$.

Let $a_1, a_2, ..., a_n \in \mathbb{R}$.

We prove there exists $k \in \{1, 2, ..., n\}$ such that $a_k \geq \frac{\sum_{i=1}^n a_i}{n}$.

Since $a_1, a_2, ..., a_n \in \mathbb{R}$ and $\mathbb{R}$ is an ordered field, then we can order the numbers so that $a_1 \leq a_2 \leq ... \leq a_n$.

Let $k = n$.

Since $n \in \{1, 2, ..., n\}$, then $k \in \{1, 2, ..., n\}$.

Since $a_1 \leq a_2 \leq ... \leq a_n$, then $a_1 \leq a_n$ and $a_2 \leq a_n$ and ..., $a_n \leq a_n$.

Adding these n inequalities, we obtain $a_1 + a_2 + ... + a_n \leq a_n + a_n + ... + a_n = na_n$.

Observe that

$$\begin{aligned} \sum_{i=1}^n a_i &= a_1 + a_2 + ... + a_n \\ &\leq na_n. \end{aligned}$$

Since $n \geq 2 > 0$, then $n > 0$, so $n \neq 0$.

Hence, we divide by n to obtain $\frac{\sum_{i=1}^n a_i}{n} \leq a_n = a_k$.

Therefore, there exists $k \in \{1, 2, ..., n\}$ such that $a_k \geq \frac{\sum_{i=1}^n a_i}{n}$, as desired. □

Exercise 45. $(\forall x, y \in \mathbb{R})(x < y \rightarrow x^3 < y^3)$.

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$.

Then $0 < y - x$, so $y - x > 0$.

Either $x \geq 0$ or $x < 0$.

We consider these cases separately.

Case 1: Suppose $x \geq 0$.

Then $0 \leq x < y$, so $0 < y$.

Hence, $y > 0$, so $y^2 > 0$.

Since $x \geq 0$, then $x^2 \geq 0$.

Since $x \geq 0$ and $y > 0$, then $xy \geq 0$.

Adding these inequalities, we obtain $y^2 + xy + x^2 > 0$.

Since $y - x > 0$ and $y^2 + xy + x^2 > 0$, then $y^3 - x^3 = (y - x)(y^2 + xy + x^2) > 0$, so $y^3 - x^3 > 0$.

Therefore, $y^3 > x^3$, so $x^3 < y^3$.

Case 2: Suppose $x < 0$.

Either $y \geq 0$ or $y < 0$.

We consider these cases separately.

Case 2a: Suppose $y \geq 0$.

Then $y^3 \geq 0$.

Since $x < 0$, then $x^3 < 0$, so $x^3 < 0 \leq y^3$.

Therefore, $x^3 < y^3$.

Case 2b: Suppose $y < 0$.

Then $x < y < 0$, so $x < 0$.

Since $x < 0$, then $x^2 > 0$.

Since $y < 0$, then $y^2 > 0$.
Since $x < 0$ and $y < 0$, then $xy > 0$.
Adding these inequalities, we obtain $y^2 + xy + x^2 > 0$.
Since $y-x > 0$ and $y^2+xy+x^2 > 0$, then $y^3-x^3 = (y-x)(y^2+xy+x^2) > 0$, so $y^3 - x^3 > 0$.
Therefore, $y^3 > x^3$, so $x^3 < y^3$. $\square$

Exercise 46. Let $a, b \in \mathbb{R}^*$.
If $a < \frac{1}{a} < b < \frac{1}{b}$, then $a < -1$.

Proof. Suppose $a < \frac{1}{a} < b < \frac{1}{b}$.
If $a > 1$, then $a^2 > a > 1 > 0$, so $a^2 > 1$.
Hence, $a > \frac{1}{a}$, which contradicts $a < \frac{1}{a}$.
Therefore, $a \leq 1$.

If $a > 0$, then $0 < a \leq 1$, so $0 < 1 \leq \frac{1}{a}$.
Since $1 \leq \frac{1}{a} < b$, then $1 < b$, so $b > 1 > 0$.
Since $0 < a < \frac{1}{b}$ and $b > 0$, then $0 < ab < 1$, so $ab < 1$.
Since $\frac{1}{a} < b$ and $a > 0$, then $1 < ab$, so $ab > 1$, which contradicts $ab < 1$.
Hence, $a \leq 0$.

Since $a \neq 0$, then this implies $a < 0$.

Suppose $a \geq -1$.
Then $-1 \leq a < 0$, so $-a \geq a^2 > 0$.
Since $a < \frac{1}{a} < 0$ and $a < 0$, then $a^2 > 1 > 0$.
Since $-a \geq a^2$ and $a^2 > 1$, then $-a > 1$, so $a < -1$.
But, this contradicts $a \geq -1$.
Therefore, $a < -1$, as desired. $\square$

Absolute value in an ordered field

Exercise 47. Find a constant M such that $|\frac{2x^2+3x+1}{2x-1}| \leq M$ for all x satisfying $2 \leq x \leq 3$.

Solution. Let $x \in \mathbb{R}$ such that $2 \leq x \leq 3$.
Then $2 \leq x$ and $x \leq 3$.
Since $x \geq 2 > 0$, then $x > 0$.
Since $x \leq 3$ and $x > 0$, then $0 < |x| = x \leq 3$.
Since $x \geq 2$ and $x > 0$, then $|x| = x \geq 2$.

Since $|2x^2 + 3x + 1| \leq |2x^2| + |3x| + |1| = 2|x^2| + 3|x| + 1 = 2|x|^2 + 3|x| + 1$, then $|2x^2 + 3x + 1| \leq 2|x|^2 + 3|x| + 1$.
Since $0 < |x| \leq 3$, then $2|x|^2 + 3|x| + 1 \leq 2 \cdot 3^2 + 3 \cdot 3 + 1 = 28$.
Since $|2x^2 + 3x + 1| \leq 2|x|^2 + 3|x| + 1 \leq 28$, then $|2x^2 + 3x + 1| \leq 28$.
Since $x > 0$, then $2x^2 + 3x + 1 > 0$, so $0 < |2x^2 + 3x + 1| \leq 28$.

Since $|2x-1| \geq |2x| - |1| = 2|x| - 1$, then $|2x-1| \geq 2|x| - 1$.
Since $|x| \geq 2$, then $2|x| - 1 \geq 2 \cdot 2 - 1 = 3$.
Since $|2x-1| \geq 2|x| - 1 \geq 3$, then $|2x-1| \geq 3$.
Since $0 < 3 \leq |2x-1|$, then $0 < \frac{1}{|2x-1|} \leq \frac{1}{3}$.

Since $0 < |2x^2+3x+1| \leq 28$ and $0 < \frac{1}{|2x-1|} \leq \frac{1}{3}$, then $0 < \frac{|2x^2+3x+1|}{|2x-1|} \leq \frac{28}{3}$, so $|\frac{2x^2+3x+1}{2x-1}| \leq \frac{28}{3} = M$. □

Exercise 48. Let $S = \{x \in \mathbb{R} : |\frac{x}{x+1}| \leq 1\}$.
Then $S = [\frac{-1}{2}, \infty)$.

Solution. Suppose $x \in \mathbb{R}$ and $|\frac{x}{x+1}| \leq 1$.
Then $-1 \leq \frac{x}{x+1} \leq 1$.
Since division by zero is not defined, then $x+1 \neq 0$, so either $x+1 > 0$ or $x+1 < 0$.
Assume $x+1 > 0$.
Then $-(x+1) \leq x \leq x+1$, so $-x-1 \leq x \leq x+1$.
Thus, $-x-1 \leq x$ and $x \leq x+1$, so $-1 \leq 2x$ and $0 \leq 1$.
Hence, $\frac{-1}{2} \leq x$, so $x \geq \frac{-1}{2}$.
Now, assume $x+1 < 0$.
Then $-(x+1) \geq x \geq x+1$, so $-(x+1) \geq x$ and $x \geq x+1$.
Thus, $x \geq x+1$, so $0 \geq 1$, a contradiction.
Hence, $x+1$ cannot be negative.
Therefore, $x \geq \frac{-1}{2}$, so $x \in [\frac{-1}{2}, \infty)$.
We conjecture that $S = [\frac{-1}{2}, \infty)$. □

Proof. To prove $S = [\frac{-1}{2}, \infty)$, we prove $S \subset [\frac{-1}{2}, \infty)$ and $[\frac{-1}{2}, \infty) \subset S$.
We first prove $S \subset [\frac{-1}{2}, \infty)$.
Since $|\frac{0}{0+1}| = 0 < 1$, then $0 \in S$, so S is not empty.
Let $x \in S$.
Then $x \in \mathbb{R}$ and $|\frac{x}{x+1}| \leq 1$, so $-1 \leq \frac{x}{x+1} \leq 1$.
Suppose $x+1 = 0$.
Then $|\frac{x}{0}| \leq 1$, so $\frac{x}{0} \in \mathbb{R}$.
But, division by zero is not defined, so $x+1 \neq 0$.
Thus, either $x+1 > 0$ or $x+1 < 0$.

Suppose $x+1 < 0$.
Since $-1 \leq \frac{x}{x+1} \leq 1$ and $x+1 < 0$, then $-(x+1) \geq x \geq x+1$, so $-x-1 \geq x \geq x+1$.
Hence, $-x-1 \geq x$ and $x \geq x+1$, so $-1 \geq 2x$ and $0 \geq 1$.
Thus, $0 \geq 1$, a contradiction.
Therefore, $x+1$ cannot be negative.
Hence, $x+1 > 0$.
Since $-1 \leq \frac{x}{x+1} \leq 1$ and $x+1 > 0$, then $-(x+1) \leq x \leq x+1$, so $-x-1 \leq x \leq x+1$.

Hence, $-x-1 \leq x$ and $x \leq x+1$, so $-1 \leq 2x$ and $0 \leq 1$.
Thus, $-1 \leq 2x$, so $2x \geq -1$.
Therefore, $x \geq \frac{-1}{2}$, so $x \in [\frac{-1}{2}, \infty)$.
Consequently, if $x \in S$, then $x \in [\frac{-1}{2}, \infty)$, so $S \subset [\frac{-1}{2}, \infty)$. □

Proof. We next prove $[\frac{-1}{2}, \infty) \subset S$.
Let $y \in [\frac{-1}{2}, \infty)$.
Then $y \geq \frac{-1}{2}$.
Thus, $y + 1 \geq \frac{1}{2} > 0$, so $y + 1 > 0$.
Hence, $|y + 1| = y + 1 > 0$.
Either $y \geq 0$ or $y < 0$.
We consider these cases separately.
Case 1: Suppose $y \geq 0$.
Then $|y| = y$.
Since $1 > 0$, then $y + 1 > y$.
Since $y + 1 > 0$, then $1 > \frac{y}{y+1} = \frac{|y|}{|y+1|} = |\frac{y}{y+1}|$, so $1 > |\frac{y}{y+1}|$.
Thus, $|\frac{y}{y+1}| < 1$.
Case 2: Suppose $y < 0$.
Then $|y| = -y$.
Since $y \geq \frac{-1}{2}$, then $-2y \leq 1$, so $-y \leq y + 1$.
Thus, $|y| \leq |y + 1|$.
Since $|y + 1| > 0$, then $\frac{|y|}{|y+1|} \leq 1$, so $|\frac{y}{y+1}| \leq 1$.
Therefore, in all cases, $|\frac{y}{y+1}| \leq 1$, so $y \in S$.
Thus, if $y \in [\frac{-1}{2}, \infty)$, then $y \in S$, so $[\frac{-1}{2}, \infty) \subset S$.

Since $S \subset [\frac{-1}{2}, \infty)$ and $[\frac{-1}{2}, \infty) \subset S$, then $S = [\frac{-1}{2}, \infty)$, as desired. □

Exercise 49. Let $a, b \in \mathbb{R}$.
If $0 \leq a < b$, then $0 \leq a^2 < b^2$.

Proof. Suppose $0 \leq a < b$.
Then $0 \leq a$ and $a < b$.
Since $a \geq 0$, then either $a > 0$ or $a = 0$.
We consider these cases separately.
Case 1: Suppose $a = 0$.
Since $a < b$, then $0 < b$.
Since $b > 0$, then $b^2 > 0$.
Since $0 = a = 0^2 = a^2 < b^2$, then $0 = a^2 < b^2$.
Case 2: Suppose $a > 0$.
Since $0 < a$ and $a < b$, then $0 < a < b$, so $0 < a^2 < b^2$.
Therefore, in all cases, $0 \leq a^2 < b^2$. □

Exercise 50. Let $B = \{x \in \mathbb{R} : |x - 1| < |x|\}$.
Then $B = (\frac{1}{2}, \infty)$.

Proof. Let $x \in B$.

Then $x \in \mathbb{R}$ and $|x-1| < |x|$.

Since $0 \leq |x-1| < |x|$, then $|x-1|^2 < |x|^2$.

Since $x^2 - 2x + 1 = (x-1)^2 = |x-1|^2 < |x|^2 = x^2$, then $x^2 - 2x + 1 < x^2$, so $-2x + 1 < 0$.

Hence, $1 < 2x$, so $\frac{1}{2} < x$.

Thus, $x > \frac{1}{2}$, so $x \in (\frac{1}{2}, \infty)$.

Therefore, $B \subset (\frac{1}{2}, \infty)$.

Let $x \in (\frac{1}{2}, \infty)$.

Then $x > \frac{1}{2}$, so either $\frac{1}{2} < x < 1$ or $x \geq 1$.

We consider these cases separately.

Case 1: Suppose $x \geq 1$.

Then $x - 1 \geq 0$.

Since $x \geq 1 > 0$, then $x > 0$.

Since $-1 < 0$, then $x - 1 < x$.

Hence, $|x-1| = x - 1 < x = |x|$.

Case 2: Suppose $\frac{1}{2} < x < 1$.

Then $\frac{1}{2} < x$ and $x < 1$.

Since $x > \frac{1}{2} > 0$, then $x > 0$.

Since $x < 1$, then $x - 1 < 0$.

Since $\frac{1}{2} < x$, then $1 < 2x$, so $1 - x < x$.

Thus, $|x-1| = 1 - x < x = |x|$.

In all cases, $|x-1| < |x|$, so $x \in B$.

Therefore, $(\frac{1}{2}, \infty) \subset B$.

Since $B \subset (\frac{1}{2}, \infty)$ and $(\frac{1}{2}, \infty) \subset B$, then $B = (\frac{1}{2}, \infty)$. ☐

Exercise 51. Let F be an ordered field.

Then $|x - y| \leq |x| + |y|$ for all $x, y \in F$.

Proof. Let $x, y \in F$.

Then

$$\begin{aligned} |x-y| &= |x+(-y)| \\ &\leq |x| + |-y| \\ &= |x| + |y|. \end{aligned}$$

Therefore, $|x-y| \leq |x| + |y|$. ☐

Exercise 52. Let a, x be elements of an ordered field F.

If $a \geq 0$ and $x \leq a$ and $-x \leq a$, then $|x| \leq a$.

Proof. Suppose $a \geq 0$ and $x \leq a$ and $-x \leq a$.

Either $x \geq 0$ or $x < 0$.

We consider these cases separately.

Case 1: Suppose $x \geq 0$.

Then $|x| = x \leq a$, so $|x| \leq a$.
Case 2: Suppose $x < 0$.
Then $|x| = -x \leq a$, so $|x| \leq a$. □

Exercise 53. Let F be an ordered field.
Let $x, y, z \in F$.
Then $d(x, y) = d(x - z, y - z)$.

Proof. Observe that

$$\begin{aligned} d(x,y) &= |x-y| \\ &= |x-y-z+z| \\ &= |x-z-y+z| \\ &= |(x-z)-(y-z)| \\ &= d(x-z, y-z). \end{aligned}$$

□

Exercise 54. Let $x, y, z \in \mathbb{R}$ with $x \leq z$.
Then $x \leq y \leq z$ iff $|x - y| + |y - z| = |x - z|$.

Proof. Suppose $x \leq y \leq z$.
Then $x \leq y$ and $y \leq z$ and $x \leq z$, so $x - y \leq 0$ and $y - z \leq 0$ and $x - z \leq 0$.
Thus,

$$\begin{aligned} |x-y| + |y-z| &= -(x-y) - (y-z) \\ &= -x + y - y + z \\ &= -x + z \\ &= -(x-z) \\ &= |x-z|. \end{aligned}$$

□

Proof. Conversely, suppose $|x - y| + |y - z| = |x - z|$.
We must prove $x \leq y$ and $y \leq z$.
Suppose $x > y$.
Then $y < x$.
Since $y < x$ and $x \leq z$, then $y < x \leq z$, so $|y - x| + |x - z| = |y - z|$.
Hence, $|x - y| = |y - x| = |y - z| - |x - z|$.
Since $|x - y| + |y - z| = |x - z|$, then $|x - y| = |x - z| - |y - z|$.
Adding these equations we obtain $2|x - y| = 0$, so $|x - y| = 0$.
Thus, $x - y = 0$, so $x = y$.
But, this contradicts the fact that $x > y$.
Therefore, $x \leq y$.

Suppose $y > z$.
Then $z < y$.
Since $x \leq z$ and $z < y$, then $x \leq z < y$, so $|x - z| + |z - y| = |x - y|$.
Hence, $|y - z| = |z - y| = |x - y| - |x - z|$.
Since $|x - y| + |y - z| = |x - z|$, then $|y - z| = |x - z| - |x - y|$.
Adding these equations we obtain $2|y - z| = 0$, so $|y - z| = 0$.
Thus, $y - z = 0$, so $y = z$.
But, this contradicts the fact that $y > z$.
Therefore, $y \leq z$.
Since $x \leq y$ and $y \leq z$, then $x \leq y \leq z$. □

Exercise 55. Let $x \in \mathbb{R}$.
Then $|x| = \max\{x, -x\}$.

Proof. Let $S = \{x, -x\}$.
We must prove $|x| = \max S$.
Either $x \geq 0$ or $x < 0$.
We consider these cases separately.
Case 1: Suppose $x \geq 0$.
Then $-x \leq 0$.
Hence, $-x \leq 0 \leq x$, so $-x \leq x$.
Since $x \in S$ and $-x \leq x$ and $x \leq x$, then $\max S = x = |x|$.
Case 2: Suppose $x < 0$.
Then $-x > 0$.
Hence, $x < 0 < -x$, so $x < -x$.
Thus, $x \leq -x$.
Since $-x \in S$ and $x \leq -x$ and $-x \leq -x$, then $\max S = -x = |x|$.
Therefore, in all cases, $\max S = |x|$, as desired. □

Exercise 56. Let F be an ordered field.
Let $a, b \in F$.
If a and b are both non-negative or both negative, then $|a + b| = |a| + |b|$.

Proof. Suppose a and b are both non-negative or both negative.
Then either a and b are both non-negative or a and b are both negative.
We consider these cases separately.
Case 1: Suppose a and b are both non-negative.
Then $a \geq 0$ and $b \geq 0$.
Hence, $a + b \geq 0$.
Therefore, $|a + b| = a + b = |a| + |b|$, as desired.
Case 2: Suppose a and b are both negative.
Then $a < 0$ and $b < 0$.
Hence, $a + b < 0$.
Therefore, $|a + b| = -(a + b) = -a + (-b) = |a| + |b|$, as desired. □

Exercise 57. Let F be an ordered field.
Let $a, b \in F$.
If $|a + b| = |a| + |b|$, then $ab \geq 0$.

Proof. Suppose $|a+b| = |a|+|b|$.
Then $|a+b|^2 = (|a|+|b|)^2$.
Thus,

$$\begin{aligned}
0 &= (|a|+|b|)^2 - |a+b|^2 \\
&= |a|^2 + 2|a||b| + |b|^2 - (a+b)^2 \\
&= a^2 + 2|ab| + b^2 - (a^2 + 2ab + b^2) \\
&= a^2 + 2|ab| + b^2 - a^2 - 2ab - b^2 \\
&= 2|ab| - 2ab \\
&= 2(|ab| - ab).
\end{aligned}$$

Since $2(|ab| - ab) = 0$, then $|ab| - ab = 0$, so $|ab| = ab$.
Either $ab \geq 0$ or $ab < 0$.

Suppose for the sake of contradiction $ab < 0$.
Then $|ab| = ab < 0$, so $|ab| < 0$.
But, this contradicts the fact that $|ab| \geq 0$.
Therefore, $ab \geq 0$. □

Exercise 58. Let $\epsilon > 0$.
Let $a, x \in \mathbb{R}$.
Then $|x-a| < \epsilon$ iff $a - \epsilon < x < a + \epsilon$.

Proof. For any real number r and $k > 0$, $|r| < k$ iff $-k < r < k$.
Since $x - a \in \mathbb{R}$ and $\epsilon > 0$, then $|x-a| < \epsilon$ iff $-\epsilon < x - a < \epsilon$.
Therefore, $|x-a| < \epsilon$ iff $a - \epsilon < x < a + \epsilon$, as desired. □

Exercise 59. Prove $|x-z| \geq |x| - |z|$ and $|x+y+z| \leq |x|+|y|+|z|$.

Proof. To prove $|x-z| \geq |x| - |z|$, we let x and z be arbitrary real numbers.
Observe that $|(x-z)+z| \leq |x-z| + |z|$, by the triangle inequality.
Hence, $|x+(-z)+z| \leq |x-z|+|z|$, so $|x+0| \leq |x-z|+|z|$.
Thus, $|x| \leq |x-z|+|z|$, so $|x|-|z| \leq |x-z|$.
Therefore, $|x-z| \geq |x|-|z|$, as desired.
To prove $|x+y+z| \leq |x|+|y|+|z|$, we let $x, y, z \in \mathbb{R}$ be arbitrary.
Then $|(x+y)+z| \leq |x+y|+|z|$, by the triangle inequality.
Hence, $|(x+y)+z| - |z| \leq |x+y|$.
By the triangle inequality, $|x+y| \leq |x|+|y|$.
Since $|(x+y)+z| - |z| \leq |x+y|$ and $|x+y| \leq |x|+|y|$, then by transitivity of $\leq$, we conclude that $|(x+y)+z| - |z| \leq |x|+|y|$.
Therefore, $|(x+y)+z| \leq |x|+|y|+|z|$.
Hence, $|x+y+z| \leq |x|+|y|+|z|$, as desired. □

Boundedness of sets in an ordered field

Exercise 60. In $\mathbb{R}$ $\sup(-\infty, 2] = 2$.

Proof. Let $S = (-\infty, 2] = \{x \in \mathbb{R} : x \leq 2\}$.
To prove $\sup S = 2$, we must prove 2 is an upper bound of S and 2 is the least upper bound of S.

We prove 2 is an upper bound of S.
Let $x \in S$.
Then $x \leq 2$, so $x \leq 2$ for all $x \in S$.
Therefore, 2 is an upper bound of S.

To prove 2 is the least upper bound of S, let $r \in \mathbb{R}$ such that $r < 2$.
Since $2 \in S$ and $2 > r$, then r is not an upper bound of S.
Since r is arbitrary, then every r less than 2 is not an upper bound of S.
Therefore, 2 is the least upper bound of S, so $\sup S = 2$. □

Exercise 61. $1 = inf(\mathbb{N})$.

Proof. We must prove 1 is the greatest lower bound of $\mathbb{N}$ in $\mathbb{R}$.
Let $n \in \mathbb{N}$.
Then $n \geq 1$, so $1 \leq n$.
Thus, $1 \leq n$ for all $n \in \mathbb{N}$.
Therefore, 1 is a lower bound of $\mathbb{N}$ in $\mathbb{R}$.
Let $\epsilon > 0$ be given.
To prove 1 is the greatest lower bound, we must prove there exists $n \in \mathbb{N}$ such that $n < 1 + \epsilon$.
Take $n = 1$.
Then $n = 1 \in \mathbb{N}$.
We prove $n < 1 + \epsilon$.
Observe that

$$\begin{aligned} 0 < \epsilon &\Leftrightarrow 1 < 1 + \epsilon \\ &\Rightarrow n < 1 + \epsilon. \end{aligned}$$

Therefore, $n < 1 + \epsilon$, as desired. □

Exercise 62. Let $a, b \in \mathbb{R}$ with $a < b$.
Then $b = lub[a, b]$.

Proof. Let $x \in [a, b]$.
Then $x \in \mathbb{R}$ and $a \leq x \leq b$.
Hence, $x \leq b$.
Thus, $x \leq b$ for all $x \in [a, b]$.
Therefore, b is an upper bound of $[a, b]$.
Let $\epsilon > 0$ be given.

To prove b is the least upper bound, we must prove there exists $y \in [a,b]$ such that $y > b - \epsilon$.

Take $y = b$.

Since $b \in [a,b]$, then $y \in [a,b]$.

Observe that

$$\begin{aligned} \epsilon > 0 &\Leftrightarrow \epsilon > b - b \\ &\Leftrightarrow \epsilon + b > b \\ &\Leftrightarrow b > b - \epsilon \\ &\Rightarrow y > b - \epsilon. \end{aligned}$$

Therefore, $y > b - \epsilon$, as desired. ☐

Exercise 63. Let $a \in \mathbb{R}$.

Then $a = glb(a, \infty)$.

Proof. Let $x \in (a, \infty)$.

Then $x \in \mathbb{R}$ and $a < x$.

Hence, $a \leq x$.

Thus, $a \leq x$ for all $x \in (a, \infty)$.

Therefore, a is a lower bound of (a, ∞).

Let $\epsilon > 0$ be given.

To prove a is the greatest lower bound, we must prove there exists $y \in (a, \infty)$ such that $y < a + \epsilon$.

Take $y = a + \frac{\epsilon}{2}$.

We prove $y \in (a, \infty)$.

Since $\epsilon > 0$, then $\frac{\epsilon}{2} > 0$, so $a + \frac{\epsilon}{2} > a$.

Hence, $y > a$, so $y \in (a, \infty)$, as desired.

We prove $y < a + \epsilon$.

Since $\frac{1}{2} < 1$ and $\epsilon > 0$, then $\frac{\epsilon}{2} < \epsilon$.

Hence, $a + \frac{\epsilon}{2} < a + \epsilon$.

Thus, $y < a + \epsilon$, as desired. ☐

Exercise 64. Let $S = (3, 4) \cup \{6\}$ in $\mathbb{R}$.

Then $3 = \inf S$ and $6 = \sup S$.

Proof. We must prove 3 is the greatest lower bound of S in $\mathbb{R}$ and 6 is the least upper bound of S in $\mathbb{R}$.

We prove 3 is the greatest lower bound of S in $\mathbb{R}$.

Let $x \in S$.

Then either $x \in (3, 4)$ or $x \in \{6\}$.

We consider these cases separately.

Case 1: Suppose $x \in (3, 4)$.

Then $3 < x < 4$, so $3 < x$.

Hence, $3 \leq x$.

Case 2: Suppose $x \in \{6\}$.

Then $x = 6$.

Since $3 < 6 = x$, then $3 < x$, so $3 \leq x$.
Hence, in all cases, $3 \leq x$.
Thus, $3 \leq x$ for all $x \in S$.
Therefore, 3 is a lower bound of S in $\mathbb{R}$.
Let $\epsilon > 0$ be given.
To prove 3 is the greatest lower bound, we must prove there exists $s \in S$ such that $s < 3 + \epsilon$.
Let $k = \min\{\frac{1}{2}, \frac{\epsilon}{2}\}$.
Then $k \leq \frac{1}{2}$ and $k \leq \frac{\epsilon}{2}$.
Let $s = 3 + k$.
To prove $s \in S$, we prove $3 + k \in (3, 4)$.
Thus, we must prove $3 < 3 + k$ and $3 + k < 4$.
Since $\epsilon > 0$, then $\frac{\epsilon}{2} > 0$.
Since $\frac{1}{2} > 0$ and $\frac{\epsilon}{2} > 0$, then $\min\{\frac{1}{2}, \frac{\epsilon}{2}\} > 0$.
Therefore, $k > 0$.
Hence, $0 < k$, so $3 < 3 + k$, as desired.
Since $k \leq \frac{1}{2}$ and $\frac{1}{2} < 1$, then $k < 1$.
Hence, $3 + k < 4$, as desired.
Therefore, $s \in S$.
We prove $s < 3 + \epsilon$.
Since $\frac{1}{2} < 1$ and $\epsilon > 0$, then $\frac{\epsilon}{2} < \epsilon$.
Observe that

$$\begin{aligned} s &= 3 + k \\ &\leq 3 + \frac{\epsilon}{2} \\ &< 3 + \epsilon. \end{aligned}$$

Therefore, $s < 3 + \epsilon$, as desired. □

Proof. We prove 6 is the least upper bound of S in $\mathbb{R}$.
Let $x \in S$.
Then either $x \in (3, 4)$ or $x \in \{6\}$.
We consider these cases separately.
Case 1: Suppose $x \in (3, 4)$.
Then $3 < x < 4$, so $x < 4$.
Since $x < 4 < 6$, then $x < 6$.
Case 2: Suppose $x \in \{6\}$.
Then $x = 6$.
Hence, in all cases, either $x < 6$ or $x = 6$, so $x \leq 6$.
Thus, $x \leq 6$ for all $x \in S$.
Therefore, 6 is an upper bound of S in $\mathbb{R}$.
Let $\epsilon > 0$ be given.
To prove 6 is the least upper bound, we must prove there exists $s \in S$ such that $s > 6 - \epsilon$.
Take $s = 6$.
Then $s = 6 \in S$.

We prove $s > 6 - \epsilon$.
Observe that

$$\begin{aligned}\epsilon > 0 &\Leftrightarrow \epsilon > 6 - 6 \\ &\Leftrightarrow 6 + \epsilon > 6 \\ &\Leftrightarrow 6 > 6 - \epsilon \\ &\Rightarrow s > 6 - \epsilon.\end{aligned}$$

Therefore, $s > 6 - \epsilon$, as desired. ☐

Exercise 65. Let $S = \{x \in \mathbb{R} : x \geq 0\}$.
Then
1. There is no upper bound of S.
2. 0 is a lower bound of S.
3. $\inf S = 0$.

Proof. We prove 1.
To prove there is no upper bound of S, we prove for every real B there exists $x \in S$ such that $x > B$.
Let $B \in \mathbb{R}$ be arbitrary.
Let $T = \{0, B\}$.
Let $x = \max T + 1$.
Since $\max T \geq 0$ and $1 > 0$, then $\max T + 1 > 0$, so $x > 0$.
Thus, $x \in S$.
Since $\max T \geq B$ and $1 > 0$, then $\max T + 1 > B$, so $x > B$.
Therefore, there exists $x \in S$ such that $x > B$, as desired. ☐

Proof. We prove 2.
To prove 0 is a lower bound of S, we prove for every $x \in S$ we have $0 \leq x$.
Let $x \in S$ be given.
Then $x \geq 0$, so $0 \leq x$ for all $x \in S$.
Therefore, 0 is a lower bound of S. ☐

Proof. We prove 3.
To prove $\inf S = 0$, we prove 0 is a lower bound of S and every real number $r > 0$ is not a lower bound of S.
Let $r > 0$ be arbitrary.
We prove r is not a lower bound of S.
Let $x = \frac{r}{2}$.
Since $r > 0$, then $\frac{r}{2} > 0$, so $x > 0$.
Thus, $x \in S$.
Since $r > 0$, then $2r > r$, so $r > \frac{r}{2}$.
Hence, $r > x$.
Therefore, there exists $x \in S$ such that $x < r$, so r is not a lower bound of S.
Consequently, every real number $r > 0$ is not a lower bound of S.

Since 0 is a lower bound of S and every real number $r > 0$ is not a lower bound of S, then 0 is the greatest lower bound of S, so $0 = \inf S$. □

Exercise 66. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.

Then

1. There is no upper bound of $\mathbb{R}^+$, so $\sup \mathbb{R}^+$ does not exist.
2. 0 is a lower bound of $\mathbb{R}^+$.
3. $\inf \mathbb{R}^+ = 0$.

Proof. We prove 1.

To prove there is no upper bound of $\mathbb{R}^+$, we prove for every real B there exists $x \in \mathbb{R}^+$ such that $x > B$.

Let $B \in \mathbb{R}$ be arbitrary.

Let $T = \{0, B\}$.

Let $x = \max T + 1$.

Since $\max T \geq 0$ and $1 > 0$, then $\max T + 1 > 0$, so $x > 0$.

Thus, $x \in \mathbb{R}^+$.

Since $\max T \geq B$ and $1 > 0$, then $\max T + 1 > B$, so $x > B$.

Therefore, there exists $x \in \mathbb{R}^+$ such that $x > B$, as desired. □

Proof. We prove 2.

To prove 0 is a lower bound of $\mathbb{R}^+$, we prove for every $x \in \mathbb{R}^+$ we have $0 \leq x$.

Let $x \in \mathbb{R}^+$ be given.

Then $x \geq 0$, so $0 \leq x$ for all $x \in \mathbb{R}^+$.

Therefore, 0 is a lower bound of $\mathbb{R}^+$. □

Proof. We prove 3.

To prove $\inf \mathbb{R}^+ = 0$, we prove 0 is a lower bound of $\mathbb{R}^+$ and every real number $r > 0$ is not a lower bound of $\mathbb{R}^+$.

Let $r > 0$ be arbitrary.

We prove r is not a lower bound of $\mathbb{R}^+$.

Let $x = \frac{r}{2}$.

Since $r > 0$, then $\frac{r}{2} > 0$, so $x > 0$.

Thus, $x \in \mathbb{R}^+$.

Since $r > 0$, then $2r > r$, so $r > \frac{r}{2}$.

Hence, $r > x$.

Therefore, there exists $x \in \mathbb{R}^+$ such that $x < r$, so r is not a lower bound of $\mathbb{R}^+$.

Consequently, every real number $r > 0$ is not a lower bound of $\mathbb{R}^+$.

Since 0 is a lower bound of $\mathbb{R}^+$ and every real number $r > 0$ is not a lower bound of S, then 0 is the greatest lower bound of $\mathbb{R}^+$, so $0 = \inf \mathbb{R}^+$. □

Lemma 67. *For every natural number n, $|(-1)^n| = 1$.*

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : |(-1)^n| = 1\}$.

Since $1 \in \mathbb{N}$ and $|-1^1| = |-1| = 1$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $|(-1)^k| = 1$.

Since $k \in \mathbb{N}$, then $k+1 \in \mathbb{N}$.

Observe that $|(-1)^{k+1}| = |(-1)^k(-1)| = |(-1)^k| \cdot |-1| = 1 \cdot 1 = 1$.

Since $k+1 \in \mathbb{N}$ and $|(-1)^{k+1}| = 1$, then $k+1 \in S$.

Thus, $k \in S$ implies $k+1 \in S$.

By the principle of mathematical induction, $|(-1)^n| = 1$ for all $n \in \mathbb{N}$. ☐

Lemma 68. *Let* $n \in \mathbb{N}$.

1. If n *is even, then* $(-1)^n = 1$.

2. If n *is odd, then* $(-1)^n = -1$.

Proof. We prove 1.

Suppose n is even.

Then $n = 2k$ for some integer k.

Thus, $(-1)^n = (-1)^{2k} = [(-1)^2]^k = 1^k = 1$. ☐

Proof. We prove 2.

Suppose n is odd.

Then $n = 2k+1$ for some integer k.

Since $2k$ is even, then $(-1)^n = (-1)^{2k+1} = (-1)^{2k} \cdot (-1)^1 = 1 \cdot (-1) = -1$. ☐

Exercise 69. Let $S = \{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\}$.

Then

1. $\sup S = 2$.
2. $\inf S = \frac{1}{2}$.

Proof. We prove 1.

We first prove 2 is an upper bound of S.

Since $2 = 1+1 = 1-(-1) = 1 - \frac{-1^1}{1}$, then $2 \in S$, so $S \neq \emptyset$.

Let $x \in S$.

Then there exists $n \in \mathbb{N}$ such that $x = 1 - \frac{(-1)^n}{n}$.

Since $n \in \mathbb{N}$, then $n \geq 1$, so $1 \geq \frac{1}{n}$.

Hence, $2 \geq 1 + \frac{1}{n}$.

Observe that

$$\begin{aligned}
|x| &= |1 - \frac{(-1)^n}{n}| \\
&= |1 + \frac{(-1)^n}{-n}| \\
&\leq |1| + |\frac{(-1)^n}{-n}| \\
&= 1 + \frac{|(-1)^n|}{|-n|} \\
&= 1 + \frac{1}{n} \\
&\leq 2.
\end{aligned}$$

Thus, $|x| \leq 2$, so $-2 \leq x \leq 2$.

Hence, $x \leq 2$, so 2 is an upper bound of S.

To prove 2 is the least upper bound of S, we prove every real number $r < 2$ is not an upper bound of S.

Let $r < 2$ be an arbitrary real number.

Since $2 \in S$ and $2 > r$, then r is not an upper bound of S.

Thus, every real number $r < 2$ is not an upper bound of S.

Since 2 is an upper bound of S and every real number $r < 2$ is not an upper bound of S, then 2 is the least upper bound of S, so $2 = \sup S$. $\square$

Proof. We prove 2.

To prove $\frac{1}{2} = \inf S$, we prove $\frac{1}{2}$ is a lower bound of S and we prove for every real number $r > \frac{1}{2}$, r is not a lower bound of S.

We first prove $\frac{1}{2}$ is a lower bound of S.

Let $x \in S$.

Then there exists $n \in \mathbb{N}$ such that $x = 1 - \frac{(-1)^n}{n}$.

Since $n \in \mathbb{N}$, then either n is even or n is odd.

We consider these cases separately.

Case 1: Suppose n is even.

Then $n \geq 2$ and $(-1)^n = 1$.

Observe that

$$\begin{aligned}
n \geq 2 &\Leftrightarrow \frac{1}{2} \geq \frac{1}{n} \\
&\Leftrightarrow \frac{1}{2} \geq \frac{(-1)^n}{n} \\
&\Leftrightarrow 1 \geq \frac{1}{2} + \frac{(-1)^n}{n} \\
&\Leftrightarrow 1 - \frac{(-1)^n}{n} \geq \frac{1}{2} \\
&\Leftrightarrow x \geq \frac{1}{2}.
\end{aligned}$$

Thus, $x \geq \frac{1}{2}$, so $\frac{1}{2} \leq x$.
Case 2: Suppose n is odd.
Then $(-1)^n = -1$.
Since $n \in \mathbb{N}$, then $n > 0$, so $\frac{1}{n} > 0$.
Since $\frac{1}{2} < 1$ and $0 < \frac{1}{n}$, then $\frac{1}{2} < 1 + \frac{1}{n} = 1 - \frac{(-1)^n}{n}$.
Thus, $\frac{1}{2} < 1 - \frac{(-1)^n}{n}$, so $\frac{1}{2} < x$.
Hence, in either case, $\frac{1}{2} \leq x$, so $\frac{1}{2}$ is a lower bound of S.

Let r be an arbitrary real number such that $r > \frac{1}{2}$.
To prove r is not a lower bound of S, we must prove there exists $x \in S$ such that $x < r$.
Since $2 \in \mathbb{N}$, then $1 - \frac{(-1)^2}{2} = 1 - \frac{1}{2} = \frac{1}{2} \in S$.
Thus, there exists $\frac{1}{2} \in S$ such that $\frac{1}{2} < r$.
Hence, r is not a lower bound of S, so every real number $r > \frac{1}{2}$ is not a lower bound of S.
Since $\frac{1}{2}$ is a lower bound of S and every real number $r > \frac{1}{2}$ is not a lower bound of S, then $\frac{1}{2}$ is the greatest lower bound of S, so $\frac{1}{2} = \inf S$. □

Exercise 70. Compute sup and inf of the set $\{x \in \mathbb{R} : |2x + \pi| < \sqrt{2}\}$.

Solution. Let $S = \{x \in \mathbb{R} : |2x + \pi| < \sqrt{2}\}$.
Since $|2(-1) + \pi| = \pi - 2 < \sqrt{2}$, then $-1 \in S$.
Hence, S is not empty.
We prove $S = (\frac{\pi+\sqrt{2}}{-2}, \frac{\pi-\sqrt{2}}{-2})$.
Observe that

$$\begin{aligned} x \in (\frac{\pi+\sqrt{2}}{-2}, \frac{\pi-\sqrt{2}}{-2}) &\Leftrightarrow \frac{\pi+\sqrt{2}}{-2} < x < \frac{\pi-\sqrt{2}}{-2} \\ &\Leftrightarrow \frac{\pi+\sqrt{2}}{-1} < 2x < \frac{\pi-\sqrt{2}}{-1} \\ &\Leftrightarrow -\pi - \sqrt{2} < 2x < -\pi + \sqrt{2} \\ &\Leftrightarrow -\sqrt{2} < 2x + \pi < \sqrt{2} \\ &\Leftrightarrow |2x + \pi| < \sqrt{2} \\ &\Leftrightarrow x \in S. \end{aligned}$$

Therefore, $(\frac{\pi+\sqrt{2}}{-2}, \frac{\pi-\sqrt{2}}{-2}) = S$.
Therefore, $\sup S = \frac{\pi-\sqrt{2}}{-2}$ and $\inf S = \frac{\pi+\sqrt{2}}{-2}$. □

Exercise 71. Let $S \subset \mathbb{R}$.
Let $r \in \mathbb{R}$.
Then $r = \sup(S)$ iff $(\forall x \in S)(x \leq r)$ and $(\forall \epsilon > 0)(\exists s \in S)(r - \epsilon < s)$.

Proof. We first prove if $(\forall x \in S)(x \leq r)$ and $(\forall \epsilon > 0)(\exists s \in S)(r - \epsilon < s)$, then $r = \sup(S)$.
Suppose $(\forall x \in S)(x \leq r)$ and $(\forall \epsilon > 0)(\exists s \in S)(r - \epsilon < s)$.

Since $x \leq r$ for all $x \in S$, then r is an upper bound of S.
Let M be an arbitrary real number less than r.
We prove M is not an upper bound of S.
Since $M < r$, then $r - M > 0$.
Hence, there exists $s \in S$ such that $r - (r - M) < s$.
Thus, there exists $s \in S$ such that $M < s$.
Since there exists $s \in S$ such that $s > M$, then M is not an upper bound of S.
Therefore, every real number $M < r$ is not an upper bound of S.
Since r is an upper bound of S and every real number $M < r$ is not an upper bound of S, then r is the least upper bound of S, so $r = \sup S$.

Conversely, we prove if $r = \sup(S)$, then $(\forall x \in S)(x \leq r)$ and $(\forall \epsilon > 0)(\exists s \in S)(r - \epsilon < s)$.
Suppose $r = \sup(S)$.
Then r is the least upper bound of S, so r is an upper bound of S and any real number $M < r$ is not an upper bound of S.
Since $r \in \mathbb{R}$ and r is an upper bound of S, then $(\forall x \in S)(x \leq r)$.
Let $\epsilon > 0$ be given.
Then $\epsilon > r - r$, so $r > r - \epsilon$.
Since any real number M less than r is not an upper bound of S, then in particular, $r - \epsilon$ is not an upper bound of S.
Hence, there exists $s \in S$ such that $s > r - \epsilon$.
Therefore, for every $\epsilon > 0$, there exists $s \in S$ such that $r - \epsilon < s$, so $(\forall \epsilon > 0)(\exists s \in S)(r - \epsilon < s)$. $\square$

Exercise 72. Let $S \subset \mathbb{R}$ be nonempty.
Let $t, u \in \mathbb{R}$.
Then u is an upper bound of S iff $t > u$ implies $t \notin S$.

Proof. We prove if u is an upper bound of S, then $t > u$ implies $t \notin S$.
Suppose u is an upper bound of S.
Then if $t \in S$, then $t \leq u$.
Hence, if $t > u$, then $t \notin S$.

Conversely, we prove if $t > u$ implies $t \notin S$, then u is an upper bound of S.
Suppose $t > u$ implies $t \notin S$.
Then if $t \in S$, then $t \leq u$.
Since $S \neq \emptyset$, let $x \in S$.
Then $x \leq u$.
Thus, $x \leq u$ for all $x \in S$, so u is an upper bound of S. $\square$

Exercise 73. Let $a \in \mathbb{R}$.
Let $S = \{s \in \mathbb{Q} : s < a\}$.
Then $\sup S = a$.

Proof. Let $s \in S$.

Then $s \in \mathbb{Q}$ and $s < a$.

Thus, $s < a$ for all $s \in S$, so a is an upper bound of S.

Let $b \in \mathbb{R}$ such that $b < a$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $q \in \mathbb{Q}$ such that $b < q < a$, so $b < q$ and $q < a$.

Since $q \in \mathbb{Q}$ and $q < a$, then $q \in S$.

Since $q \in S$ and $q > b$, then b is not an upper bound of S.

Hence, every real number b less than a is not an upper bound of S.

Therefore, $a = \sup S$. □

Exercise 74. Let $S \subset \mathbb{R}$ such that $B \in S$ and B is an upper bound of S.

Then $B = \sup S$.

Proof. Since B is an upper bound of S, then S has at least one upper bound in $\mathbb{R}$.

Let M be an arbitrary upper bound of S in $\mathbb{R}$.

Since $B \in S$ and M is an upper bound of S, then $B \leq M$.

Thus, $B \leq M$ for any upper bound M of S.

Since B is an upper bound of S and $B \leq M$ any upper bound M of S, then B is the least upper bound of S.

Therefore, $B = \sup S$. □

Lemma 75. *Let* $S \subset \mathbb{R}$.

If $\sup S$ *exists, then every real number* $r > \sup S$ *is an upper bound of* S.

Proof. Suppose $\sup S$ exists.

Let r be an arbitrary real number such that $r > \sup S$.

Since $\sup S$ exists, then $S \neq \emptyset$.

Let $x \in S$.

Since $\sup S$ is an upper bound of S, then $x \leq \sup S$.

Since $x \leq \sup S$ and $\sup S < r$, then $x < r$.

Thus, $x < r$ for all $x \in S$, so r is an upper bound of S. □

Exercise 76. Let $S \subset \mathbb{R}$ such that $\sup S$ exists. Then

1. $\sup S - \frac{1}{n}$ is not an upper bound of S for all $n \in \mathbb{N}$.
2. $\sup S + \frac{1}{n}$ is an upper bound of S for all $n \in \mathbb{N}$.

Proof. We prove 1.

Let $n \in \mathbb{N}$ be arbitrary.

Then $n > 0$, so $\frac{1}{n} > 0$.

Thus, $\frac{1}{n} > \sup S - \sup S$, so $\sup S + \frac{1}{n} > \sup S$.

Hence, $\sup S > \sup S - \frac{1}{n}$.

Since $\sup S$ is the least upper bound of S, then for every real number $r < \sup S$, r is not an upper bound of S.

Since $\sup S - \frac{1}{n} < \sup S$, then we conclude $\sup S - \frac{1}{n}$ is not an upper bound of S. □

Proof. We prove 2.

Let $n \in \mathbb{N}$ be arbitrary.

Then $n > 0$, so $\frac{1}{n} > 0$.

Thus, $\sup S + \frac{1}{n} > \sup S$.

Since $\sup S$ exists, then every real number $r > \sup S$ is an upper bound of S.

Since $\sup S + \frac{1}{n} > \sup S$, then we conclude $\sup S + \frac{1}{n}$ is an upper bound of S. □

Exercise 77. Let S be a subset of an ordered field F.

If $b \in F$ and b is an upper bound for S, then $\sup S \leq b$.

Proof. Suppose $b \in F$ and b is an upper bound for S.

Since $\sup S$ is the least upper bound of S, then for every $b \in F$ such that $b < \sup S$, b is not an upper bound of S.

Thus, if $b \in F$ and $b < \sup S$, then b is not an upper bound of S.

Hence, if $b \in F$ and b is an upper bound of S, then $b \geq \sup S$.

Since $b \in F$ and b is an upper bound of S, then we conclude $b \geq \sup S$, so $\sup S \leq b$. □

Exercise 78. Let A and B be subsets of an ordered field F.

If A is unbounded above in F and $(\forall x \in A)(\exists y \in B)(x \leq y)$, then B is unbounded above in F.

Proof. Suppose A is unbounded above in F and $(\forall x \in A)(\exists y \in B)(x \leq y)$.

Let $b \in F$ be arbitrary.

Since A is unbounded above in F, then there exists $x \in A$ such that $x > b$.

Since $x \in A$, then there exists $y \in B$ such that $x \leq y$.

Thus, $b < x \leq y$, so $b < y$.

Hence, there exists $y \in B$ such that $y > b$.

Therefore, B is unbounded above in F. □

Exercise 79. Let $S = \{ \sqrt[n]{n} : n \in \mathbb{N}\}$.

Compute $\max S, \min S, \sup S, \inf S$, if they exist.

Solution. We prove $\max S = \sqrt[3]{3}$.

Since $3 \in \mathbb{N}$, then $\sqrt[3]{3} \in S$.

We prove $\sqrt[3]{3}$ is an upper bound of S.

Let $x \in S$.

Then $x \in \mathbb{R}$ and there exists $n \in \mathbb{N}$ such that $x = \sqrt[n]{n}$.

To prove $\sqrt[3]{3}$ is an upper bound of S, we must prove $x \leq \sqrt[3]{3}$.

Suppose for the sake of contradiction that $x > \sqrt[3]{3}$.

Then $\sqrt[n]{n} > \sqrt[3]{3}$, so $n^{\frac{1}{n}} > 3^{\frac{1}{3}}$.

Since $n^{\frac{1}{n}} > 3^{\frac{1}{3}} > 0$ and $3n \in \mathbb{N}$, then $(n^{\frac{1}{n}})^{3n} > (3^{\frac{1}{3}})^{3n}$, so $n^3 > 3^n$.

Thus, there exists $n \in \mathbb{N}$ such that $n^3 > 3^n$.

Since $n \in \mathbb{N}$, then either $n = 1$ or $n = 2$ or $n = 3$ or $n > 3$.

Since $1 = 1^3 < 3^1 = 3$, then $n \neq 1$.

Since $8 = 2^3 < 3^2 = 9$, then $n \neq 2$.

Since $3^3 \not> 3^3$, then $n \neq 3$.
Hence, n cannot be 1 or 2 or 3.
We prove $n^3 < 3^n$ for all natural numbers $n > 3$ by induction on n.
Define predicate $p(n) : n^3 < 3^n$ for all natural numbers $n > 3$.
Since $64 = 4^3 < 3^4 = 81$, then $p(4)$ is true.
Suppose $n > 3$ such that $p(n)$ is true.
Then $n^3 < 3^n$.
To prove $p(n+1)$, we must prove $(n+1)^3 < 3^{n+1}$.
Since $n > 3$, then $2n > 6$, so $2n - 3 > 3$.
Since $n > 3$ and $2n - 3 > 3$, then $n(2n-3) > 3 \cdot 3$, so $2n^2 - 3n > 9 > 4$.
Hence, $2n^2 - 3n > 4$, so $2n^2 - 3n - 3 > 1$.
Since $n > 1$ and $2n^2 - 3n - 3 > 1$, then $n(2n^2 - 3n - 3) > 1 \cdot 1$, so $2n^3 - 3n^2 - 3n > 1$.
Thus, $2n^3 > 3n^2 + 3n + 1$, so $3n^3 > n^3 + 3n^2 + 3n + 1 = (n+1)^3$.
Hence, $3n^3 > (n+1)^3$, so $(n+1)^3 < 3n^3$.
Since $n^3 < 3^n$, then $3n^3 < 3^{n+1}$.
Thus, $(n+1)^3 < 3n^3$ and $3n^3 < 3^{n+1}$, so $(n+1)^3 < 3^{n+1}$.
Therefore, $p(n+1)$ is true, as desired.
Hence, by induction, $n^3 < 3^n$ for all $n > 3$.
Thus, $n^3 \not> 3^n$ for all $n > 3$.
Therefore, n cannot be greater than 3.
Thus, there is no $n \in \mathbb{N}$ such that $n^3 > 3^n$.
Therefore, $x \leq \sqrt[3]{3}$.
Hence $\sqrt[3]{3}$ is an upper bound of S.
Since $\sqrt[3]{3} \in S$ and $\sqrt[3]{3}$ is an upper bound of S, then $\sqrt[3]{3} = \max S = \sup S$.
We prove $\min S = 1$.
Since $1 \in \mathbb{N}$ and $1 = 1^{\frac{1}{n}}$, then $1 \in S$.
We prove 1 is a lower bound of S.
Let $y \in S$.
Then $y \in \mathbb{R}$ and there exists $n \in \mathbb{N}$ such that $y = \sqrt[n]{n}$.
To prove 1 is a lower bound of S, we must prove $1 \leq y$.
Suppose for the sake of contradiction that $1 > y$.
Then $1 > \sqrt[n]{n}$.
Since $n > 0$, then $\sqrt[n]{n} > 0$.
Since $1 > \sqrt[n]{n} > 0$ and $n \in \mathbb{N}$, then $1^n > (\sqrt[n]{n})^n$.
Hence, $1 > n$, so $n < 1$.
But, $n \in \mathbb{N}$, so $n \geq 1$.
Thus, we have $n < 1$ and $n \geq 1$, a violation of trichotomy.
Therefore, $1 \leq y$, so 1 is a lower bound of S.
Since $1 \in S$ and 1 is a lower bound of S, then $1 = \min S = \inf S$. □

Exercise 80. Let S be a set of positive real numbers.
Let $T = \{x^2 : x \in S\}$.
If $\sup S$ exists, then $\sup T = (\sup S)^2$.

Proof. We first prove $(\sup S)^2$ is an upper bound of T.

Since $\sup S$ exists, then $S \neq \emptyset$.
Let $x \in S$.
Then $x^2 \in T$, so $T \neq \emptyset$.
Let $t \in T$ be arbitrary.
Then $t = s^2$ for some $s \in S$.
Since $s \in S$, then $s > 0$.
Since $s \in S$ and $\sup S$ is an upper bound of S, then $s \leq \sup S$.
Thus, $0 < s \leq \sup S$, so $0 < s^2 = t \leq (\sup S)^2$ and $0 < \sup S$.
Hence, $t \leq (\sup S)^2$, so $(\sup S)^2$ is an upper bound of T.

We next prove $(\sup S)^2$ is the least upper bound of T.
Let $\epsilon > 0$ be given.
Since $\sup S > 0$, then $\frac{\epsilon}{2\sup S} > 0$.
Since $\sup S$ is the least upper bound of S, then there exists $x \in S$ such that $x > \sup S - \frac{\epsilon}{2\sup S}$.
Hence, $\frac{\epsilon}{2\sup S} > \sup S - x$.
Since $x \in S$, then $x > 0$.
Since $x \in S$ and $\sup S$ is an upper bound of S, then $x \leq \sup S$.
Thus, $0 < x \leq \sup S$.
Therefore, $0 < 2x \leq \sup S + x \leq 2\sup S$ and $0 \leq \sup S - x$.
Hence, $0 < \sup S + x \leq 2\sup S$ and $0 \leq \sup S - x < \frac{\epsilon}{2\sup S}$, so $(\sup S + x)(\sup S - x) < \epsilon$.
Consequently, $(\sup S)^2 - x^2 < \epsilon$, so $(\sup S)^2 - \epsilon < x^2$.
Let $t = x^2$.
Since $x \in S$, then $t \in T$, so $(\sup S)^2 - \epsilon < t$.
Therefore, $t > (\sup S)^2 - \epsilon$, as desired. □

Exercise 81. Let $A \subset \mathbb{R}$.
Let $B = \{x^2 : x \in A\}$.
1. If $\sup A$ and $\sup B$ exist, then $\sup B \geq (\sup A)^2$.
2. If $\inf A$ and $\inf B$ exist, then $\inf B \leq (\inf A)^2$.

Proof. We prove 1.
We prove if $\sup A$ and $\sup B$ exist, then $\sup B \geq (\sup A)^2$.
Suppose $\sup A$ and $\sup B$ exist in $\mathbb{R}$.
Either $\sup A \geq 0$ or $\sup A < 0$.
We consider these cases separately.
Case 1: Suppose $\sup A \leq 0$.
Since $\sup A$ exists, then A is not empty.
Let $x \in A$.
Since $\sup A$ is an upper bound of A, then $x \leq \sup A$.
Thus, $x \leq \sup A \leq 0$, so $-x \geq -\sup A \geq 0$.
Hence, $0 \leq -\sup A \leq -x$, so $0 \leq (\sup A)^2 \leq x^2$.
Thus, $(\sup A)^2 \leq x^2$.
Since $x^2 \in B$ and $\sup B$ is an upper bound of B, then $x^2 \leq \sup B$.

Hence, $(\sup A)^2 \leq x^2 \leq \sup B$, so $(\sup A)^2 \leq \sup B$.
Therefore, $\sup B \geq (\sup A)^2$, as desired.
Case 2: Suppose $\sup A > 0$.
Since $\sup A$ exists, then A is not empty.
Let $x \in A$.
Then $x^2 \in B$.
Since $\sup B$ is an upper bound of B, then $x^2 \leq \sup B$.
Since $x^2 \geq 0$ and $x^2 \leq \sup B$, then $0 \leq x^2 \leq \sup B$, so $0 \leq \sup B$.
Suppose for the sake of contradiction $\sup B < (\sup A)^2$.
Then $0 \leq \sup B < (\sup A)^2$.
Hence, $0 \leq \sqrt{\sup B} < \sup A$, so $\sqrt{\sup B} < \sup A$.
Thus, $\sup A - \sqrt{\sup B} > 0$.
Since $\sup A$ is the least upper bound of A, then there exists $a \in A$ such that $a > \sup A - (\sup A - \sqrt{\sup B})$.
Hence, there exists $a^2 \in B$ such that $a > \sqrt{\sup B}$.
Since $a > \sqrt{\sup B} \geq 0$, then $a^2 > \sup B$.
Therefore, there exists $a^2 \in B$ such that $a^2 > \sup B$.
But, this contradicts the fact that $\sup B$ is an upper bound of B.
Therefore, $\sup B \geq (\sup A)^2$, as desired. □

Proof. We prove 2.
We prove if $\inf A$ and $\inf B$ exist, then $\inf B \leq (\inf A)^2$.
Suppose $\inf A$ and $\inf B$ exist in $\mathbb{R}$.
Suppose for the sake of contradiction $\inf B > (\inf A)^2$.
Either $\inf A \geq 0$ or $\inf A < 0$.
We consider these cases separately.
Case 1: Suppose $\inf A < 0$.
Then $-\inf A > 0$.
Thus, $0 \leq (\inf A)^2 < \inf B$, so $0 < \inf B$.
Hence, $\sqrt{\inf B} > 0$.
Since $\sqrt{\inf B} > 0$ and $-\inf A > 0$, then $\sqrt{\inf B} - \inf A > 0$.
Since $\inf A$ is the greatest lower bound of A, then there exists $a \in A$ such that $a < \inf A + (\sqrt{\inf B} - \inf A)$.
Thus, there exists $a \in A$ such that $a < \sqrt{\inf B}$.
Either $a \geq 0$ or $a < 0$.
Suppose $a \geq 0$.
Then $0 \leq a < \sqrt{\inf B}$.
Hence, $0 \leq a^2 < \inf B$.
Therefore, there exists $a^2 \in B$ such that $a^2 < \inf B$.
Suppose $a < 0$.
Since $\inf A$ is a lower bound of A and $a \in A$, then $\inf A \leq a$.
Hence, $\inf A \leq a < 0$, so $-\inf A \geq -a > 0$.
Thus, $0 < -a \leq -\inf A$, so $0 < a^2 \leq (\inf A)^2$.
Therefore, $0 < a^2 \leq (\inf A)^2 < \inf B$, so $0 < a^2 < \inf B$.
Hence, there exists $a^2 \in B$ such that $a^2 < \inf B$.
In either case, there exists $a^2 \in B$ such that $a^2 < \inf B$.

But, this contradicts the fact that $\inf B$ is a lower bound of B.

Case 2: Suppose $\inf A \geq 0$.

Then $0 \leq (\inf A)^2 < \inf B$, so $0 \leq \inf A < \sqrt{\inf B}$.

Thus, $\inf A < \sqrt{\inf B}$, so $\sqrt{\inf B} - \inf A > 0$.

Since $\inf A$ is the greatest lower bound of A, then there exists $a \in A$ such that $a < \inf A + (\sqrt{\inf B} - \inf A)$.

Hence, there exists $a^2 \in B$ such that $a < \sqrt{\inf B}$.

Since $\inf A$ is a lower bound of A and $a \in A$, then $\inf A \leq a$.

Thus, $0 \leq \inf A \leq a < \sqrt{\inf B}$, so $0 \leq a < \sqrt{\inf B}$.

Hence, $0 \leq a^2 < \inf B$, so $a^2 < \inf B$.

Therefore, there exists $a^2 \in B$ such that $a^2 < \inf B$.

But, this contradicts the fact that $\inf B$ is a lower bound of B.

Since a contradiction arises in all cases, then $\inf B \leq (\inf A)^2$, as desired. $\square$

Exercise 82. Let $S \subset \mathbb{R}$.

Here is a definition of least upper bound of S.

A real number u is called a least upper bound of S iff

1. $(\forall x \in S)(x \leq u)$.
2. $(\forall \epsilon > 0)(\exists y \in S)(y > u - \epsilon)$.

Using the above definition of least upper bound of S, prove that there is at most one least upper bound of S.

Solution. This is a more elegant solution.

The statement there is at most one least upper bound of S means that

if x and y are upper bounds of S, then $x = y$.

Define predicate: $A(x)$: x is a least upper bound of S over domain of discourse $\mathbb{R}$.

Then the statement means $A(x) \wedge A(y) \Rightarrow x = y$, so we must prove $(\forall x)(\forall y)(A(x) \wedge A(y) \rightarrow x = y)$.

To prove $(\forall x)(\forall y)(A(x) \wedge A(y) \rightarrow x = y)$, we assume arbitrary $a, b \in \mathbb{R}$ such that $A(a) \wedge A(b)$.

We must prove $a = b$.

To prove $a = b$, assume $a \neq b$ and use proof by contradiction.

Since $a \neq b$, then either $a < b$ or $a > b$.

Without loss of generality, we may assume $a < b$.

How can we derive the desired contradiction?

We must use the fact that a and b are lubs of S.

Thus we have:

1. $(\forall x \in S)(x \leq a)$
2. $(\forall \epsilon > 0)(\exists y \in S)(y > a - \epsilon)$
3. $(\forall x \in S)(x \leq b)$
4. $(\forall \epsilon > 0)(\exists y \in S)(y > b - \epsilon)$

To derive a contradiction among the 4 statements, we need to find a suitable $\epsilon > 0$.

How should we choose ϵ?

Since $a < b$, then $0 < b - a$, so $b - a > 0$.

Let's try $\epsilon = b - a$.

Can we derive a contradiction?

We consider the 4 facts given and see if any logical contradictions arise.

Since $\epsilon > 0$ is a particular object, by universal elimination, $(\exists y \in S)(y > a - \epsilon)$ and $(\exists y \in S)(y > b - \epsilon)$.

By existential elimination, let y_1, y_2 be some elements of S.

Then $y_1 > a - \epsilon$ and $y_2 > b - \epsilon$.

Hence, $y_1 > a - (b - a)$, so $y_1 > 2a - b$ and $y_2 > b - (b - a)$, so $y_2 > a$.

By universal elimination, since $(\forall x \in S)(x \leq a)$ and $y_2 \in S$, then $y_2 \leq a$.

Thus, we have $y_2 > a$ and $y_2 \leq a$.

Since $y_2 \in S$ and $S \subset \mathbb{R}$, then $y_2 \in \mathbb{R}$.

Hence, we have a violation of trichotomy of $\mathbb{R}$.

Thus, a cannot be less than b, so it cannot be that $a \neq b$.

Therefore, $a = b$, as desired. □

Proof. Let S be a subset of $\mathbb{R}$.

Assume arbitrary real numbers a and b such that a and b are least upper bounds of S.

To prove $a = b$, suppose for the sake of contradiction that $a \neq b$.

Since $a \neq b$, then either $a < b$ or $a > b$.

Without loss of generality, we may assume $a < b$.

Since a and b are least upper bounds of S, then each element of S is less than or equal to a and for each positive real ϵ, there corresponds $x \in S$ such that $x > b - \epsilon$.

Let $\epsilon = b - a$.

Since $a < b$, then $b > a$, so $b - a > 0$.

Hence, $\epsilon > 0$.

Thus, there is $x \in S$ such that $x > b - \epsilon$.

Since $x > b - (b - a)$, then $x > a$.

Since $x \in S$, then $x \leq a$.

Hence, $x > a$ and $x \leq a$.

Thus, we have a violation of trichotomy of $\mathbb{R}$.

Consequently, a cannot be less than b, so it cannot be that $a \neq b$.

Therefore, $a = b$, as desired. □

Exercise 83. Let A and B be subsets of $\mathbb{R}$ such that $(\forall a \in A)(\exists b \in B)(a \leq b)$.

If $\sup A$ and $\sup B$ exist, then $\sup A \leq \sup B$.

Proof. Suppose $\sup A$ and $\sup B$ exist.

Since $\sup A$ exists, then $A \neq \emptyset$.

Let $a \in A$ be given.

Then there exists $b \in B$ such that $a \leq b$.

Since $b \in B$ and $\sup B$ is an upper bound of B, then $b \leq \sup B$.

Thus, $a \leq b \leq \sup B$, so $a \leq \sup B$.

Hence, $\sup B$ is an upper bound of A.

Since $\sup A$ is the least upper bound of A and $\sup B$ is an upper bound of A, then $\sup A \leq \sup B$. □

Exercise 84. Let A and B be subsets of $\mathbb{R}$ such that $(\forall a \in A)(\forall b \in B)(a \leq b)$.
If $\sup A$ and $\inf B$ exist, then $\sup A \leq \inf B$.

Proof. Suppose $\sup A$ and $\inf B$ exist.
Since $\inf B$ exists, then $B \neq \emptyset$.
Let $b \in B$.
Then $a \leq b$ for all $a \in A$, so b is an upper bound of A.
Since $\sup A$ is the least upper bound of A, then $\sup A \leq b$.
Since b is arbitrary, then $\sup A \leq b$ for all $b \in B$, so $\sup A$ is a lower bound of B.
Since $\inf B$ is the greatest lower bound of B, then $\sup A \leq \inf B$. □

Proposition 85. *Let A and B be subsets of $\mathbb{R}$ such that $(\forall a \in A)(\forall b \in B)(a \leq b)$.*
If $\sup A$ and $\sup B$ exist, then $\sup A \leq \sup B$.

Proof. Suppose $\sup A$ and $\sup B$ exist.
Since $\sup B$ exists, then $B \neq \emptyset$, so there exists $b \in B$.
Thus, $a \leq b$ for all $a \in A$, so b is an upper bound of A.
Since $\sup A$ is the least upper bound of A, then $\sup A \leq b$.
Since $b \in B$ and $\sup B$ is an upper bound of B, then $b \leq \sup B$.
Therefore, $\sup A \leq b \leq \sup B$, so $\sup A \leq \sup B$. □

Exercise 86. Let A and B be nonempty sets of real numbers.
Let $\delta(A, B) = \inf\{|a - b| : a \in A, b \in B\}$.
We call $\delta(A, B)$ the distance betweens sets A and B.
1. Let $A = \mathbb{N}$ and $B = \mathbb{R} - \mathbb{N}$. What is $\delta(A, B)$?
2. If A and B are finite sets, what does $\delta(A, B)$ represent?

Solution. We compute $\delta(A, B)$ when $A = \mathbb{N}$ and $B = \mathbb{R} - \mathbb{N}$.
Let $S = \{|a - b| : a \in A, b \in B\}$.
Then $\delta(A, B) = \inf S$.
Since A and B are not empty, then there is at least one element in A and B.
Let $x \in S$.
Then there exists $a \in A$ and $b \in B$ such that $x = |a - b|$.
Since $a \in A$ and $A = \mathbb{N} \subset \mathbb{R}$, then $a \in \mathbb{R}$.
Since $b \in B$ and $B = \mathbb{R} - \mathbb{N}$, then $b \in \mathbb{R}$ and $b \notin \mathbb{N}$.
Since $a \in \mathbb{R}$ and $b \in \mathbb{R}$, then $a - b \in \mathbb{R}$, so $|a - b| \in \mathbb{R}$.
Since $|x| \geq 0$ for any $x \in \mathbb{R}$, then $|a - b| \geq 0$.
Hence, $x \geq 0$, so $0 \leq x$.
Therefore, 0 is a lower bound of S.
We prove $0 = \inf S$.
Let $\epsilon > 0$.
To prove 0 is the greatest lower bound of S, we must prove there exists $s \in S$ such that $s < \epsilon$.
Either $\epsilon > 1$ or $\epsilon = 1$ or $\epsilon < 1$.
We consider these cases separately.

Case 1: Suppose $\epsilon > 1$.
Then $1 < \epsilon$.
Since $0 \in \mathbb{R}$ and $0 \notin \mathbb{N}$, then $0 \in B$.
Since $1 \in A$ and $0 \in B$, then $|1 - 0| = 1 \in S$.
Thus, there exists $1 \in S$ such that $1 < \epsilon$.
Case 2: Suppose $\epsilon = 1$.
Since $\frac{1}{2} \in \mathbb{R}$ and $\frac{1}{2} \notin \mathbb{N}$, then $\frac{1}{2} \in B$.
Since $1 \in A$ and $\frac{1}{2} \in B$, then $|1 - \frac{1}{2}| = \frac{1}{2} \in S$.
Thus, there exists $\frac{1}{2} \in S$ such that $\frac{1}{2} < 1 = \epsilon$.
Case 3: Suppose $\epsilon < 1$.
Then $0 < \epsilon < 1$.
Hence, $\frac{\epsilon}{2} \in \mathbb{R}$ and $0 < \frac{\epsilon}{2} < 1$.
Suppose $1 - \frac{\epsilon}{2} \in \mathbb{N}$.
Then there exists $n \in \mathbb{N}$ such that $n = 1 - \frac{\epsilon}{2}$.
Hence, $2n = 2 - \epsilon$, so $\epsilon = 2 - 2n$.
Since $n \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{Z}$, then $n \in \mathbb{Z}$.
Thus, $2 - 2n \in \mathbb{Z}$, so $\epsilon \in \mathbb{Z}$.
But, $0 < \epsilon < 1$, so $\epsilon \notin \mathbb{Z}$.
Therefore, $1 - \frac{\epsilon}{2} \notin \mathbb{N}$.
Since $1 - \frac{\epsilon}{2} \in \mathbb{R}$ and $1 - \frac{\epsilon}{2} \notin \mathbb{N}$, then $1 - \frac{\epsilon}{2} \in B$.
Since $\epsilon > 0$ and $1 \in A$ and $1 - \frac{\epsilon}{2} \in B$, then $|1 - (1 - \frac{\epsilon}{2})| = \frac{\epsilon}{2} \in S$.
Since $\epsilon > 0$ and $\frac{1}{2} < 1$, then $\frac{\epsilon}{2} < \epsilon$.
Thus, there exists $\frac{\epsilon}{2} \in S$ such that $\frac{\epsilon}{2} < \epsilon$.
Therefore, in all cases, there exists $s \in S$ such that $s < \epsilon$, as desired.
Hence, $0 = \inf S$.
Therefore, $\delta(A, B) = \inf S = 0$.

We answer 2.
We try various examples of A and B as nonempty finite sets.
It turns out that $\delta(A, B)$ represents the distance of the element in A that is closest to an element of B.
In addition, since A and B are finite sets, then so is S. □

Exercise 87. Let $X = Y = (0, 1)$.
Let $h : X \times Y \to \mathbb{R}$ be a function defined by $h(x, y) = 2x + y$.
1. If $f(x) = \sup\{h(x, y) : y \in Y\}$ for each $x \in X$, then $f(x) = 2x + 1$ and $\inf\{f(x) : x \in X\} = 1$.
2. If $g(y) = \inf\{h(x, y) : x \in X\}$ for each $y \in Y$, then $g(y) = y$ and $\sup\{g(y) : y \in Y\} = 1$.

Proof. We prove 1.
Let $x \in X$ be given.

Then

$$\begin{aligned}
f(x) &= \sup\{h(x,y) : y \in Y\} \\
&= \sup\{2x + y : y \in Y\} \\
&= 2x + \sup Y \\
&= 2x + \sup(0,1) \\
&= 2x + 1.
\end{aligned}$$

Observe that

$$\begin{aligned}
\inf\{f(x) : x \in X\} &= \inf\{2x + 1 : x \in X\} \\
&= \inf\{1 + 2x : x \in X\} \\
&= 1 + \inf\{2x : x \in X\} \\
&= 1 + 2\inf X \\
&= 1 + 2\inf(0,1) \\
&= 1 + 2 \cdot 0 \\
&= 1.
\end{aligned}$$

□

Proof. We prove 2.

Let $y \in Y$ be given.

Then

$$\begin{aligned}
g(y) &= \inf\{h(x,y) : x \in X\} \\
&= \inf\{2x + y : x \in X\} \\
&= \inf\{y + 2x : x \in X\} \\
&= y + \inf\{2x : x \in X\} \\
&= y + 2\inf X \\
&= y + 2\inf(0,1) \\
&= y + 2 \cdot 0 \\
&= y.
\end{aligned}$$

Observe that

$$\begin{aligned}
\sup\{g(y) : y \in Y\} &= \sup\{y : y \in Y\} \\
&= \sup Y \\
&= \sup(0,1) \\
&= 1.
\end{aligned}$$

□

Exercise 88. Let $X = Y = (0, 1)$.
Let $h : X \times Y \to \mathbb{R}$ be a function defined by

$$h(x, y) = \begin{cases} 0 & \text{if } x < y \\ 1 & \text{if } x \geq y. \end{cases}$$

1. If $f(x) = \sup\{h(x, y) : y \in Y\}$ for each $x \in X$, then $f(x) = 1$.
2. If $g(y) = \inf\{h(x, y) : x \in X\}$ for each $y \in Y$, then $g(y) = 0$.

Proof. We prove 1.
Suppose $f(x) = \sup\{h(x, y) : y \in Y\}$ for each $x \in X$.
Let $x \in X$ be arbitrary.
Then $x \in (0, 1)$, so $0 < x < 1$.
Hence, $0 < x$ and $x < 1$.
Let $S = \{h(x, y) : y \in Y\}$.

We prove $0 \in S$.
Let $y = \frac{x+1}{2}$.
Since $-1 < 0 < x < 1$, then $-1 < x < 1$, so $0 < x + 1 < 2$.
Thus, $0 < \frac{x+1}{2} < 1$, so $0 < y < 1$.
Hence, $y \in (0, 1)$, so $y \in Y$.
Since $x < 1$, then $2x < x + 1$, so $x < \frac{x+1}{2}$.
Thus, $x < y$.
Since $y \in Y$ and $x < y$, then $h(x, y) = 0$, so $0 \in S$.

We prove $1 \in S$.
Let $y' = \frac{x}{2}$.
Since $0 < x < 1 < 2$, then $0 < x < 2$, so $0 < \frac{x}{2} < 1$.
Thus, $0 < y' < 1$, so $y' \in (0, 1)$.
Therefore, $y' \in Y$.
Since $0 < x$, then $x < 2x$, so $\frac{x}{2} < x$.
Thus, $y' < x$.
Since $y' \in Y$ and $x > y'$, then $h(x, y') = 1$, so $1 \in S$.

Since $0 \in S$ and $1 \in S$, then $\{0, 1\} \subset S$.

We prove $S \subset \{0, 1\}$.
Suppose $s \in S$.
Then there exists $t \in Y$ such that $s = h(x, t)$.
Since $t \in Y$ and $Y \subset \mathbb{R}$, then $t \in \mathbb{R}$, so either $t > x$ or $t \leq x$.
We consider these cases separately.
Case 1: Suppose $t > x$.
Since $t \in Y$ and $x < t$, then $s = h(x, t) = 0$.
Case 2: Suppose $t \leq x$.

Since $t \in Y$ and $x \geq t$, then $s = h(x,t) = 1$.
Thus, either $s = 0$ or $s = 1$, so either $s \in \{0\}$ or $s \in \{1\}$.
Hence, $s \in \{0\} \cup \{1\}$, so $s \in \{0,1\}$.
Therefore, if $s \in S$, then $s \in \{0,1\}$, so $S \subset \{0,1\}$.

Since $S \subset \{0,1\}$ and $\{0,1\} \subset S$, then $S = \{0,1\}$.
Therefore,

$$\begin{aligned} f(x) &= \sup\{h(x,y) : y \in Y\} \\ &= \sup S \\ &= \sup\{0,1\} \\ &= 1. \end{aligned}$$

Thus, $f(x) = 1$ for all $x \in X$. $\square$

Proof. We prove 2.
Suppose $g(y) = \inf\{h(x,y) : x \in X\}$ for each $y \in Y$.
Let $y \in Y$ be arbitrary.
Then $y \in (0,1)$, so $0 < y < 1$.
Hence, $0 < y$ and $y < 1$.
Let $S = \{h(x,y) : x \in X\}$.

We prove $0 \in S$.
Let $x = \frac{y}{2}$.
Since $0 < y < 1 < 2$, then $0 < y < 2$, so $0 < \frac{y}{2} < 1$.
Thus, $0 < x < 1$, so $x \in (0,1)$.
Hence, $x \in X$.
Since $0 < y$, then $y < 2y$, so $\frac{y}{2} < y$.
Thus, $x < y$.
Since $x \in X$ and $x < y$, then $h(x,y) = 0$, so $0 \in S$.

We prove $1 \in S$.
Let $x' = \frac{y+1}{2}$.
Since $-1 < 0 < y < 1$, then $-1 < y < 1$, so $0 < y + 1 < 2$.
Thus, $0 < \frac{y+1}{2} < 1$, so $x' \in (0,1)$.
Therefore, $x' \in X$.
Since $y < 1$, then $2y < y + 1$, so $y < \frac{y+1}{2}$.
Thus, $y < x'$.
Since $x' \in X$ and $x' > y$, then $h(x',y) = 1$, so $1 \in S$.

Since $0 \in S$ and $1 \in S$, then $\{0,1\} \subset S$.

We prove $S \subset \{0,1\}$.

Suppose $s \in S$.

Then there exists $t \in X$ such that $s = h(t,y)$.

Since $t \in X$ and $X \subset \mathbb{R}$, then $t \in \mathbb{R}$, so either $t > y$ or $t \leq y$.

We consider these cases separately.

Case 1: Suppose $t > y$.

Since $t \in X$ and $t > y$, then $s = h(t,y) = 1$.

Case 2: Suppose $t \leq y$.

Since $t \in X$ and $t \leq y$, then $s = h(t,y) = 0$.

Thus, either $s = 0$ or $s = 1$, so either $s \in \{0\}$ or $s \in \{1\}$.

Hence, $s \in \{0\} \cup \{1\}$, so $s \in \{0,1\}$.

Therefore, if $s \in S$, then $s \in \{0,1\}$, so $S \subset \{0,1\}$.

Since $S \subset \{0,1\}$ and $\{0,1\} \subset S$, then $S = \{0,1\}$.

Therefore,

$$\begin{aligned} g(y) &= \inf\{h(x,y) : x \in X\} \\ &= \inf S \\ &= \inf\{0,1\} \\ &= 0. \end{aligned}$$

Thus, $g(y) = 0$ for all $y \in Y$. $\square$

Complete ordered fields

Exercise 89. Analyze boundedness of $\mathbb{Q}$.

Solution. $\square$

Exercise 90. Analyze boundedness of $\mathbb{R}$.

Solution. To prove there is no upper bound of $\mathbb{R}$, we prove for every $r \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $x > r$.

Let $r \in \mathbb{R}$ be arbitrary.

Let $x = r + 1$.

Then $x \in \mathbb{R}$ by closure of $\mathbb{R}$ under addition.

Since $1 > 0$, then $r + 1 > r$.

Hence, $x > r$.

Thus, there exists a real number greater than r.

Therefore, $\mathbb{R}$ is unbounded above, so there is no upper bound of $\mathbb{R}$.

Since there is no upper bound of $\mathbb{R}$, then there can be no greatest element of $\mathbb{R}$.

Therefore, $\max \mathbb{R}$ does not exist.

Since there is no upper bound of $\mathbb{R}$, then there can be no least upper bound of $\mathbb{R}$.

Therefore, $\sup \mathbb{R}$ does not exist.

To prove there is no lower bound of $\mathbb{R}$, we prove for every $r \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $x < r$.

Let $r \in \mathbb{R}$.

Let $x = r - 1$.

Then $x \in \mathbb{R}$ by closure of $\mathbb{R}$ under subtraction.

Since $1 > 0$, then $-1 < 0$, so $r - 1 < r$.

Hence, $x < r$.

Thus, there exists a real number less than r.

Therefore, $\mathbb{R}$ is unbounded below, so there is no lower bound of $\mathbb{R}$.

Since there is no lower bound of $\mathbb{R}$, then there can be no least element of $\mathbb{R}$.

Therefore, $\min \mathbb{R}$ does not exist.

Since there is no lower bound of $\mathbb{R}$, then there can be no greatest lower bound of $\mathbb{R}$.

Therefore, $\inf \mathbb{R}$ does not exist. □

Exercise 91. $\mathbb{N}$ is unbounded above in $\mathbb{R}$

For every real number x, there exists a natural number n such that $n > x$.

Proof. To prove $\mathbb{N}$ is unbounded above in $\mathbb{R}$, we must prove there is no upper bound of $\mathbb{N}$ in $\mathbb{R}$.

We prove by contradiction.

Suppose there is an upper bound of $\mathbb{N}$ in $\mathbb{R}$.

Then $\mathbb{N}$ is bounded above in $\mathbb{R}$.

Since $1 \in \mathbb{N}$, then $\mathbb{N}$ is not empty.

Since $\mathbb{N} \subset \mathbb{R}$, then $\mathbb{N}$ is a subset of $\mathbb{R}$.

Thus, $\mathbb{N}$ is a nonempty subset of $\mathbb{R}$ that is bounded above in $\mathbb{R}$.

Hence, by the completeness of $\mathbb{R}$, $\mathbb{N}$ has a least upper bound in $\mathbb{R}$.

Let b be the least upper bound of $\mathbb{N}$ in $\mathbb{R}$.

Then $b \in \mathbb{R}$ and b is an upper bound of $\mathbb{N}$.

Since $b - 1 < b$, then $b - 1$ is not an upper bound of $\mathbb{N}$.

Hence, there exists $n \in \mathbb{N}$ such that $n > b - 1$.

Thus, $n + 1 > b$.

Therefore, there exists $n + 1 \in \mathbb{N}$ such that $n + 1 > b$.

This contradicts the fact that b is an upper bound of $\mathbb{N}$.

Therefore, there is no upper bound of $\mathbb{N}$ in $\mathbb{R}$. □

Exercise 92. Let $E \neq \emptyset$.

Let $f : E \to \mathbb{R}$ be a function with bounded range.

Let $a \in \mathbb{R}$.

1. Then $\sup\{a + f(x) : x \in E\} = a + \sup\{f(x) : x \in E\}$.
2. Then $\inf\{a + f(x) : x \in E\} = a + \inf\{f(x) : x \in E\}$.

Proof. Let $f(E) = \{f(x) : x \in E\}$.

Since $E \neq \emptyset$, let $x \in E$.

Then $f(x) \in f(E)$, so $f(E) \neq \emptyset$.

Since the range of f is bounded, then $f(E)$ is bounded, so $f(E)$ is bounded above and below in $\mathbb{R}$.

Since $f(E)$ is not empty and bounded above in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\sup f(E)$ exists.

Since $f(E)$ is not empty and bounded below in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\inf f(E)$ exists.

Let $a + f(E) = \{a + f(x) : x \in E\}$.

We must prove $\sup(a+f(E)) = a+\sup f(E)$ and $\inf(a+f(E)) = a+\inf f(E)$.

Since $\sup f(E)$ exists, then $\sup(a + f(E)) = a + \sup(f(E))$.

Since $\inf f(E)$ exists, then $\inf(a + f(E)) = a + \inf(f(E))$. □

Exercise 93. Let $E \neq \emptyset$.

Let $f : E \to \mathbb{R}$ be a function with bounded range.

Let $g : E \to \mathbb{R}$ be a function with bounded range.

1. Then $\sup\{f(x)+g(x) : x \in E\} \leq \sup\{f(x) : x \in E\}+\sup\{g(x) : x \in E\}$.
2. Then $\inf\{f(x) : x \in E\} + \inf\{g(x) : x \in E\} \leq \inf\{f(x) + g(x) : x \in E\}$.

Proof. Let $f(E) = \{f(x) : x \in E\}$.

Since the range of f is bounded, then $f(E)$ is bounded, so $f(E)$ is bounded above and below in $\mathbb{R}$.

Let $g(E) = \{g(x) : x \in E\}$.

Since the range of g is bounded, then $g(E)$ is bounded, so $g(E)$ is bounded above and below in $\mathbb{R}$.

Let $f(E) + g(E) = \{f(x) + g(x) : x \in E\}$.

Since $E \neq \emptyset$, let $x \in E$.

Then $f(x) \in f(E)$ and $g(x) \in g(E)$, so $f(x) + g(x) \in f(E) + g(E)$.

Since $f(x) \in f(E)$, then $f(E) \neq \emptyset$.

Since $g(x) \in g(E)$, then $g(E) \neq \emptyset$.

Since $f(x) + g(x) \in f(E) + g(E)$, then $f(E) + g(E) \neq \emptyset$.

Since $f(E) \neq \emptyset$ and $f(E)$ is bounded above in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\sup f(E)$ exists.

Since $g(E) \neq \emptyset$ and $g(E)$ is bounded above in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\sup g(E)$ exists.

Since $f(x) \in f(E)$ and $\sup f(E)$ is an upper bound of $f(E)$, then $f(x) \leq \sup f(E)$.

Since $g(x) \in g(E)$ and $\sup g(E)$ is an upper bound of $g(E)$, then $g(x) \leq \sup g(E)$.

Hence, $f(x) + g(x) \leq \sup f(E) + \sup g(E)$.

Thus, $\sup f(E)+\sup g(E)$ is an upper bound of $f(E)+g(E)$, so $f(E)+g(E)$ is bounded above in $\mathbb{R}$.

Since $f(E) + g(E) \neq \emptyset$ and $f(E) + g(E)$ is bounded above in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\sup(f(E) + g(E))$ exists.

Since $\sup(f(E)+g(E))$ is the least upper bound of $f(E)+g(E)$ and $\sup f(E)+\sup g(E)$ is an upper bound of $f(E)+g(E)$, then $\sup(f(E)+g(E)) \leq \sup f(E)+\sup g(E)$, as desired.

Since $f(E) \neq \emptyset$ and $f(E)$ is bounded below in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\inf f(E)$ exists.

Since $g(E) \neq \emptyset$ and $g(E)$ is bounded below in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\inf g(E)$ exists.

Since $f(x) \in f(E)$ and $\inf f(E)$ is a lower bound of $f(E)$, then $\inf f(E) \leq f(x)$.

Since $g(x) \in g(E)$ and $\inf g(E)$ is a lower bound of $g(E)$, then $\inf g(E) \leq g(x)$.

Hence, $\inf f(E) + \inf g(E) \leq f(x) + g(x)$.

Thus, $\inf f(E) + \inf g(E)$ is a lower bound of $f(E) + g(E)$, so $f(E) + g(E)$ is bounded below in $\mathbb{R}$.

Since $f(E) + g(E) \neq \emptyset$ and $f(E) + g(E)$ is bounded below in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\inf(f(E) + g(E))$ exists.

Since $\inf(f(E) + g(E))$ is the greatest lower bound of $f(E) + g(E)$ and $\inf f(E) + \inf g(E)$ is a lower bound of $f(E) + g(E)$, then $\inf f(E) + \inf g(E) \leq \inf(f(E) + g(E))$, as desired. □

Archimedean ordered fields

Exercise 94. Prove $\sup\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 1$.

Proof. Let $A = \{1\}$.
Then $\sup A = 1$.
Let $B = \{\frac{1}{n} : n \in \mathbb{N}\}$.
Let $A - B = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$.
We must prove $\sup(A - B) = 1$.
Since $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$, then $\inf B = 0$.
Therefore, $\sup(A - B) = \sup A - \inf B = 1 - 0 = 1$. □

Proof. Let $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$.
We must prove $1 = \sup S$.

We first prove 1 is an upper bound of S.
Since $1 \in \mathbb{N}$ and $1 - \frac{1}{1} = 1 - 1 = 0$, then $0 \in S$, so $S \neq \emptyset$.
Let $x \in S$.
Then there exists $n \in \mathbb{N}$ such that $x = 1 - \frac{1}{n}$.
Since $n \in \mathbb{N}$, then $n > 0$, so $\frac{1}{n} > 0$.
Thus, $\frac{1}{n} > 1 - 1$, so $1 + \frac{1}{n} > 1$.
Hence, $1 > 1 - \frac{1}{n}$, so $1 > x$.
Thus, $x < 1$ for all $x \in S$, so 1 is an upper bound of S.

To prove 1 is the least upper bound of S, we prove for every $\epsilon > 0$, there exists $x \in S$ such that $x > 1 - \epsilon$.
Let $\epsilon > 0$ be given.
By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Thus, $\frac{-1}{n} > -\epsilon$, so $1 - \frac{1}{n} > 1 - \epsilon$.
Let $x = 1 - \frac{1}{n}$.
Then $x \in S$ and $x > 1 - \epsilon$.
Therefore, 1 is the least upper bound of S, so $1 = \sup S$. $\square$

Exercise 95. Let $S = \{\frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N}\}$.
Then $\sup S = 1$ and $\inf S = -1$.

Proof. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$.
Then $\sup A = 1$ and $\inf A = 0$.
Let $B = \{\frac{1}{m} : m \in \mathbb{N}\}$.
Then $\sup B = 1$ and $\inf B = 0$.
Let $A - B = \{\frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N}\}$.
We must prove $\sup(A - B) = 1$ and $\inf(A - B) = -1$.
Observe that $\sup(A - B) = \sup A - \inf B = 1 - 0 = 1$.
Since $\sup(B - A) = \sup B - \inf A = 1 - 0 = 1$, then $\inf(A - B) = -\sup(B - A) = -1$. $\square$

Proof. Since $1 \in \mathbb{N}$ and $\frac{1}{1} - \frac{1}{1} = 1 - 1 = 0$, then $0 \in S$, so $S \neq \emptyset$.
Let $x \in S$.
Then there exist $m, n \in \mathbb{N}$ such that $x = \frac{1}{n} - \frac{1}{m}$.
Since $m, n \in \mathbb{N}$, then $m \geq 1 > 0$ and $n \geq 1 > 0$, so $m \geq 1$ and $n \geq 1$ and $m > 0$ and $n > 0$.
Since $1 \leq m$ and $m > 0$, then $\frac{1}{m} \leq 1$.
Since $m > 0$, then $\frac{1}{m} > 0$.
Thus, $0 < \frac{1}{m} \leq 1$.
Since $1 \leq n$ and $n > 0$, then $\frac{1}{n} \leq 1$.
Since $n > 0$, then $\frac{1}{n} > 0$.
Thus, $0 < \frac{1}{n} \leq 1$.
Since $0 < \frac{1}{n} \leq 1$ and $0 < \frac{1}{m} \leq 1$, then by a previous exercise, we have $|\frac{1}{n} - \frac{1}{m}| \leq 1 - 0 = 1$, so $|x| \leq 1$.
Hence, $-1 \leq x \leq 1$, so $-1 \leq x$ and $x \leq 1$.
Thus, $-1 \leq x$ and $x \leq 1$ for all $x \in S$, so $-1 \leq x$ for all $x \in S$ and $x \leq 1$ for all $x \in S$.
Since $x \leq 1$ for all $x \in S$, then 1 is an upper bound of S.
Since $-1 \leq x$ for all $x \in S$, then -1 is a lower bound of S.

To prove 1 is the least upper bound of S, let $\epsilon > 0$ be given.
We must prove there exists $x \in S$ such that $x > 1 - \epsilon$.
Since $\epsilon > 0$, then by the Archimedean property of $\mathbb{R}$, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$.
Let $x = 1 - \frac{1}{m}$.
Since $1 \in \mathbb{N}$ and $m \in \mathbb{N}$ and $\frac{1}{1} - \frac{1}{m} = 1 - \frac{1}{m} = x$, then $x \in S$.
Since $\frac{1}{m} < \epsilon$, then $\frac{-1}{m} > -\epsilon$, so $1 - \frac{1}{m} > 1 - \epsilon$.
Thus, $x > 1 - \epsilon$.
Since 1 is an upper bound of S and for every $\epsilon > 0$, there exists $x \in S$ such that $x > 1 - \epsilon$, then 1 is the least upper bound of S, so $1 = \sup S$.

To prove -1 is the greatest lower bound of S, let $\epsilon > 0$ be given.

We must prove there exists $x \in S$ such that $x < -1 + \epsilon$.

Since $\epsilon > 0$, then by the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Let $x = \frac{1}{n} - 1$.

Since $1 \in \mathbb{N}$ and $n \in \mathbb{N}$ and $\frac{1}{n} - \frac{1}{1} = \frac{1}{n} - 1 = x$, then $x \in S$.

Since $\frac{1}{n} < \epsilon$, then $\frac{1}{n} - 1 < \epsilon - 1$, so $x < \epsilon - 1$.

Thus, $x < -1 + \epsilon$.

Since -1 is a lower bound of S and for every $\epsilon > 0$, there exists $x \in S$ such that $x < -1 + \epsilon$, then -1 is the greatest lower bound of S, so $-1 = \inf S$. $\square$

Exercise 96. Let $S \subset \mathbb{R}$.

Let $B \in \mathbb{R}$.

If $B - \frac{1}{n}$ is not an upper bound of S for all $n \in \mathbb{N}$ and $B + \frac{1}{n}$ is an upper bound of S for all $n \in \mathbb{N}$, then $B = \sup S$.

Proof. Suppose that $B - \frac{1}{n}$ is not an upper bound of S for all $n \in \mathbb{N}$ and $B + \frac{1}{n}$ is an upper bound of S for all $n \in \mathbb{N}$.

We must prove $B = \sup S$.

Since $1 \in \mathbb{N}$, then $B - \frac{1}{1} = B - 1$ is not an upper bound of S.

Hence, there exists $s \in S$ such that $s > B - 1$.

Suppose $s > B$.

Then $s - B > 0$.

By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < s - B$, so $B + \frac{1}{n} < s$.

Since $n \in \mathbb{N}$, then $B + \frac{1}{n}$ is an upper bound of S.

But, $s \in S$ and $s > B + \frac{1}{n}$ contradicts the fact that $B + \frac{1}{n}$ is an upper bound of S.

Hence, there does not exist $s \in S$ such that $s > B$.

Therefore, for every $s \in S$, we have $s \leq B$, so B is an upper bound of S.

To prove B is the least upper bound of S, let $\epsilon > 0$ be given.

We must prove there exists $x \in S$ such that $x > B - \epsilon$.

Since $\epsilon > 0$, then by the Archimedean property of $\mathbb{R}$, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$.

Hence, $\frac{-1}{m} > -\epsilon$, so $B - \frac{1}{m} > B - \epsilon$.

Since $m \in \mathbb{N}$, then $B - \frac{1}{m}$ is not an upper bound of S.

Thus, there exists $x \in S$ such that $x > B - \frac{1}{m}$.

Since $x > B - \frac{1}{m}$ and $B - \frac{1}{m} > B - \epsilon$, then $x > B - \epsilon$.

Therefore, for every $\epsilon > 0$ there exists $x \in S$ such that $x > B - \epsilon$.

Since B is an upper bound of S and for every $\epsilon > 0$ there exists $x \in S$ such that $x > B - \epsilon$, then $B = \sup S$. $\square$

Exercise 97. For every rational number $\epsilon > 0$, there exists a nonnegative rational number x such that $x^2 < 2 < (x + \epsilon)^2$.

Proof. Let $\epsilon > 0$ be rational.

Suppose for the sake of contradiction there does not exist a nonnegative rational number x such that $x^2 < 2 < (x + \epsilon)^2$.

Then for every nonnegative rational number x, if $x^2 < 2$, then $(x + \epsilon)^2 \leq 2$.

Let x be a nonnegative rational number such that $x^2 < 2$.

Then $(x + \epsilon)^2 \leq 2$.

Since $x, \epsilon \in \mathbb{Q}$, then $x + \epsilon \in \mathbb{Q}$, so $(x + \epsilon)^2 \in \mathbb{Q}$.

Since there is no rational number whose square is two, then $(x + \epsilon)^2 \neq 2$, so $(x + \epsilon)^2 < 2$.

Therefore, for every nonnegative rational number x, if $x^2 < 2$, then $(x+\epsilon)^2 < 2$.

Thus, for $x = 0$, we have $0^2 = 0 < 2$, so $(0 + \epsilon)^2 < 2$.

Hence, $\epsilon^2 < 2$.

We prove $(n\epsilon)^2 < 2$ for all $n \in \mathbb{N}$ by induction on n.

Basis:

Since $(1\epsilon)^2 = \epsilon^2 < 2$, then the statement holds for $n = 1$.

Induction:

Let $k \in \mathbb{N}$ such that $(k\epsilon)^2 < 2$.

Since $k\epsilon \in \mathbb{Q}$ and $k\epsilon > 0$ and $(k\epsilon)^2 < 2$, then $(k\epsilon + \epsilon)^2 < 2$.

Thus, $((k + 1)\epsilon)^2 < 2$.

Therefore, by PMI, $(n\epsilon)^2 < 2$ for all $n \in \mathbb{N}$.

Since $\frac{2}{\epsilon} \in \mathbb{R}$ and $\mathbb{N}$ is unbounded in $\mathbb{R}$, then there exists $N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$.

Thus, $N\epsilon > 2$, so $(N\epsilon)^2 > 4 > 2$.

Hence, $(N\epsilon)^2 > 2$.

Therefore, there exists $N \in \mathbb{N}$ such that $(N\epsilon)^2 > 2$.

This contradicts the statement $(n\epsilon)^2 < 2$ for all $n \in \mathbb{N}$.

Thus, there does exist a nonnegative rational number x such that $x^2 < 2 < (x + \epsilon)^2$, as desired. □

Exercise 98. Given the statement $(\forall\epsilon > 0)(\exists n \in \mathbb{N})(\frac{1}{n} < \epsilon)$, prove that $\mathbb{N}$ has no upper bound in $\mathbb{R}$.

Proof. To prove $\mathbb{N}$ has no upper bound in $\mathbb{R}$, we must prove for each $r \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $n > r$.

Let $r \in \mathbb{R}$.

Then either $r > 0$ or $r \leq 0$.

We consider these cases separately.

Case 1: Suppose $r \leq 0$.

Since $1 > 0$ and $0 \geq r$, then $1 > r$.

Therefore, 1 is a natural number and $1 > r$, as desired.

Case 2: Suppose $r > 0$.

Then $\frac{1}{r} > 0$.

Thus, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{1}{r}$.

Since $\frac{1}{n} < \frac{1}{r}$, then $\frac{1}{r} - \frac{1}{n} > 0$, so $\frac{n-r}{rn} > 0$.
Since n and r are positive, then rn is positive.
We multiply by rn to get $n - r > 0$.
Thus, $n > r$.
Therefore, there is a natural number n such that $n > r$, as desired. $\square$

Exercise 99. Assume $\mathbb{N}$ has no upper bound in $\mathbb{R}$.
Prove:
1. $(\forall r \in \mathbb{R})(\exists n \in \mathbb{N})(n > r)$.
2. $(\forall a \in \mathbb{R}, b > 0)(\exists n \in \mathbb{N})(nb > a)$.
3. $(\forall \epsilon > 0)(\exists n \in \mathbb{N})(\frac{1}{n} < \epsilon)$.

Proof. Since $\mathbb{N}$ has no upper bound in $\mathbb{R}$, then the statement there is $r \in \mathbb{R}$ such that $n \leq r$ for all $n \in \mathbb{N}$ is false.
Therefore, the statement for all $r \in \mathbb{R}$ there is $n \in \mathbb{N}$ such that $n > r$ is true.
Let $a \in \mathbb{R}$ and $b > 0$.
We must prove there is a natural number n such that $nb > a$.
Since $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and $b \neq 0$, then $\frac{a}{b} \in \mathbb{R}$.
Hence, there is a natural number n such that $n > \frac{a}{b}$.
Since $b > 0$, then $nb > a$.
Therefore, there is a natural number n for which $nb > a$, as desired.
Let $\epsilon > 0$.
Then $\frac{1}{\epsilon} > 0$.
We must prove there is a natural number n such that $\frac{1}{n} < \epsilon$.
Since $\frac{1}{\epsilon} \in \mathbb{R}$, then there is a natural number n such that $n > \frac{1}{\epsilon}$.
Since $n > \frac{1}{\epsilon}$ and $\epsilon > 0$, then $n\epsilon > 1$.
Since $n > 0$, then $\epsilon > \frac{1}{n}$.
Therefore, there is a natural number n for which $\frac{1}{n} < \epsilon$, as desired. $\square$

Exercise 100. Analyze boundedness of $\mathbb{N}$.

Solution. By the Archimedean property of $\mathbb{N}$ in $\mathbb{R}$ the set $\mathbb{N}$ has no upper bound in $\mathbb{R}$.
Since there is no upper bound of $\mathbb{N}$, then there can be no greatest element of $\mathbb{N}$ in $\mathbb{R}$.
Therefore, $\max \mathbb{N}$ does not exist in $\mathbb{R}$.
Since there is no upper bound of $\mathbb{N}$ in $\mathbb{R}$, then there can be no least upper bound of $\mathbb{N}$ in $\mathbb{R}$.
Therefore, $\sup \mathbb{N}$ does not exist in $\mathbb{R}$.
Since $1 \in \mathbb{N}$ and $1 \leq n$ for all $n \in \mathbb{N}$, then 1 is the least element of $\mathbb{N}$.
Hence, $\min \mathbb{N} = 1$.
The set $\mathbb{N}$ has many lower bounds in $\mathbb{R}$.
For example, -3 is a lower bound of $\mathbb{N}$.
Let $n \in \mathbb{N}$.
Then $n \geq 1$.
Since $-3 \leq 1$ and $1 \leq n$, then $-3 \leq n$.
Hence, since $-3 \leq n$ for all $n \in \mathbb{N}$.

Therefore, -3 is a lower bound of $\mathbb{N}$.
We prove $\inf \mathbb{N} = 1$.
Since $1 \leq n$ for all $n \in \mathbb{N}$, then 1 is a lower bound of $\mathbb{N}$ in $\mathbb{R}$.
Let $\epsilon > 0$ be given.
To prove 1 is the greatest lower bound, we must find $n \in \mathbb{N}$ such that $n < 1 + \epsilon$.
Take $n = 1$.
Clearly, $n \in \mathbb{N}$.
Since $0 < \epsilon$, then $1 < 1 + \epsilon$.
Hence, $n < 1 + \epsilon$, as desired.
Therefore, $1 = \inf \mathbb{N}$. □

Exercise 101. Analyze boundedness of $\mathbb{Z}$.

Solution. The set of integers $\mathbb{Z}$ is unbounded above in $\mathbb{R}$.
To prove $\mathbb{Z}$ has no upper bound in $\mathbb{R}$, we prove for all $r \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $n > r$.
Let $r \in \mathbb{R}$.
By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > r$.
Since $n \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{Z}$, then $n \in \mathbb{Z}$.
Hence, there exists an integer n greater than r.
Therefore, $\mathbb{Z}$ has no upper bound in $\mathbb{R}$, so $\mathbb{Z}$ is unbounded above in $\mathbb{R}$.
Since there is no upper bound of $\mathbb{Z}$, then there can be no greatest element of $\mathbb{Z}$.
Therefore, $\max \mathbb{Z}$ does not exist in $\mathbb{R}$.
Since there is no upper bound of $\mathbb{Z}$ in $\mathbb{R}$, then there can be no least upper bound of $\mathbb{Z}$ in $\mathbb{R}$.
Therefore, $\sup \mathbb{Z}$ does not exist in $\mathbb{R}$.
To prove $\mathbb{Z}$ has no lower bound in $\mathbb{R}$, we prove for every $r \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $n < r$.
Let $r \in \mathbb{R}$.
Either $r \geq 0$ or $r < 0$.
We consider these cases separately.
Case 1: Suppose $r \geq 0$.
Since -1 is an integer and $-1 < 0 \leq r$, then $-1 < r$.
Hence, there exists an integer less than r.
Case 2: Suppose $r < 0$.
Then $-r > 0$.
Since $-r \in \mathbb{R}$, then by the Archimedean property, there exists $n \in \mathbb{N}$ such that $n > -r$.
We multiply by -1 to get $-n < r$.
Since $n \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{Z}$, then $n \in \mathbb{Z}$.
Hence, $-n \in \mathbb{Z}$.
Thus, there exists an integer less than r.
Hence, in all cases, there exists an integer less than r.
Therefore, there is no lower bound of $\mathbb{Z}$ in $\mathbb{R}$.

Since there is no lower bound of $\mathbb{Z}$ in $\mathbb{R}$, then there can be no least element of $\mathbb{Z}$.

Therefore, $\min \mathbb{Z}$ does not exist in $\mathbb{R}$.

Since there is no lower bound of $\mathbb{Z}$ in $\mathbb{R}$, then there can be no greatest lower bound of $\mathbb{Z}$ in $\mathbb{R}$.

Therefore, $\inf \mathbb{Z}$ does not exist in $\mathbb{R}$. $\square$

Exercise 102. There is no smallest positive rational number.

Proof. We prove by contradiction.

Suppose there is a smallest positive rational number.

Let m be a smallest positive rational number.

Then $m \in \mathbb{Q}$ and $m > 0$ and $m \leq q$ for every positive rational number q.

Since $m \in \mathbb{Q}$, then $\frac{m}{2} \in \mathbb{Q}$.

Since $m > 0$, then $\frac{m}{2} > 0$.

Since $\frac{m}{2} \in \mathbb{Q}$ and $\frac{m}{2} > 0$, then $\frac{m}{2}$ is a positive rational number.

Since $0 < m$, then $m < 2m$, so $\frac{m}{2} < m$.

Thus, there exists a positive rational number $\frac{m}{2}$ such that $m > \frac{m}{2}$.

This contradicts the fact that $m \leq q$ for every positive rational number q.

Therefore, there is no smallest positive rational number. $\square$

Exercise 103. There is no smallest positive real number.

Solution. If s is a smallest positive real number, then half of s is even smaller. This implies s cannot be the smallest positive real number. $\square$

Proof. Suppose there is a smallest positive real number.

Then there exists a positive real number s such that $s \leq x$ for all $x \in \mathbb{R}$.

Since s is a positive real number, then $s \in \mathbb{R}$ and $s > 0$.

Since $0 < s$, then $s < 2s$, so $\frac{s}{2} < s$.

Since $s \in \mathbb{R}$, then $\frac{s}{2} \in \mathbb{R}$.

Hence, there exists $\frac{s}{2} \in \mathbb{R}$ such that $s > \frac{s}{2}$.

This contradicts the fact that $s \leq x$ for all $x \in \mathbb{R}$.

Therefore, there is no smallest positive real number. $\square$

Exercise 104. Disprove the assertion that there is a positive real number that is smaller than all positive rational numbers.

Solution. The assertion states there exists a real number $r > 0$ such that $r < q$ for all $q \in \mathbb{Q}^+$.

In symbols this is:

$(\exists r > 0)(\forall q \in \mathbb{Q}^+)(r < q)$.

The negation is:

$(\forall r > 0)(\exists q \in \mathbb{Q}^+)(r \geq q)$.

Therefore, to disprove the assertion we must prove its negation. $\square$

Proof. Suppose there is a positive real number that is smaller than all positive rational numbers.

Let r be some positive real number that is smaller than all positive rational numbers.

Then $r \in \mathbb{R}$ and $r > 0$ and $r < q$ for all positive rational q.

Since $r > 0$, then $\frac{1}{r} > 0$, so by the Archimedean property of $\mathbb{R}$, there is $n \in \mathbb{N}$ such that $n > \frac{1}{r}$.

Hence, $r > \frac{1}{n}$, so $\frac{1}{n}$ is a positive rational number such that $r > \frac{1}{n}$.

This contradicts the fact that r is smaller than all positive rational numbers.

Therefore, there is no positive real number that is smaller than all positive rational numbers. □

Proof. Let r be a positive real number.

To disprove the assertion, we must prove there exists $q \in \mathbb{Q}^+$ such that $r \geq q$.

Since $r > 0$, then by the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < r$.

Since $\frac{1}{n} < r$, then $\frac{1}{n} \leq r$, so $r \geq \frac{1}{n}$.

Since $\frac{1}{n}$ is a positive rational number, let $q = \frac{1}{n}$.

Then $r \geq q$, as desired. □

Lemma 105. *For all $n \in \mathbb{N}$, $2^n > n$.*

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : 2^n > n\}$.

Basis:

Since $1 \in \mathbb{N}$ and $2^1 = 2 > 1$, then $1 \in S$.

Induction:

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $2^k > k$.

Since $k \in \mathbb{N}$, then $k \geq 1$ and $k + 1 \in \mathbb{N}$.

Since $2^{k+1} = 2^k \cdot 2 > 2k = k + k \geq k + 1$, then $2^{k+1} > k + 1$.

Since $k + 1 \in \mathbb{N}$ and $2^{k+1} > k + 1$, then $k + 1 \in S$.

Thus, by induction, $k \in S$ implies $k + 1 \in S$, so $S = \mathbb{N}$.

Therefore, $2^n > n$ for all $n \in \mathbb{N}$. □

Exercise 106. Let $x > 0$.

Then there exists $n \in \mathbb{N}$ such that $2^n > \frac{1}{x}$.

Proof. Since $x > 0$, then $\frac{1}{x} > 0$.

By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{x}$.

Since $2^n > n$ for all $n \in \mathbb{N}$, then $2^n > n$.

Thus, we have $2^n > n > \frac{1}{x}$, so $2^n > \frac{1}{x}$.

Therefore, there exists $n \in \mathbb{N}$ such that $2^n > \frac{1}{x}$. □

Exercise 107. Let $\epsilon > 0$.

Let $x, y \in \mathbb{R}$ such that $x < y$.

Then there exists $q \in \mathbb{Q}$ such that $x < q\epsilon < y$.

(Therefore, the set $\{q\epsilon : q \in \mathbb{Q}\}$ is dense in $\mathbb{R}$).

Proof. Since $x < y$ and $\epsilon > 0$, then $\frac{x}{\epsilon} < \frac{y}{\epsilon}$.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $q \in \mathbb{Q}$ such that $\frac{x}{\epsilon} < q < \frac{y}{\epsilon}$.
Since $\epsilon > 0$, then $x < q\epsilon < y$.
Therefore, there exists $q \in \mathbb{Q}$ such that $x < q\epsilon < y$. □

Exercise 108. Let $t \in \mathbb{R}$ and $t \neq 0$.
Let $S = \{qt : q \in \mathbb{Q}\}$.
Then S is dense in $\mathbb{R}$.

Proof. To prove S is dense in $\mathbb{R}$, let $a, b \in \mathbb{R}$ with $a < b$.
We must prove there exists $s \in S$ such that $a < s < b$.
Since $t \neq 0$, then either $t > 0$ or $t < 0$.
We consider these two cases separately.
Case 1: Suppose $t > 0$.
Since $a < b$, then $\frac{a}{t} < \frac{b}{t}$.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $q \in \mathbb{Q}$ such that $\frac{a}{t} < q < \frac{b}{t}$, so $a < qt < b$.
Let $s = qt$.
Since $q \in \mathbb{Q}$, then $s \in S$, so $a < s < b$.
Hence, there exists $s \in S$ such that $a < s < b$.
Case 2: Suppose $t < 0$.
Then $-t > 0$.
Since $a < b$, then $\frac{a}{-t} < \frac{b}{-t}$.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $q \in \mathbb{Q}$ such that $\frac{a}{-t} < q < \frac{b}{-t}$, so $a < -qt < b$.
Let $s = -qt$.
Since $q \in \mathbb{Q}$, then $-q \in \mathbb{Q}$, so $s \in S$.
Thus, $a < s < b$.
Hence, there exists $s \in S$ such that $a < s < b$.
Therefore, in all cases, there exists $s \in S$ such that $a < s < b$, as desired. □

Existence of square roots in $\mathbb{R}$

Exercise 109. Let $x, y \in \mathbb{R}$.
If $0 \leq x < y$, then $0 \leq \sqrt{x} < \sqrt{y}$.

Proof. Suppose $0 \leq x < y$.
Then $0 \leq x$ and $x < y$.
Since $x \geq 0$, then either $x > 0$ or $x = 0$.
We consider these cases separately.
Case 1: Suppose $x > 0$.
Since $0 < x$ and $x < y$, then $0 < x < y$.
Hence, $0 < \sqrt{x} < \sqrt{y}$.
Case 2: Suppose $x = 0$.
Since $y > x$ and $x = 0$, then $y > 0$, so $\sqrt{y} > 0$.
Since $\sqrt{x} = \sqrt{0} = 0$ and $0 < \sqrt{y}$, then $\sqrt{x} = 0 < \sqrt{y}$.
Therefore, $0 = \sqrt{x} < \sqrt{y}$. □

Exercise 110. Let $a, b \in \mathbb{R}$.
If $0 < a < b$, then $a < \sqrt{ab} < b$.

Proof. Suppose $0 < a < b$.
Then $0 < a$ and $a < b$, so $0 < b$.
Since $a > 0$ and $b > 0$, then $ab > 0$.
Since $a < b$ an $b > 0$, then $ab < b^2$.
Thus, $0 < ab$ and $ab < b^2$, so $0 < ab < b^2$.
Therefore, $0 < \sqrt{ab} < \sqrt{b^2} = |b| = b$, so $0 < \sqrt{ab} < b$, as desired. $\square$

Exercise 111. another proof of triangle inequality
Let $a, b \in \mathbb{R}$.
Then $|a + b| \leq |a| + |b|$.

Proof. Since $ab \leq |ab| = |a||b|$, then $2ab \leq 2|a||b|$.
Since $0 \leq (a + b)^2 = a^2 + 2ab + b^2 \leq a^2 + 2|a||b| + b^2 = |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2$, then $0 \leq (a + b)^2 \leq (|a| + |b|)^2$, so $0 \leq |a + b| \leq ||a| + |b|| = |a| + |b|$.
Therefore, $|a + b| \leq |a| + |b|$. $\square$

Lemma 112. *Let $a, b \in \mathbb{R}$.*
Then $2ab \leq a^2 + b^2$ and $ab \leq (\frac{a+b}{2})^2$.
Furthermore, if $a = b$, then $2ab = a^2 + b^2$ and $ab = (\frac{a+b}{2})^2$.

Proof. We first prove $2ab \leq a^2 + b^2$.
Since $0 \leq (a - b)^2 = a^2 - 2ab + b^2$, then $2ab \leq a^2 + b^2$.
Suppose $a = b$.
Then $2ab = 2a^2 = a^2 + a^2 = a^2 + b^2$.
We next prove $ab \leq (\frac{a+b}{2})^2$.
Since $2ab \leq a^2 + b^2$, then $4ab \leq a^2 + 2ab + b^2 = (a + b)^2$, so $ab \leq \frac{(a+b)^2}{4} = (\frac{a+b}{2})^2$.
Suppose $a = b$.
Then $ab = a^2 = (\frac{2a}{2})^2 = (\frac{a+a}{2})^2 = (\frac{a+b}{2})^2$. $\square$

Proposition 113. ***arithmetic-geometric mean inequality***
Let $a, b \in \mathbb{R}$.
If $a \geq 0$ and $b \geq 0$, then $\sqrt{ab} \leq \frac{a+b}{2}$.
Furthermore, if $a = b$, then $\sqrt{ab} = \frac{a+b}{2}$.

Proof. Suppose $a \geq 0$ and $b \geq 0$.
Then $\sqrt{a} \geq 0$ and $\sqrt{b} \geq 0$, so $(\sqrt{a} - \sqrt{b})^2 \geq 0$.
Thus, $0 \leq (\sqrt{a} - \sqrt{b})^2 = (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 = a - 2\sqrt{ab} + b$, so $0 \leq a - 2\sqrt{ab} + b$.
Hence, $2\sqrt{ab} \leq a + b$, so $\sqrt{ab} \leq \frac{a+b}{2}$.

Suppose $a = b$.

Then $\sqrt{ab} = \sqrt{a^2} = |a| = a = \frac{2a}{2} = \frac{a+a}{2} = \frac{a+b}{2}$. □

Proof. Here is an alternate proof based on the previous lemma.

Since $a \geq 0$ and $b \geq 0$, then $ab \geq 0$ and $a + b \geq 0$.

Thus, $0 \leq ab \leq (\frac{a+b}{2})^2$, so $\sqrt{ab} \leq |\frac{a+b}{2}| = \frac{|a+b|}{2} = \frac{a+b}{2}$. □

Corollary 114. *Let $a, b \in \mathbb{R}$.*

If $a > 0$ and $b > 0$, then $\frac{2ab}{a+b} \leq \sqrt{ab}$.

Furthermore, if $a = b$, then $\frac{2ab}{a+b} = \sqrt{ab}$.

Solution. We call the expression $\frac{2ab}{a+b}$ the **harmonic mean of** a **and** b.

Thus, if $a > 0$ and $b > 0$, then $\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2}$.

Therefore, for any positive real numbers a and b, the harmonic mean is smaller than the geometric mean which is smaller than the arithmetic mean of a and b. □

Proof. Suppose $a > 0$ and $b > 0$.

Then $a + b > 0$ and $ab > 0$, so $\sqrt{ab} > 0$.

Hence, $\frac{2\sqrt{ab}}{a+b} > 0$.

Since $\sqrt{ab} \leq \frac{a+b}{2}$, then $\frac{2ab}{a+b} = \frac{2(\sqrt{ab})^2}{a+b} = \frac{2\sqrt{ab}}{a+b} \cdot \sqrt{ab} \leq \frac{2\sqrt{ab}}{a+b} \cdot \frac{a+b}{2} = \sqrt{ab}$.

Suppose $a = b$.

Then $\frac{2ab}{a+b} = \frac{2a^2}{2a} = a = (\sqrt{a})^2 = \sqrt{a}\sqrt{a} = \sqrt{aa} = \sqrt{ab}$. □

Exercise 115. Given 400 meters of fence, the largest rectangular area that can fence in from three sides along a straight river using the river as the fourth side is 100 x 200 meters.

Proof. Let the rectangular fence be composed of two smaller equal sized rectangular pieces such that each rectangular piece of the fence has length l and width w.

Then the dimension of the rectangular fence is $2l$ by w.

The perimeter of the fence is $400 = 2l + 2w = 2(l + w)$, so $200 = l + w$.

Since $l > 0$ and $w > 0$, then by AGM, $0 < \sqrt{lw} \leq \frac{l+w}{2}$, so $lw \leq (\frac{l+w}{2})^2$.

The maximum area occurs when each smaller rectangle piece is a square, so $l = w$.

Thus, the maximum area is $lw = ww = w^2 = (\frac{l+w}{2})^2 = (\frac{200}{2})^2 = 100^2$, so $w^2 = 100^2$.

Hence, $w = 100$ and $l = 100$.

Therefore, the fence has dimensions 200 by 100. □

Exercise 116. Let $c \in \mathbb{R}$ with $c > 0$.

Then the function given by $f(x) = x(c - x)$ is maximized when $x = \frac{c}{2}$.

Suppose $a > 0$.

What value of x will maximize $x(c - ax)$?

Proof. Let $f : [0, c] \to \mathbb{R}$ be the function defined by $f(x) = x(c - x)$.

To prove f is maximized when $x = \frac{c}{2}$, we must prove $f(x) \leq f(\frac{c}{2})$ for every $x \in domf$.

Let $x \in domf = [0, c]$.

Then $0 \leq x \leq c$, so $0 \leq x$ and $x \leq c$.

Since $x \leq c$, then $0 \leq c - x$.

Since $x \geq 0$ and $c - x \geq 0$, then by AGM, $\sqrt{x(c-x)} \leq \frac{x+(c-x)}{2} = \frac{c}{2}$.

Since $0 \leq \sqrt{x(c-x)} \leq \frac{c}{2}$, then $f(x) = x(c-x) \leq (\frac{c}{2})^2 = \frac{c}{2} \cdot \frac{c}{2} = \frac{c}{2}(c - \frac{c}{2}) = f(\frac{c}{2})$, so $f(x) \leq f(\frac{c}{2})$, as desired. ☐

Solution. Suppose $a > 0$.

Let g be a real valued function defined by $g(x) = x(c - ax)$.

We must find a value of x that will maximize g.

Observe that $g(x) = xc - ax^2 = ax(\frac{c}{a} - x)$.

Let h be a real valued function defined by $h(x) = x(\frac{c}{a} - x)$.

Then $g(x) = a \cdot h(x)$.

Since a is a constant scalar, then g is maximized when h is maximized.

Since $a > 0$ and $c > 0$, then $\frac{c}{a} > 0$, so h is maximized when $x = \frac{c/a}{2} = \frac{c}{2a}$.

Therefore, g is maximized when $x = \frac{c}{2a}$. ☐

Exercise 117. Let $x, y, z \in \mathbb{R}$ such that $x \geq 0$ and $y \geq 0$ and $z \geq 0$ and $y + z \geq 2$.

Then $(x + y + z)^2 \geq 4x + 4yz$.

Proof. Since $y \geq 0$ and $z \geq 0$, then by AGM, $\sqrt{yz} \leq \frac{y+z}{2}$.

Since $0 \leq \sqrt{yz} \leq \frac{y+z}{2}$, then $yz \leq \frac{(y+z)^2}{4}$, so $4yz \leq (y + z)^2$.

Hence, $(y + z)^2 \geq 4yz$.

Since $y + z \geq 2$ and $x \geq 0$, then $2x(y + z) \geq 4x$.

Since $x^2 \geq 0$, then $x^2 + 2x(y + z) \geq 4x$.

Observe that

$$\begin{aligned}
(x + y + z)^2 &= [x + (y + z)]^2 \\
&= x^2 + 2x(y + z) + (y + z)^2 \\
&\geq x^2 + 2x(y + z) + 4yz \\
&\geq 4x + 4yz.
\end{aligned}$$

Therefore, $(x + y + z)^2 \geq 4x + 4yz$. ☐

Exercise 118. Let $x, y, u, v \in \mathbb{R}$.

Then $(xu + yv)^2 \leq (x^2 + y^2)(u^2 + v^2)$.

Proof. Since $(xv)^2 \geq 0$ and $(yu)^2 \geq 0$, then by AGM, $|xuyv| = |xvyu| = \sqrt{(xvyu)^2} = \sqrt{(xv)^2(yu)^2} \leq \frac{(xv)^2+(yu)^2}{2}$, so $2|xuyv| \leq (xv)^2 + (yu)^2$.

Hence, $(xu)^2 + 2|xuyv| + (yv)^2 \leq (xu)^2 + (xv)^2 + (yu)^2 + (yv)^2$, so $|xu|^2 + 2|xuyv| + |yv|^2 \leq x^2u^2 + x^2v^2 + y^2u^2 + y^2v^2$.

Thus, $(|xu| + |yv|)^2 \leq (x^2 + y^2)(u^2 + v^2)$.

Since $0 \leq |xu+yv| \leq |xu|+|yv|$, then $(xu+yv)^2 = |xu+yv|^2 \leq (|xu|+|yv|)^2$.

Since $(xu + yv)^2 \leq (|xu| + |yv|)^2$ and $(|xu| + |yv|)^2 \leq (x^2 + y^2)(u^2 + v^2)$, then $(xu + yv)^2 \leq (x^2 + y^2)(u^2 + v^2)$, as desired. ☐

Proposition 119. ***generalized arithmetic-geometric mean inequality***

Let $n \in \mathbb{Z}^+$.

Let $a_1, a_2, ..., a_n \in \mathbb{R}^+$.

Then $\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{1}{n} \sum_{k=1}^{n} a_k$.

Proof. We prove by induction on n. ☐

Exercise 120. If x is irrational, then $x + y$ is irrational for all $y \in \mathbb{Q}$.

Proof. We prove by contrapositive.

Suppose there exists $y \in \mathbb{Q}$ such that $x + y$ is rational.

Since $x + y$ is rational, then $x + y \in \mathbb{Q}$.

Since $\mathbb{Q}$ is closed under subtraction and $x+y \in \mathbb{Q}$ and $y \in \mathbb{Q}$, then $(x+y)-y \in \mathbb{Q}$.

Therefore, $x \in \mathbb{Q}$, so x is rational, as desired. ☐

Exercise 121. The number $\sqrt{3}$ is irrational.

Proof. Suppose for the sake of contradiction that $\sqrt{3}$ is not irrational.

Then $\sqrt{3}$ is rational, so there are integers a and b for which

$$\sqrt{3} = \frac{a}{b}. \tag{1}$$

Let this fraction be reduced to lowest terms.

This means, in particular, that a and b are not both multiples of 3, for if they were, the fraction could be further reduced by factoring the 3's from the numerator and denominator and canceling.

Since $3 = (\frac{a}{b})^2 = \frac{a^2}{b^2}$, then $a^2 = 3b^2$, so a^2 is a multiple of 3.

Thus, 3 divides a^2.

Since $3|a^2$ and 3 is prime, then by Euclid's lemma, $3|a$, so a is a multiple of 3.

Hence, $a = 3k$ for some integer k.

Thus, $3b^2 = (3k)^2 = 9k^2$, so $b^2 = 3k^2$.

Therefore, b^2 is a multiple of 3, so $3|b^2$.

Since $3|b^2$ and 3 is prime, then by Euclid's lemma, $3|b$, so b is a multiple of 3.

Hence, a and b are both multiples of 3 which contradicts the assumption a and b are not both multiples of 3.

Therefore, $\sqrt{3}$ is irrational. ☐

Proof. Suppose for the sake of contradiction that $\sqrt{3}$ is not irrational.

Then $\sqrt{3}$ is rational, so there are integers a and b for which

$$\sqrt{3} = \frac{a}{b}. \tag{2}$$

Let this fraction be reduced to lowest terms.

This means, in particular, that a and b are not both even, for if they were, the fraction could be further reduced by factoring the 2's from the numerator and denominator and canceling.

Squaring both sides of Equation 2 we get

$$a^2 = 3b^2. \tag{3}$$

Either b is even or b is odd.

We consider these two cases separately.

Case 1: Suppose b is even.

Since a and b are not both even and b is even, then it immediately follows that a is odd.

Since b is even, then there is an integer c for which $b = 2c$.

Substituting this into Equation 3 we get $a^2 = 3(2c)^2 = 12c^2 = 2(6c^2)$.

Thus a^2 is even, and therefore a is even.

But we previously deduced that a is odd, so we now have a contradiction a is even and a is odd.

Thus b cannot be even.

Case 2: Suppose b is odd.

Then there is an integer c for which $b = 2c + 1$.

Substituting this into Equation 3 we get $a^2 = 3(2c+1)^2 = 3(4c^2+4c+1) = 12c^2 + 12c + 3 = 2(6c^2 + 6c + 1) + 1$.

Therefore a^2 is odd, and consequently a is odd.

This implies there is an integer d for which $a = 2d + 1$.

Substituting into Equation 3 we get

$$\begin{aligned}
(2d+1)^2 &= 3(2c+1)^2 \\
4d^2 + 4d + 1 &= 3(4c^2 + 4c + 1) \\
4d^2 + 4d + 1 &= 12c^2 + 12c + 3 \\
4d^2 + 4d - 12c^2 - 12c &= 2 \\
2d^2 + 2d - 6c^2 - 6c &= 1 \\
2(d^2 + d - 3c^2 - 3c) &= 1
\end{aligned}$$

Since $d^2 + d - 3c^2 - 3c \in \mathbb{Z}$ then the last equation means that 1 is even, a contradiction.

Both cases show that a contradiction results when we assume that $\sqrt{3}$ is rational. Thus $\sqrt{3}$ must be irrational. □

Exercise 122. The number $\sqrt[3]{2}$ is irrational.

Proof. We prove by contradiction.

Suppose $\sqrt[3]{2}$ is rational.

Then there exist integers a, b with $b \neq 0$ such that $\sqrt[3]{2} = \frac{a}{b}$.

We may assume $\frac{a}{b}$ is in lowest terms; that is, we assume $\gcd(a, b) = 1$.

Observe that $(\frac{a}{b})^3 = 2$, so $a^3 = 2b^3$.
Since $b^3 \in \mathbb{Z}$ and $a^3 = 2b^3$, then a^3 is even.
Thus, a is even, so there exists an integer k such that $a = 2k$.
Thus, $2b^3 = (2k)^3 = 8k^3$, so $b^3 = 4k^3 = 2(2k^3)$.
Since $2k^3 \in \mathbb{Z}$ and $b^3 = 2(2k^3)$, then b^3 is even.
Thus, b is even, so there exists an integer m such that $b = 2m$.
Since $a = 2k$ and $b = 2m$, then $2|a$ and $2|b$, so 2 is a common divisor of a and b.
By definition of gcd, any common divisor of a and b divides $\gcd(a, b)$.
Hence, $2|1$, a contradiction.
Therefore, $\sqrt[3]{2}$ is irrational. □

Proof. Suppose $\sqrt[3]{2}$ is rational.
Then there are integers a and b for which $\sqrt[3]{2} = \frac{a}{b}$.
Let this fraction be fully reduced. In particular, this means a and b are not both even, for if they were, the fraction could be further reduced by factoring the 2's from the numerator and denominator and canceling.
Cubing gives $2 = \frac{a^3}{b^3}$ and therefore $a^3 = 2b^3$.
Thus a^3 is even. It follows that a is even since we proved proposition (which implies that its contrapositive is true, namely, that if x^3 is even, then x is even).
Since a is even and a and b are not both even, then it follows that b is not even, ie, b is odd.
Since a is even, then there is some integer c for which $a = 2c$.
Then $(2c)^3 = 2b^3$. Dividing by 2 gives $4c^3 = b^3$.
Since $4c^3 = 2(2c^3)$ then b^3 is even and it follows that b is even.
But we previously deduced that b is odd.
Thus we have a contradiction that b is even and b is odd. □

Exercise 123. The number $\sqrt[3]{3}$ is irrational.

Proof. We prove by contradiction.
Suppose $\sqrt[3]{3}$ is not irrational.
Then $\sqrt[3]{3}$ is rational, so there exist integers m and n with $n \neq 0$ such that $\frac{m}{n} = \sqrt[3]{3}$.
Assume $\frac{m}{n}$ is in lowest terms, so that $\gcd(m, n) = 1$.
Since $\frac{m}{n} = \sqrt[3]{3}$, then $m^3 = 3n^3$.
Since $n^3 \in \mathbb{Z}$ and $m^3 = 3n^3$, then $3|m^3$.
Since 3 is prime and $3|m \cdot m \cdot m$, then by corollary to Euclid's lemma, $3|m$.
Thus, there exists an integer k such that $m = 3k$, so $3n^3 = m^3 = (3k)^3 = 27k^3$, so $n^3 = 9k^3$.
Since k^3 is an integer and $n^3 = 9k^3$, then $9|n^3$.
Since $3|9$ and $9|n^3$, then by transitivity of the divides relation, $3|n^3$.
Since 3 is prime and $3|n \cdot n \cdot n$, then by corollary to Euclid's lemma, $3|n$.
Since $3|m$ and $3|n$, then 3 is a common divisor of m and n.
Since 1 is the greatest common divisor of m and n, then any positive integer that is a common divisor of m and n must be less than or equal to 1.

Since 3 is a positive common divisor of m and n, then $3 \leq 1$, a contradiction.
Therefore, $\sqrt[3]{3}$ is rational. ∎

Exercise 124. For any real number x, either $\sqrt{2}+x$ or $\sqrt{2}-x$ is irrational.

Proof. Suppose for the sake of contradiction that there is a real number x such that $\sqrt{2}+x$ is rational and $\sqrt{2}-x$ is rational.

Thus, $\sqrt{2}+x \in \mathbb{Q}$ and $\sqrt{2}-x \in \mathbb{Q}$.

By closure of $\mathbb{Q}$ under addition, we have $(\sqrt{2}+x)+(\sqrt{2}-x) \in \mathbb{Q}$, so $2\sqrt{2} \in \mathbb{Q}$.

Since 2 is rational and $\sqrt{2}$ is irrational, then the product $2\sqrt{2}$ is irrational, so $2\sqrt{2} \notin \mathbb{Q}$.

Hence, we have $2\sqrt{2} \in \mathbb{Q}$ and $2\sqrt{2} \notin \mathbb{Q}$, a contradiction.

Therefore, there is no real number x such that $\sqrt{2}+x$ is rational and $\sqrt{2}-x$ is rational.

Thus, for every real number x, either $\sqrt{2}+x$ or $\sqrt{2}-x$ is irrational, as desired. ∎

We ask under what conditions is the square root of a natural number a rational number?

Exercise 125. Let n be a positive integer.
Then $\sqrt{n} \in \mathbb{Q}$ iff n is a perfect square.

Proof. Suppose $\sqrt{n} \in \mathbb{Q}$.

Since $n > 0$, then there exist positive integers a and b such that $\sqrt{n} = \frac{a}{b}$.

We may assume $\frac{a}{b}$ is reduced to lowest terms; i.e. a and b have no common factor greater than 1.

Thus, a and b are relatively prime, so $\gcd(a, b) = 1$.

Hence, $\gcd(b, a) = 1$.

Since $\sqrt{n} = \frac{a}{b}$, then $n = (\frac{a}{b})^2 = \frac{a^2}{b^2}$, so $nb^2 = a^2$.

Since $a^2 = nb^2 = nbb = bnb = b(nb)$ and $nb \in \mathbb{Z}$, then $b|a^2$.

Since $b|a^2$ and $\gcd(b, a) = 1$, then $b|a$, so there exists an integer k such that $a = bk$.

Since $b > 0$, then $b \neq 0$, so $\frac{a}{b} = k$.

Therefore, $n = k^2$, so n is a perfect square, as desired. ∎

Proof. Conversely, we prove if n is a perfect square, then $\sqrt{n} \in \mathbb{Q}$.

Suppose n is a perfect square.

Then there exists an integer k such that $n = k^2$.

Thus, $\sqrt{n} = \sqrt{k^2} = |k|$.

Since $k \in \mathbb{Z}$, then $|k| \in \mathbb{Z}$.

Since $\mathbb{Z} \subset \mathbb{Q}$, then $|k| \in \mathbb{Q}$.

Therefore, $\sqrt{n} \in \mathbb{Q}$, as desired. ∎

Therefore, $\sqrt{n} \notin \mathbb{Q}$ iff n is not a perfect square.

Hence, $\sqrt{n}$ is irrational iff n is not a perfect square.

Exercise 126. The number $\sqrt{6}$ is irrational.

Proof. Suppose for the sake of contradiction that $\sqrt{6}$ is not irrational.

Then $\sqrt{6}$ is rational, so there are integers a and b for which $\sqrt{6} = \frac{a}{b}$.

Let this fraction be reduced to lowest terms which means that a and b have no common factors > 1.

In particular, a and b are not both even, for if they were, then 2's could be factored out of the numerator and denominator and canceled.

Squaring both sides we get $6 = (\frac{a}{b})^2$ which implies $a^2 = 6b^2$.

Since $6b^2 = 2(3b^2)$ we know that a^2 must be even. We immediately conclude that a must also be even since we previously proved this.

Since a and b are not both even and a is even, then b must be odd.

Since a is even, then there is an integer c for which $a = 2c$.

Substituting this into the equation $a^2 = 2(3b^2)$ and dividing by 2 gives $2c^2 = 3b^2$.

Hence $3b^2$ must be even. Since $3b^2$ is even and 3 is odd, then it follows that b^2 must be even since we proved this.

Since b^2 is even, we immediately deduce that b is even.

But previously we deduced that b is odd.

Thus we have the contradiction that b is even and b is odd. □

Exercise 127. The number $\sqrt{2} + \sqrt{3}$ is irrational.

Proof. We prove by contradiction.

Suppose $\sqrt{2} + \sqrt{3}$ is rational.

Then there exists $q \in \mathbb{Q}$ such that $\sqrt{2} + \sqrt{3} = q$.

Hence, $q^2 = (\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{2}\sqrt{3} + 3 = 2\sqrt{6} + 5$, so $q^2 - 5 = 2\sqrt{6}$.

Thus, $\sqrt{6} = \frac{q^2-5}{2}$.

Since $q \in \mathbb{Q}$, then $\frac{q^2-5}{2} \in \mathbb{Q}$, so $\sqrt{6}$ is rational.

But, this contradicts the fact that $\sqrt{6}$ is irrational.

Therefore, $\sqrt{2} + \sqrt{3}$ is irrational. □

Exercise 128. The number $3\sqrt{2} - 1$ is irrational.

Proof. Since 3 is a nonzero rational and $\sqrt{2}$ is irrational, then the product $3\sqrt{2}$ is irrational.

Since -1 is rational and $3\sqrt{2}$ is irrational, then the sum $-1 + 3\sqrt{2} = 3\sqrt{2} - 1$ is irrational. □

Exercise 129. If r is irrational, then $\sqrt{r}$ is irrational.

Proof. We prove by contrapositive.

Suppose $\sqrt{r}$ is rational.

Then $\sqrt{r} = \frac{m}{n}$ for some integers m and n.

Since $r = (\sqrt{r})^2 = (\frac{m}{n})^2 = \frac{m^2}{n^2}$ and m^2 and n^2 are integers, then r is rational, as desired. □

Proposition 130. *Every nonzero rational number can be expressed as a product of two irrational numbers.*

Proof. This proposition can be reworded as follows:

If r is a nonzero rational number, then r is a product of two irrational numbers.

Suppose r is a nonzero rational number.

Then $r = \frac{a}{b}$ for nonzero integers a and b.

Also, r can be written as a product of two numbers as follows

$$r = \sqrt{2} \cdot \frac{r}{\sqrt{2}}.$$

Since we know $\sqrt{2}$ is irrational(we previously proved this fact), we must prove that $r/\sqrt{2}$ is also irrational.

To show this, assume for the sake of contradiction that $r/\sqrt{2}$ is rational.

This means

$$\frac{r}{\sqrt{2}} = \frac{c}{d}$$

for nonzero integers c and d, so

$$\sqrt{2} = r\frac{d}{c}.$$

But we know $r = a/b$, so combining this with the above equation we get

$$\sqrt{2} = r\frac{d}{c} = \frac{a}{b}\frac{d}{c} = \frac{ad}{bc}.$$

This means $\sqrt{2}$ is rational (since ad and bc are nonzero integers), which is a contradiction because we know $\sqrt{2}$ is irrational.

Therefore $r/\sqrt{2}$ is irrational.

Consequently $r = \sqrt{2} \cdot r/\sqrt{2}$ is a product of two irrational numbers. □

Exercise 131. There are two irrational numbers a and b such that a^b is rational.

Solution. Let a, b be any arbitrary irrational numbers.

Define predicate $P(a, b) : a^b$ is rational.

We must find concrete values for a, b with $a \neq b$ such that $P(a, b)$ is true.

By law of excluded middle we know $P(a, b) \vee \neg P(a, b) \Leftrightarrow T$ (no third possibility exists).

We know $\sqrt{2}$ is irrational.

If we think about various ways to combine $\sqrt{2}$ to become 2, that would help. □

Proof. Observe that $\sqrt{2}$ is irrational.

Consider the number $(\sqrt{2})^{\sqrt{2}}$.

By the law of excluded middle either $(\sqrt{2})^{\sqrt{2}}$ is rational or $(\sqrt{2})^{\sqrt{2}}$ is irrational.

We must prove $(\sqrt{2})^{\sqrt{2}}$ is rational.

Suppose $(\sqrt{2})^{\sqrt{2}}$ is rational.

Then we are done and $a = \sqrt{2}$ and $b = \sqrt{2}$.
Suppose $(\sqrt{2})^{\sqrt{2}}$ is irrational.
Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.
Then $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = (\sqrt{2})^2 = 2$ and 2 is rational.
The proof is complete. $\square$

Exercise 132. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(1) = 2$ and $f(n) = \sqrt{3 + f(n-1)}$ for all $n \geq 2$.
Then $f(n) < 2.4$ for all $n \in \mathbb{N}$.

Proof. Let $S = \{n \in \mathbb{N} : f(n) < 2.4\}$.
To prove $f(n) < 2.4$ for all $n \in \mathbb{N}$, we prove $S = \mathbb{N}$ by induction on n.
Since $1 \in \mathbb{N}$ and $f(1) = 2 < 2.4$, then $1 \in S$.
Basis:
Since $5 < 5.76 = 2.4^2$, then $\sqrt{5} < 2.4$.
Since $2 \in \mathbb{N}$ and $f(2) = \sqrt{3 + f(1)} = \sqrt{3 + 2} = \sqrt{5} < 2.4$, then $2 \in S$.
Induction:
Let $k \in \mathbb{N}$ with $k \geq 2$ such that $k \in S$.
Since $k \in \mathbb{N}$, then $k + 1 \in \mathbb{N}$.
Since $k \in S$, then $f(k) < 2.4$, so $3 + f(k) < 5.4$.
Since $5.4 < 5.76 = 2.4^2$, then $\sqrt{5.4} < 2.4$.
Since $k \geq 2$, then $k + 1 \geq 3 > 2$, so $k + 1 > 2$.
Thus, $f(k + 1) = \sqrt{3 + f(k)} < \sqrt{5.4} < 2.4$.
Since $k + 1 \in \mathbb{N}$ and $f(k + 1) < 2.4$, then $k + 1 \in S$.
Hence, $k \in S$ implies $k + 1 \in S$, so by induction $S = \mathbb{N}$.
Therefore, $f(n) < 2.4$ for all $n \in \mathbb{N}$. $\square$

Exercise 133. For all $n \in \mathbb{N}$ with $n \geq 2$, $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$.

Proof. Let $p(n)$ be the predicate $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$ defined over $\mathbb{N}$.
To prove $p(n)$ is true for all $n \geq 2$, we prove by induction on n.
Basis:
Since $\sqrt{2} > 1$, then $\sqrt{2} + 1 > 2$, so $\frac{\sqrt{2}+1}{\sqrt{2}} > \frac{2}{\sqrt{2}}$.
Hence, $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$.
Since $\sum_{k=1}^{2} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}} > \sqrt{2}$, then $p(2)$ is true.
Induction:
Let $m \in \mathbb{N}$ with $m \geq 2$ such that $p(m)$ is true.
Then $\sum_{k=1}^{m} \frac{1}{\sqrt{k}} > \sqrt{m}$.
To prove $p(m + 1)$ is true, we must prove $\sum_{k=1}^{m+1} \frac{1}{\sqrt{k}} > \sqrt{m + 1}$.

We first prove $\sqrt{m} + \frac{1}{\sqrt{m+1}} > \sqrt{m + 1}$.
Since $m \geq 2 > 0$, then $m > 0$, so $m^2 + m > m^2 > 0$.
Hence, $m(m + 1) > m^2 > 0$, so $\sqrt{m(m + 1)} > m > 0$.
Thus, $\sqrt{m(m + 1)} > m$, so $0 > m - \sqrt{m(m + 1)}$.

Hence, $1 > m+1-\sqrt{m(m+1)} = \sqrt{m+1}(\sqrt{m+1}-\sqrt{m})$.
Therefore, $1 > \sqrt{m+1}(\sqrt{m+1}-\sqrt{m})$.
Since $\sqrt{m+1} > 0$, then $\frac{1}{\sqrt{m+1}} > \sqrt{m+1}-\sqrt{m}$.
Thus, $\sqrt{m} + \frac{1}{\sqrt{m+1}} > \sqrt{m+1}$.
Observe that

$$\begin{aligned} \sum_{k=1}^{m+1} \frac{1}{\sqrt{k}} &= \sum_{k=1}^{m} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{m+1}} \\ &> \sqrt{m} + \frac{1}{\sqrt{m+1}} \\ &> \sqrt{m+1}. \end{aligned}$$

Since $m+1 \in \mathbb{N}$ and $\sum_{k=1}^{m+1} \frac{1}{\sqrt{k}} > \sqrt{m+1}$, then $p(m+1)$ is true.

Therefore, $p(m)$ implies $p(m+1)$, so by PMI, $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$ for all $n \in \mathbb{N}$, as desired. ☐

Exercise 134. Let $K = \{s + t\sqrt{2} : s, t \in \mathbb{Q}\}$.

1. If $x_1, x_2 \in K$, then $x_1 + x_2 \in K$. (closure under addition)
2. If $x_1, x_2 \in K$, then $x_1 x_2 \in K$. (closure under multiplication)
3. If $x \in K$ and $x \neq 0$, then $\frac{1}{x} \in K$. (multiplicative inverse exists for nonzero elements of K)

This shows that K is a subfield of $\mathbb{R}$ and lies between $\mathbb{Q}$ and $\mathbb{R}$.

Proof. We prove 1.
Suppose $x_1, x_2 \in K$.
Then there exist $s_1, t_1 \in \mathbb{Q}$ such that $x_1 = s_1 + t_1\sqrt{2}$ and there exist $s_2, t_2 \in \mathbb{Q}$ such that $x_2 = s_2 + t_2\sqrt{2}$.
Let $s = s_1 + s_2$ and let $t = t_1 + t_2$.
Since $s_1, s_2 \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under addition, then $s \in \mathbb{Q}$.
Since $t_1, t_2 \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under addition, then $t \in \mathbb{Q}$.
Thus,

$$\begin{aligned} x_1 + x_2 &= (s_1 + t_1\sqrt{2}) + (s_2 + t_2\sqrt{2}) \\ &= s_1 + (t_1\sqrt{2} + s_2) + t_2\sqrt{2} \\ &= s_1 + (s_2 + t_1\sqrt{2}) + t_2\sqrt{2} \\ &= (s_1 + s_2) + (t_1\sqrt{2} + t_2\sqrt{2}) \\ &= (s_1 + s_2) + (t_1 + t_2)\sqrt{2} \\ &= s + t\sqrt{2}. \end{aligned}$$

Since there exist $s, t \in \mathbb{Q}$ such that $x_1 + x_2 = s + t\sqrt{2}$, then $x_1 + x_2 \in K$, as desired. ☐

Proof. We prove 2.
Suppose $x_1, x_2 \in K$.

Then there exist $s_1, t_1 \in \mathbb{Q}$ such that $x_1 = s_1 + t_1\sqrt{2}$ and there exist $s_2, t_2 \in \mathbb{Q}$ such that $x_2 = s_2 + t_2\sqrt{2}$.

Let $s = s_1s_2 + 2t_1t_2$ and let $t = t_1s_2 + s_1t_2$.

Since $s_1, s_2, t_1, t_2 \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under addition and multiplication, then $s, t \in \mathbb{Q}$.

Thus,

$$\begin{aligned} x_1x_2 &= (s_1 + t_1\sqrt{2})(s_2 + t_2\sqrt{2}) \\ &= (s_1 + t_1\sqrt{2})s_2 + (s_1 + t_1\sqrt{2})t_2\sqrt{2} \\ &= s_1s_2 + t_1s_2\sqrt{2} + s_1t_2\sqrt{2} + 2t_1t_2 \\ &= s_1s_2 + t_1s_2\sqrt{2} + 2t_1t_2 + s_1t_2\sqrt{2} \\ &= s_1s_2 + 2t_1t_2 + t_1s_2\sqrt{2} + s_1t_2\sqrt{2} \\ &= (s_1s_2 + 2t_1t_2) + (t_1s_2 + s_1t_2)\sqrt{2} \\ &= s + t\sqrt{2}. \end{aligned}$$

Since there exist $s, t \in \mathbb{Q}$ such that $x_1x_2 = s + t\sqrt{2}$, then $x_1x_2 \in K$, as desired. □

Proof. We prove 3.

Suppose $x \in K$ and $x \neq 0$.

Since $x \in K$, then there exist $s, t \in \mathbb{Q}$ such that $x = s + t\sqrt{2}$.

We first prove $x = 0$ iff $s = 0$ and $t = 0$.

Suppose $s = 0$ and $t = 0$.

Then $x = 0 + 0\sqrt{2} = 0$.

Conversely, suppose $x = 0$.

Then $0 = s + t\sqrt{2}$, so $-s = t\sqrt{2}$.

Suppose $t \neq 0$.

Then $\frac{-s}{t} = \sqrt{2}$.

Since $s, t \in \mathbb{Q}$ and $t \neq 0$, then $\frac{s}{t} \in \mathbb{Q}$, so $\frac{-s}{t} \in \mathbb{Q}$.

Hence, $\sqrt{2} \in \mathbb{Q}$, so $\sqrt{2}$ is rational.

But, this contradicts the fact that $\sqrt{2}$ is irrational.

Thus, $t = 0$, so $-s = 0\sqrt{2} = 0$.

Hence, $s = 0$.

Therefore, if $x = 0$, then $s = 0$ and $t = 0$.

Since $x \neq 0$, then either $s \neq 0$ or $t \neq 0$.

Thus, either $s \neq 0$ and $t = 0$ or $s = 0$ and $t \neq 0$ or $s \neq 0$ and $t \neq 0$.

We consider these cases separately.

Case 1: Suppose $s \neq 0$ and $t = 0$.

Let $s' = \frac{1}{s}$ and $t' = t$.

Since $s \in \mathbb{Q}$ and $s \neq 0$, then $s' \in \mathbb{Q}$.

Since $t \in \mathbb{Q}$, then $t' \in \mathbb{Q}$.

Since $x \neq 0$, then

$$
\begin{aligned}
\frac{1}{x} &= \frac{1}{s+t\sqrt{2}} \\
&= \frac{1}{s+0\sqrt{2}} \\
&= \frac{1}{s} \\
&= \frac{1}{s}+0 \\
&= \frac{1}{s}+0\sqrt{2} \\
&= s'+t'\sqrt{2}.
\end{aligned}
$$

Since there exist $s', t' \in \mathbb{Q}$ such that $\frac{1}{x} = s' + t'\sqrt{2}$, then $x' \in K$.
Case 2: Suppose $s = 0$ and $t \neq 0$.
Let $s' = s$ and $t' = \frac{1}{2t}$.
Since $s \in \mathbb{Q}$, then $s' \in \mathbb{Q}$.
Since $t \in \mathbb{Q}$ and $t \neq 0$, then $2t \in \mathbb{Q}$ and $2t \neq 0$, so $t' \in \mathbb{Q}$.
Since $x \neq 0$, then

$$
\begin{aligned}
\frac{1}{x} &= \frac{1}{s+t\sqrt{2}} \\
&= \frac{1}{0+t\sqrt{2}} \\
&= \frac{1}{t\sqrt{2}} \\
&= \frac{1}{t\sqrt{2}} \cdot 1 \\
&= \frac{1}{t\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{2t} \\
&= 0+\frac{\sqrt{2}}{2t} \\
&= s'+\frac{1}{2t}\sqrt{2} \\
&= s'+t'\sqrt{2}.
\end{aligned}
$$

Since there exist $s', t' \in \mathbb{Q}$ such that $\frac{1}{x} = s' + t'\sqrt{2}$, then $x' \in K$.
Case 3: Suppose $s \neq 0$ and $t \neq 0$.
Let $s' = \frac{s}{s^2-2t^2}$ and $t' = \frac{-t}{s^2-2t^2}$.
We first prove $s^2 - 2t^2 \neq 0$.
Suppose for the sake of contradiction $s^2 - 2t^2 = 0$.

Then $s^2 = 2t^2$.
Since $t \neq 0$, then $t^2 \neq 0$, so $2 = \frac{s^2}{t^2} = (\frac{s}{t})^2$.
Since $s, t \in \mathbb{Q}$ and $t \neq 0$, then $\frac{s}{t} \in \mathbb{Q}$.
Thus, there is a rational number whose square is 2.
This contradicts the fact that there is no rational number whose square is 2.
Therefore, $s^2 - 2t^2 \neq 0$.
Since $s, t \in \mathbb{Q}$, then $s^2 - 2t^2 \in \mathbb{Q}$ and $-t \in \mathbb{Q}$, so $s', t' \in \mathbb{Q}$.
We prove $s - t\sqrt{2} \neq 0$.
Suppose for the sake of contradiction $s - t\sqrt{2} = 0$.
Then $s = t\sqrt{2}$.
Since $t \neq 0$, then $\frac{s}{t} = \sqrt{2}$.
Since $s, t \in \mathbb{Q}$ and $t \neq 0$, then $\frac{s}{t} \in \mathbb{Q}$.
Thus, $\sqrt{2}$ is rational, a contradiction.
Therefore $s - t\sqrt{2} \neq 0$.
Since $x \neq 0$, then

$$\begin{aligned}
\frac{1}{x} &= \frac{1}{s+t\sqrt{2}} \\
&= \frac{1}{s+t\sqrt{2}} \cdot 1 \\
&= \frac{1}{s+t\sqrt{2}} \cdot \frac{s-t\sqrt{2}}{s-t\sqrt{2}} \\
&= \frac{s-t\sqrt{2}}{s^2-2t^2} \\
&= \frac{s}{s^2-2t^2} - \frac{t\sqrt{2}}{s^2-2t^2} \\
&= s' + t'\sqrt{2}.
\end{aligned}$$

Since there exist $s', t' \in \mathbb{Q}$ such that $\frac{1}{x} = s' + t'\sqrt{2}$, then $x' \in K$.
Therefore, in all cases, $x' \in K$, as desired. $\square$