Real Number System Notes

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Sets of Numbers

$\mathbb{N} =$ set of all natural numbers $= \{1, 2, 3, ...\}$
$\mathbb{Z} =$ set of all integers $= \{0, 1, -1, 2, -2, 3, -3, ...\}$
$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z} \wedge n \neq 0\} =$ set of all rational numbers
$\mathbb{R} =$ set of all real numbers
$\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty) =$ set of all nonzero real numbers
$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} = (0, \infty) =$ set of all positive real numbers
$\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\} = [0, \infty) = [0, \infty[=$ set of all nonnegative real numbers

Number system relationships
$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Construction of $\mathbb{Q}$

Proposition 1. *Let $\sim$ be a relation defined for all $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}^*$ by $(a, b) \sim (c, d)$ iff $ad = bc$.*
Then $\sim$ is an equivalence relation over $\mathbb{Z} \times \mathbb{Z}^*$.

Let $(m, n), (p, q), (r, s) \in \mathbb{Z} \times \mathbb{Z}^*$.
Then $m, n, p, q, r, s \in \mathbb{Z}$ and $n, q, s \neq 0$ and
1. reflexive $(m, n) \sim (m, n)$.
2. symmetric if $(m, n) \sim (p, q)$ then $(p, q) \sim (m, n)$.
3. transitive if $(m, n) \sim (p, q)$ and $(p, q) \sim (r, s)$, then $(m, n) \sim (r, s)$.

Definition 2. rational number
Let $(m, n) \in \mathbb{Z} \times \mathbb{Z}^*$.
Then $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ and $n \neq 0$.
The **equivalence class of** (m, n), denoted $\frac{m}{n}$, is the set of all ordered pairs of integers to which (m, n) is equivalent.

Therefore $\frac{m}{n} = \{(p, q) \in \mathbb{Z} \times \mathbb{Z}^* : (m, n) \sim (p, q)\}$.

$$\begin{aligned}\frac{m}{n} &= \{(p, q) \in \mathbb{Z} \times \mathbb{Z}^* : (m, n) \sim (p, q)\} \\ &= \{(p, q) \in \mathbb{Z} \times \mathbb{Z}^* : mq = np\} \\ &= \{(p, q) \in \mathbb{Z} \times \mathbb{Z} : mq = np \wedge n, q \neq 0\}.\end{aligned}$$

$\frac{m}{n}$ is called a **rational number** (ie, fraction).

Since $\sim$ is an equivalence relation, then $\frac{a}{b} = \frac{c}{d}$ iff $(a,b) \sim (c,d)$.

Therefore, $\frac{a}{b} = \frac{c}{d}$ iff $(a,b) \sim (c,d)$ iff $ad = bc$ and $b, d \neq 0$.

Example 3. $\frac{1}{2} = \{(p,q) \in \mathbb{Z} \times \mathbb{Z}^* : (1,2) \sim (p,q)\} = \{(p,q) \in \mathbb{Z} \times \mathbb{Z}^* : q = 2p\}$.
Since $4 = 2 * 2$, then $(2,4) \in \frac{1}{2}$.
Since $6 = 2 * 3$, then $(3,6) \in \frac{1}{2}$.
Since $8 = 2 * 4$, then $(4,8) \in \frac{1}{2}$.
Since $-2 = 2 * (-1)$, then $(-1,-2) \in \frac{1}{2}$.
Since $-4 = 2 * (-2)$, then $(-2,-4) \in \frac{1}{2}$.
Therefore, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{-1}{-2} = \frac{-2}{-4}$.

Since $\sim$ is an equivalence relation defined over the set $\mathbb{Z} \times \mathbb{Z}^*$, then the set of all equivalence classes of $\sim$, called the quotient set of $\mathbb{Z} \times \mathbb{Z}^*$ by $\sim$, is $\frac{\mathbb{Z} \times \mathbb{Z}^*}{\sim}$, which we denote by $\mathbb{Q}$.

Definition 4. rational numbers $\mathbb{Q}$
The collection of all equivalence classes $\frac{m}{n}$ is the set of rational numbers $\mathbb{Q}$.

Therefore, $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z} \wedge n \neq 0\}$ where $\frac{m}{n}$ is the class of ordered pairs (p,q) in $\mathbb{Z} \times \mathbb{Z}$ such that $(m,n) \sim (p,q)$ iff $mq = np$ and $n, q \neq 0$.

We want to define addition of fractions so that if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then the sum should be the same.
Hence, we want $\frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} + \frac{c'}{d'}$ regardless of the particular class representatives chosen.
For example, since $\frac{1}{2} = \frac{2}{4}$ and $\frac{3}{5} = \frac{6}{10}$, we want $\frac{1}{2} + \frac{3}{5} = \frac{2}{4} + \frac{6}{10}$.
Therefore, any definition of addition in $\mathbb{Q}$ must be defined to be independent of a particular representative of the equivalence class.
Therefore, we want addition to be well defined.

Definition 5. addition over $\mathbb{Q}$
Let $+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ be a relation defined by $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ for all $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$.

Proposition 6. *Addition is a binary operation on $\mathbb{Q}$.*

Therefore, addition over $\mathbb{Q}$ is well defined.
$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ means $(a,b) + (c,d) = (ad + bc, bd)$.

Theorem 7. *algebraic properties of addition over $\mathbb{Q}$*
1. $\frac{m}{n} + (\frac{p}{q} + \frac{r}{s}) = (\frac{m}{n} + \frac{p}{q}) + \frac{r}{s}$ *for all* $\frac{m}{n}, \frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$. *(associative)*
2. $\frac{m}{n} + \frac{p}{q} = \frac{p}{q} + \frac{m}{n}$ *for all* $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$. *(commutative)*
3. $\frac{m}{n} + 0 = 0 + \frac{m}{n} = \frac{m}{n}$ *for all* $\frac{m}{n} \in \mathbb{Q}$. *(additive identity)*
4. $\frac{m}{n} + \frac{-m}{n} = \frac{-m}{n} + \frac{m}{n} = 0$ *for all* $\frac{m}{n} \in \mathbb{Q}$. *(additive inverses)*

We want to define multiplication of fractions so that if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then the product should be the same.

Hence, we want $\frac{a}{b} \cdot \frac{c}{d} = \frac{a'}{b'} \cdot \frac{c'}{d'}$ regardless of the particular class representatives chosen.

For example, since $\frac{1}{2} = \frac{2}{4}$ and $\frac{3}{5} = \frac{6}{10}$, we want $\frac{1}{2} \cdot \frac{3}{5} = \frac{2}{4} \cdot \frac{6}{10}$.

Therefore, any definition of multiplication in $\mathbb{Q}$ must be defined to be independent of a particular representative of the equivalence class.

Therefore, we want multiplication to be well defined.

Definition 8. multiplication over $\mathbb{Q}$

Let $\cdot : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ be a relation defined by $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ for all $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$.

Proposition 9. *Multiplication is a binary operation on $\mathbb{Q}$.*

Therefore, multiplication over $\mathbb{Q}$ is well defined.

$\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$ means $(a,b)(c,d) = (ac,bd)$.

Theorem 10. ***algebraic properties of multiplication over $\mathbb{Q}$***

1. $\frac{m}{n} \cdot (\frac{p}{q} \cdot \frac{r}{s}) = (\frac{m}{n} \cdot \frac{p}{q}) \cdot \frac{r}{s}$ *for all* $\frac{m}{n}, \frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$. *(associative)*
2. $\frac{m}{n} \cdot \frac{p}{q} = \frac{p}{q} \cdot \frac{m}{n}$ *for all* $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$. *(commutative)*
3. $\frac{m}{n} \cdot 1 = 1 \cdot \frac{m}{n} = \frac{m}{n}$ *for all* $\frac{m}{n} \in \mathbb{Q}$. *(multiplicative identity)*
4. $\frac{m}{n} \cdot 0 = 0 \cdot \frac{m}{n} = 0$ *for all* $\frac{m}{n} \in \mathbb{Q}$.
5. $\frac{m}{n} \cdot (\frac{p}{q} + \frac{r}{s}) = \frac{m}{n} \cdot \frac{p}{q} + \frac{m}{n} \cdot \frac{r}{s}$ *for all* $\frac{m}{n}, \frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$. *(left distributive)*
6. $(\frac{m}{n} + \frac{p}{q}) \cdot \frac{r}{s} = \frac{m}{n} \cdot \frac{r}{s} + \frac{p}{q} \cdot \frac{r}{s}$ *for all* $\frac{m}{n}, \frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$. *(right distributive)*

Proposition 11. ***$\mathbb{Q}$ extends $\mathbb{Z}$.***

Let $\mathbb{Q} = \{\frac{m}{n} : n \neq 0\}$ *where* $\frac{m}{n}$ *is the class of ordered pairs* (p,q) *in* $\mathbb{Z} \times \mathbb{Z}$ *such that* $(p,q) \sim (m,n)$ *iff* $pn = qm$ *and* $q,n \neq 0$.

$\mathbb{Q}$ *extends* $\mathbb{Z}$.

Let $S = \{\frac{n}{1} : n \in \mathbb{Z}\}$.

Then $S \subset \mathbb{Q}$ and S is a subring of $\mathbb{Q}$ and $\mathbb{Z}$ is isomorphic to S.

Hence, $\mathbb{Q}$ contains a copy of $\mathbb{Z}$, so $\mathbb{Z}$ is embedded in $\mathbb{Q}$.

Therefore, $\mathbb{Q}$ extends $\mathbb{Z}$.

Ordered Fields

Definition 12. ordered field

An **ordered field** (F,P) is a field $(F,+,\cdot)$ and nonempty subset P of F such that the following axioms hold:

OF1. P is closed under addition defined over F.

$(\forall a,b \in P)(a+b \in P)$.

OF2. P is closed under multiplication defined over F.

$(\forall a,b \in P)(ab \in P)$.

OF3. Trichotomy.

For every $a \in F$ exactly one of the following statements is true:

i. $a \in P$.

ii. $a = 0$.
iii. $-a \in P$.

The subset P is the **positive** part of F.
An element a of F is **positive** iff $a \in P$.
An element a of F is **negative** iff $-a \in P$.
OF1 implies the sum of two positive elements is positive.
OF2 implies the product of two positive elements is positive.
OF3 implies $0 \notin P$.

Example 13. $(\mathbb{C}, +, \cdot)$ is not an ordered field.

Proof. Suppose $(\mathbb{C}, +, \cdot)$ is an ordered field.
Then there is a subset P of positive elements of $\mathbb{C}$ and $1 \in P$.
Since $i \in \mathbb{C}$ and $i \neq 0$, then $i^2 \in P$.
Since $i^2 = -1$, then $-1 \in P$.
Hence, we have $1 \in P$ and $-1 \in P$, a violation of trichotomy.
Therefore, $(\mathbb{C}, +, \cdot)$ is not an ordered field. ☐

Example 14. $(\mathbb{Z}_5, +, \cdot)$ is not an ordered field.

Proof. Suppose $(\mathbb{Z}_5, +, \cdot)$ is an ordered field.
Then the subset P of positive elements of $\mathbb{Z}_5$ is closed under addition.
Since $[1] \in P$, then $5 * [1] = [1] + [1] + [1] + [1] + [1] \in P$.
Since $5 \cdot [1] = [0]$, then $[0] \in P$.
But $[0] \notin P$ in an ordered field.
Therefore, $(\mathbb{Z}_5, +, \cdot)$ is not an ordered field. ☐

Definition 15. positive and negative rational number
A number $q \in \mathbb{Q}$ is said to be a **positive rational number** iff there exist $a, b \in \mathbb{Z}^+$ such that $q = \frac{a}{b}$.
A number $q \in \mathbb{Q}$ is said to be a **negative rational number** iff $-q$ is positive.

Proposition 16. *Positivity of $\mathbb{Q}$ is well defined.*

Let $\frac{m}{n}, \frac{m'}{n'} \in \mathbb{Q}$.
Then $m, n \in \mathbb{Z}$ and $n \neq 0$ and $m', n' \in \mathbb{Z}$ and $n' \neq 0$.
Therefore, if $(m, n) \sim (m', n')$, then $\frac{m}{n}$ is positive iff $\frac{m'}{n'}$ is positive.

Proposition 17. $(\mathbb{Q}, +, \cdot)$ *is an ordered field.*

Let $\mathbb{Q}^+$ be the positive subset of the ordered field $(\mathbb{Q}, +, \cdot)$.
Then $\mathbb{Q}^+ = \{\frac{a}{b} \in \mathbb{Q} : a, b \in \mathbb{Z}^+\}$, so $\mathbb{Q}^+ \subset \mathbb{Q}$.
The set $\mathbb{Q}^+$ is the set of all positive rational numbers.

Proposition 18. *Let F be an ordered field with positive subset P. Then*
1. $1 \in P$.
2. *if $x \in P$, then $x^{-1} \in P$.*
3. *if $x, y \in P$, then $\frac{x}{y} \in P$.*
4. *if $x \in F$ and $x \neq 0$, then $x^2 \in P$.*
5. *if $x \in P$, then $nx \in P$ for all $n \in \mathbb{N}$.*

Definition 19. relation $<$ over an ordered field

Let F be an ordered field with positive subset P.

Define a relation "is less than", denoted $<$, on F by $a < b$ iff $b - a \in P$ for all $a, b \in F$.

Define a relation "is greater than", denoted $>$, on F by $a > b$ iff $b < a$ for all $a, b \in F$.

We denote the ordered field F with relation $<$ defined over F by $(F, +, \cdot, <)$.

Example 20. Let $\mathbb{Q}^+$ be the set of all positive rational numbers.

Define the relation $<$ on $\mathbb{Q}$ by $a < b$ iff $b - a \in \mathbb{Q}^+$ for all $a, b, \in \mathbb{Q}$.

Define the relation $>$ on $\mathbb{Q}$ by $a > b$ iff $b < a$ for all $a, b \in \mathbb{Q}$.

Then $(\mathbb{Q}, +, \cdot, <)$ denotes the ordered field $(\mathbb{Q}, +, \cdot)$ with the relation $<$ defined over $\mathbb{Q}$.

Proposition 21. *Let F be an ordered field with positive subset P. Then for all $a, b \in F$*

1. *$a > 0$ iff $a \in P$.*
2. *$a < 0$ iff $-a \in P$.*
3. *$a < b$ iff $b - a > 0$.*

Let F be an ordered field with positive subset P.

Since $1 \in P$ and $1 > 0$ iff $1 \in P$, then $1 > 0$.

Therefore, $1 > 0$ in any ordered field.

Let $x \in F$.

Since $x \in P$ iff $x > 0$, then x is positive iff $x > 0$.

Since $-x \in P$ iff $x < 0$, then x is negative iff $x < 0$.

Let $F^+ = \{x \in F : x \text{ is positive }\} = \{x \in F : x > 0\}$.

Let $F^- = \{x \in F : x \text{ is negative }\} = \{x \in F : x < 0\}$.

Let $F^* = \{x \in F : x \neq 0\} = F^+ \cup F^-$.

Thus, if $x \in F^*$ then either x is positive or x is negative.

The set $\{F^+, F^-, \{0\}\}$ is a partition of F.

Therefore, $F = F^* \cup \{0\} = F^+ \cup \{0\} \cup F^-$.

Let $x \in F$.

Then either x is positive or x is zero or x is negative.

Therefore an element of an ordered field is either positive or zero or negative.

Let $a, b \in F$.

Then $a < b$ iff $b - a \in F^+$ iff $b - a > 0$ iff $b - a$ is positive.

Let $x \in F$.
Since $x > 0$ iff $x \in F^+$ iff $-(-x) \in F^+$ iff $-x < 0$, then $x > 0$ iff $-x < 0$.
Therefore, $x > 0$ iff $-x < 0$.
Hence, x is positive iff $-x$ is negative.
Since $1 > 0$ and $1 > 0$ iff $-1 < 0$, then $-1 < 0$.
Therefore, $-1 < 0$ in any ordered field.

Let F be an ordered field.
Let F^+ be the set of all positive elements of F.
Then $F^+ = \{x \in F : x > 0\}$.
Since $1 \in F$ and $1 > 0$, then $1 \in F^+$.
Let $x \in F$.
If $x \in F^+$, then $x^{-1} \in F^+$.
Thus, if $x > 0$, then $x^{-1} = \frac{1}{x} > 0$.
Therefore, in an ordered field, if x is positive, then its reciprocal $\frac{1}{x}$ is positive.

If $x, y \in F^+$, then $\frac{x}{y} \in F^+$.
Thus, if $x > 0$ and $y > 0$, then $\frac{x}{y} > 0$.
Therefore, in an ordered field, if x is positive and y is positive, then the ratio $\frac{x}{y}$ is positive.

If $x \in F$ and $x \neq 0$, then $x^2 \in F^+$.
Thus, if $x \neq 0$, then $x^2 > 0$.
Therefore, in an ordered field, if x is nonzero, then its square x^2 is positive.

If $x \in F^+$, then $nx \in F^+$ for all $n \in \mathbb{N}$.
Thus, if $x > 0$, then $nx > 0$ for all $n \in \mathbb{N}$.
Therefore, in an ordered field, if x is positive, then every positive integer multiple of x is positive.
Moreover, if $x > 0$, then $0 < x < 2x < 3x < 4x < 5x < ....$

Lemma 22. *Let $(F, +, \cdot, <)$ be an ordered field with $a, b \in F$.*
If $a > 0$ and $b < 0$, then $ab < 0$.

Proposition 23. ***positivity of a product in an ordered field***
Let $(F, +, \cdot, <)$ be an ordered field with $a, b \in F$. Then
1. $ab > 0$ iff either $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.
2. $ab < 0$ iff either $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.

Let F be an ordered field with $a, b \in F$. Then
1. ab is positive iff a and b are either both positive or both negative.
2. ab is negative iff either a is positive and b is negative or a is negative and b is positive.
$(+)(+) = +$
$(+)(-) = -$
$(-)(+) = -$

$(-)(-) = +$
Therefore, ab is positive iff a and b have the same sign and ab is negative iff a and b have opposite signs.

Corollary 24. *Let $(F, +, \cdot, <)$ be an ordered field.*
Let $a, b \in F$.
Then $\frac{a}{b} > 0$ iff $ab > 0$.

Arithmetic Properties
In $\mathbb{Z}$, $m < n$ iff $n - m$ is positive. ie, in the set $\{1, 2, 3, 4, 5, ...\}$.
In $\mathbb{Z} \times \mathbb{Z}$, $(a, b) < (c, d)$ iff $a < c$ or ($a = c \wedge b < d$).
This is dictionary order.

Theorem 25. ***ordered fields satisfy transitivity and trichotomy laws***
Let $(F, +, \cdot, <)$ be an ordered field. Then
1. $a < a$ is false for all $a \in F$. (Therefore, $<$ is not reflexive.)
2. For all $a, b, c \in F$, if $a < b$ and $b < c$, then $a < c$. ($<$ is transitive)
3. For every $a \in F$, exactly one of the following is true (trichotomy):
i. $a > 0$
ii. $a = 0$
iii. $a < 0$
4. For every $a, b \in F$, exactly one of the following is true (trichotomy):
i. $a > b$
ii. $a = b$
iii. $a < b$

Corollary 26. *Let $(F, +, \cdot, <)$ be an ordered field.*
Let $a, b \in F$.
If $0 < a < b$, then $0 < \frac{1}{b} < \frac{1}{a}$.

Theorem 27. ***order is preserved by the field operations in an ordered field***
Let $(F, +, \cdot, <)$ be an ordered field.
Let $a, b, c, d \in F$.
1. If $a < b$, then $a + c < b + c$. (preserves order for addition)
2. If $a < b$, then $a - c < b - c$. (preserves order for subtraction)
3. If $a < b$ and $c > 0$, then $ac < bc$. (preserves order for multiplication by a positive element)
4. If $a < b$ and $c < 0$, then $ac > bc$. (reverses order for multiplication by a negative element)
5. If $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$. (preserves order for division by a positive element)

Proposition 28. *Let $(F, +, \cdot, <)$ be an ordered field.*
Let $a, b, c, d \in F$.
1. If $a < b$ and $c < d$, then $a + c < b + d$. (adding inequalities is valid)
2. If $0 < a < b$ and $0 < c < d$, then $0 < ac < bd$.

Proposition 29. *Let $(F, +, \cdot, <)$ be an ordered field.*
Let $\frac{a}{b}, \frac{c}{d} \in F$ with $b, d > 0$.
Then $\frac{a}{b} < \frac{c}{d}$ iff $ad < bc$.

Definition 30. Let $(F, +, \cdot, <)$ be an ordered field.
Let $a, b, c \in F$.
We say that b is **between a and c** iff $a < b$ and $b < c$ and we write $a < b < c$.

Theorem 31. ***density of ordered fields***
Between any two distinct elements of an ordered field is a third element.

Let F be an ordered field with $a, b \in F$ such that $a < b$.
Then there exists $c \in F$ such that $a < c < b$.

Let F be an ordered field with $a, b \in F$.
If $a < b$, then $a < \frac{a+b}{2} < b$.

Example 32. density of $\mathbb{Q}$
Since $(\mathbb{Q}, +, \cdot, <)$ is an ordered field, then between any two distinct rational numbers is another rational number.
Therefore, if $a, b \in \mathbb{Q}$ and $a < b$, then there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Corollary 33. ***ordered fields are infinite***
An ordered field contains an infinite number of elements.

Example 34. $\mathbb{Q}$ is infinite
Since $(\mathbb{Q}, +, \cdot, <)$ is an ordered field, then $\mathbb{Q}$ contains an infinite number of elements.
Therefore, there are infinitely many rational numbers.

Definition 35. relation $\leq$ over an ordered field
Let $(F, +, \cdot, <)$ be an ordered field.
Define a relation "is less than or equal to", denoted $\leq$, on F by $a \leq b$ iff either $a < b$ or $a = b$ for all $a, b \in F$.
Define a relation "is greater than or equal to", denoted $\geq$, on F by $a \geq b$ iff $b \leq a$ for all $a, b \in F$.
We denote the ordered field $(F, +, \cdot, <)$ with relation $\leq$ defined over F by $(F, +, \cdot, \leq)$.

Example 36. Let $(\mathbb{Q}, +, \cdot, <)$ be the ordered field of rational numbers.
Define the relation $\leq$ on $\mathbb{Q}$ by $a \leq b$ iff either $a < b$ or $a = b$ for all $a, b \in \mathbb{Q}$.
Define the relation $\geq$ on $\mathbb{Q}$ by $a \geq b$ iff $b \leq a$ for all $a, b \in \mathbb{Q}$.
Then $(\mathbb{Q}, +, \cdot, \leq)$ denotes the ordered field $(\mathbb{Q}, +, \cdot, <)$ with the relation $\leq$ defined over $\mathbb{Q}$.

Theorem 37. ***ordered fields are totally ordered***
Let $(F, +, \cdot, \leq)$ be an ordered field. Then
1. $\leq$ is a partial order over F. Therefore, $(F, \leq)$ is a poset.
2. $\leq$ is a total order over F.

Therefore, for any elements a, b, c of an ordered field F
1. Reflexive: $a \leq a$.
2. Antisymmetric: if $a \leq b$ and $b \leq a$, then $a = b$.
3. Transitive: if $a \leq b$ and $b \leq c$, then $a \leq c$.
4. Comparable: for every $a, b \in F$, either $a \leq b$ or $b \leq a$.

Any total order is a linear chain, so an ordered field is a linear chain.

Proposition 38. *Let $(F, +, \cdot, \leq)$ be an ordered field. Then*
1. $x^2 = 0$ iff $x = 0$.
2. $x^2 > 0$ iff $x \neq 0$.
3. $x^2 \geq 0$ for all $x \in F$.

Since $1 \neq 0$ in every ordered field, then $1^2 > 0$.
Therefore, $1 > 0$ in every ordered field.

Absolute value in an ordered field

The absolute value of an element in an ordered field measures size(magnitude).

Definition 39. absolute value in an ordered field
Let F be an ordered field.
Let $x \in F$.
The **absolute value** of x, denoted $|x|$, is defined by the rule

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The absolute value in an ordered field F is a function from F to F.
Observe that $|0| = 0$.
Since $1 > 0$, then $|1| = 1$.

Lemma 40. *Let F be an ordered field. Let $x \in F$.*
1. If $x < 0$, then $\frac{1}{x} < 0$.
2. If $x \neq 0$, then $|\frac{1}{x}| = \frac{1}{|x|}$.

Theorem 41. ***arithmetic operations and absolute value***
Let F be an ordered field. For all $a, b \in F$
1. $|ab| = |a||b|$.
2. if $b \neq 0$, then $|\frac{a}{b}| = \frac{|a|}{|b|}$.
3. $|a|^2 = a^2$.
4. if $a \neq 0$, then $|a^n| = |a|^n$ for all $n \in \mathbb{Z}$.

Theorem 42. ***properties of the absolute value function***
Let $(F, +, \cdot, \leq)$ be an ordered field.
Let $a, k \in F$ and $k > 0$. Then
1. $|a| \geq 0$.
2. $|a| = 0$ iff $a = 0$.

3. $|-a| = |a|$.
4. $-|a| \leq a \leq |a|$.
5. $|a| < k$ *iff* $-k < a < k$.
6. $|a| > k$ *iff* $a > k$ *or* $a < -k$.
7. $|a| = k$ *iff* $a = k$ *or* $a = -k$.

Theorem 43. ***triangle inequality***
Let $(F, +, \cdot, \leq)$ *be an ordered field.*
Let $a, b \in F$*. Then* $|a + b| \leq |a| + |b|$.

This statement indicates that the length of a side of a triangle is less than the sum of the lengths of the other two sides.

Corollary 44. *Let* $(F, +, \cdot, \leq)$ *be an ordered field. Then*
1. $|a - b| \geq |a| - |b|$ *and* $|a - b| \geq |b| - |a|$ *for all* $a, b \in F$.
2. $||a| - |b|| \leq |a - b| \leq |a| + |b|$ *for all* $a, b \in F$.

Let F be an ordered field.
Then $|a - b| \geq |a| - |b|$ and $|a - b| \geq |b| - |a|$ for all $a, b \in F$.
This statement indicates that the length of a side of a triangle is greater than the difference of the lengths of the other two sides.

Corollary 45. ***generalized triangle inequality***
Let $(F, +, \cdot, \leq)$ *be an ordered field.*
Let $n \in \mathbb{N}$.
Let $x_1, x_2, ..., x_n \in F$*. Then*
$|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n|$.

Ordered field properties of $\mathbb{R}$

We assume there exists a complete ordered field and call it $\mathbb{R}$.

Axiom 46. $(\mathbb{R}, +, \cdot, \leq)$ ***is a complete ordered field.***
The set of real numbers $\mathbb{R}$ *with the operations of addition and multiplication and the relation* $\leq$ *defined over* $\mathbb{R}$ *is defined to be a complete ordered field.*

Therefore, $(\mathbb{R}, +, \cdot, \leq)$ is defined to be a complete ordered field.

Since $(\mathbb{R}, +, \cdot, \leq)$ is a field, then the field axioms hold for $\mathbb{R}$.

Field axioms of $(\mathbb{R}, +, \cdot, \leq)$
A1. $x + y \in \mathbb{R}$ for all $x, y \in \mathbb{R}$. (closure under addition)
A2. $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{R}$. (addition is associative)
A3. $x + y = y + x$ for all $x, y \in \mathbb{R}$. (addition is commutative)
A4. $(\exists 0 \in \mathbb{R})(\forall x \in \mathbb{R})(0 + x = x + 0 = x)$. (existence of additive identity)
A5. $(\forall x \in \mathbb{R})(\exists -x \in \mathbb{R})(x + (-x) = -x + x = 0)$. (existence of additive inverses)

M1. $xy \in \mathbb{R}$ for all $x, y \in \mathbb{R}$. (closure under multiplication)
M2. $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{R}$. (multiplication is associative)
M3. $xy = yx$ for all $x, y \in \mathbb{R}$. (multiplication is commutative)
M4. $(\exists 1 \in \mathbb{R})(\forall x \in \mathbb{R})(1 \cdot x = x \cdot 1 = x)$. (existence of multiplicative identity)
M5. $(\forall x \in \mathbb{R}^*)(\exists x^{-1} \in \mathbb{R})(xx^{-1} = x^{-1}x = 1)$. (existence of multiplicative inverses)
D1. $x(y+z) = xy + xz$ for all $x, y, z \in \mathbb{R}$. (multiplication is left distributive over addition)
D2. $(y+z)x = yx + zx$ for all $x, y, z \in \mathbb{R}$. (multiplication is right distributive over addition)
F1. $1 \neq 0$. (multiplicative identity is distinct from additive identity)
The additive identity of $\mathbb{R}$ is 0.
The additive inverse of $x \in \mathbb{R}$ is $-x$.
The multiplicative identity of $\mathbb{R}$ is 1.
The multiplicative inverse of $x \in \mathbb{R}^*$ is $\frac{1}{x} \in \mathbb{R}^*$.

Since $(\mathbb{R}, +, \cdot, \leq)$ is a field, then $\mathbb{R}$ is an integral domain.
Therefore, $xy = 0$ iff $x = 0$ or $y = 0$ for all $x, y \in \mathbb{R}$.
Equivalently, $xy \neq 0$ iff $x \neq 0$ and $y \neq 0$ for all $x, y \in \mathbb{R}$.
Therefore, the product of any two nonzero elements of $\mathbb{R}$ is nonzero.

Since $(\mathbb{R}, +, \cdot, \leq)$ is a field, then $\mathbb{R}$ satisfies the multiplicative cancellation laws.
Therefore, if $xz = yz$ and $z \neq 0$, then $x = y$ for all $x, y, z \in \mathbb{R}$.

Since $(\mathbb{R}, +, \cdot), \leq)$ is an ordered field, then there exists a nonempty subset $\mathbb{R}^+$ of $\mathbb{R}$ such that
OF1. $\mathbb{R}^+$ is closed under addition. $(\forall a, b \in \mathbb{R}^+)(a + b \in \mathbb{R}^+)$.
OF2. $\mathbb{R}^+$ is closed under multiplication. $(\forall a, b \in \mathbb{R}^+)(ab \in \mathbb{R}^+)$.
OF3. For every $r \in \mathbb{R}^+$ exactly one of the following is true:
i. $r \in \mathbb{R}^+$
ii. $r = 0$
iii. $-r \in \mathbb{R}^+$.

Definition 47. Let $\mathbb{R}^+$ be the set of all positive real numbers.
Define the relation $<$ on $\mathbb{R}$ by $a < b$ iff $b - a \in \mathbb{R}^+$ for all $a, b, \in \mathbb{R}$.
Define the relation $>$ on $\mathbb{R}$ by $a > b$ iff $b < a$ for all $a, b \in \mathbb{R}$.
Then $(\mathbb{R}, +, \cdot, <)$ denotes the ordered field $(\mathbb{R}, +, \cdot)$ with the relation $<$ defined over $\mathbb{R}$.

Definition 48. Let $(\mathbb{R}, +, \cdot, <)$ be the ordered field of real numbers.
Define the relation $\leq$ on $\mathbb{R}$ by $a \leq b$ iff either $a < b$ or $a = b$ for all $a, b \in \mathbb{R}$.
Define the relation $\geq$ on $\mathbb{R}$ by $a \geq b$ iff $b \leq a$ for all $a, b \in \mathbb{R}$.
Then $(\mathbb{R}, +, \cdot, \leq)$ denotes the ordered field $(\mathbb{R}, +, \cdot, <)$ with the relation $\leq$ defined over $\mathbb{R}$.

Definition 49. sign of a real number

Let $x \in \mathbb{R}$.
x is **nonzero** iff $x \neq 0$.
x is **positive** iff $x > 0$.
x is **negative** iff $x < 0$.
x is **non-negative** iff $x \geq 0$.
x is **non-positive** iff $x \leq 0$.

$\mathbb{R}^+ = \{x \in \mathbb{R} : x \text{ is positive } \} = \{x \in \mathbb{R} : x > 0\} = (0, \infty)$.
$\mathbb{R}^- = \{x \in \mathbb{R} : x \text{ is negative } \} = \{x \in \mathbb{R} : x < 0\} = (-\infty, 0)$.
$\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\} = \mathbb{R}^+ \cup \mathbb{R}^- = (0, \infty) \cup (-\infty, 0)$.
Thus, if $x \in \mathbb{R}^*$ then either x is positive or x is negative.
$\{\mathbb{R}^+, \mathbb{R}^-, \{0\}\}$ is a partition of $\mathbb{R}$.
$\{\mathbb{R}^+, \mathbb{R}^-\}$ is a partition of $\mathbb{R}^*$.
Therefore, $\mathbb{R} = \mathbb{R}^* \cup \{0\} = \mathbb{R}^+ \cup \{0\} \cup \mathbb{R}^-$.
Hence, an element $x \in \mathbb{R}$ is either positive or zero or negative.
Therefore, every real number is either positive, zero, or negative.

Since $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field, the following are true:
1. If $x, y \in \mathbb{R}^+$, then $x + y \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under $+$)
2. If $x, y \in \mathbb{R}^+$, then $xy \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under $\cdot$)
3. For every $x, y \in \mathbb{R}$, exactly one of the following is true (trichotomy):
$x > y$, $x = y$, $x < y$.
4. If $x < y$ and $y < z$, then $x < z$. ($<$ is transitive)
5. If $x < y$, then $x + z < y + z$. (preserves order for addition)
6. If $x < y$ and $z > 0$, then $xz < yz$. (preserves order for multiplication by a positive element)

Example 50. density of $\mathbb{R}$

Since $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field, then between any two distinct real numbers is another real number.

Therefore, if $a, b \in \mathbb{R}$ and $a < b$, then there exists $r \in \mathbb{R}$ such that $a < r < b$.

Example 51. $\mathbb{R}$ is infinite

Since $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field, then $\mathbb{R}$ contains an infinite number of elements.

Therefore, there are infinitely many real numbers.

Since $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field, then $\leq$ is a total order on $\mathbb{R}$.
Therefore, $(\mathbb{R}, \leq)$ is a total order, so $(\mathbb{R}, \leq)$ is a poset.
Since $\leq$ is a total order over $\mathbb{R}$, then the following are true:
1. $(\forall x \in \mathbb{R})(x \leq x)$ (reflexive)
2. $(\forall x, y \in \mathbb{R})([x \leq y \wedge y \leq x) \rightarrow (x = y)]$ (anti-symmetric)
3. $(\forall x, y, z \in \mathbb{R})(x \leq y \wedge y \leq z \rightarrow x \leq z)$ (transitive)
4. $(\forall x, y \in \mathbb{R})(x \leq y \vee y \leq x)$. (comparable)

Since $\leq$ is a total order on $\mathbb{R}$, then $(\mathbb{R}, \leq)$ is a linearly ordered set.

Since $\mathbb{R}$ is complete, then the Hasse diagram is a straight line with no holes, i.e., the real number line.

Therefore, $(\mathbb{R}, +, \cdot, \leq)$ is a complete linearly ordered field.

Boundedness of sets in an ordered field

Definition 52. upper bound of a subset of an ordered field

Let F be an ordered field.

Let $S \subset F$.

An element $b \in F$ is an **upper bound of** S **in** F iff $(\forall x \in S)(x \leq b)$.

The set S is **bounded above in** F iff S has an upper bound in F.

Therefore, S is bounded above in F iff $(\exists b \in F)(\forall x \in S)(x \leq b)$.

The statement 'S has an upper bound in F' means: $(\exists b \in F)(\forall x \in S)(x \leq b)$.

Observe that

$$\begin{aligned} \neg(\exists b \in F)(\forall x \in S)(x \leq b) &\Leftrightarrow (\forall b \in F)(\exists x \in S)(x \not\leq b) \\ &\Leftrightarrow (\forall b \in F)(\exists x \in S)(x > b). \end{aligned}$$

Therefore, the statement 'S has no upper bound in F' means: $(\forall b \in F)(\exists x \in S)(x > b)$.

Therefore S has no upper bound in F iff for each $b \in F$ there is some $x \in S$ such that $x > b$.

An element $b \in F$ is not an upper bound for S iff there exists $x \in S$ such that $x > b$.

Definition 53. lower bound of a subset of an ordered field

Let F be an ordered field.

Let $S \subset F$.

An element $b \in F$ is a **lower bound of** S **in** F iff $(\forall x \in S)(b \leq x)$.

The set S is **bounded below in** F iff S has a lower bound in F.

Therefore, S is bounded below in F iff $(\exists b \in F)(\forall x \in S)(b \leq x)$.

The statement 'S has a lower bound in F' means: $(\exists b \in F)(\forall x \in S)(b \leq x)$.

Observe that

$$\begin{aligned} \neg(\exists b \in F)(\forall x \in S)(b \leq x) &\Leftrightarrow (\forall b \in F)(\exists x \in S)(b \not\leq x) \\ &\Leftrightarrow (\forall b \in F)(\exists x \in S)(b > x). \end{aligned}$$

Therefore, the statement 'S has no lower bound in F' means: $(\forall b \in F)(\exists x \in S)(x < b)$.

Therefore S has no lower bound in F iff for each $b \in F$ there is some $x \in S$ such that $x < b$.

An element $b \in F$ is not a lower bound for S iff there exists $x \in S$ such that $x < b$.

Definition 54. bounded subset of an ordered field

Let F be an ordered field.

Let $S \subset F$.

The set S is **bounded in** F iff there exists $b \in F$ such that $|x| \leq b$ for all $x \in S$.

The set S is **unbounded in** F iff S is not bounded in F.

In symbols, S is bounded in F iff $(\exists b \in F)(\forall x \in S)(|x| \leq b)$.

Therefore, S is unbounded in F iff $(\forall b \in F)(\exists x \in S)(|x| > b)$.

Let S be a subset of an ordered field F.

Suppose S is bounded in F.

Then there exists $B \in F$ such that $|x| \leq B$ for all $x \in S$.

Let $x \in S$.

Then $|x| \leq B$.

Since $|x| \geq 0$ and $B + 1 > B$, then $0 \leq |x| \leq B < B + 1$.

Hence, $|x| < B + 1$ and $0 < B + 1$.

Therefore, there exists $B + 1 > 0$ such that $|x| < B + 1$ for all $x \in S$.

Let $b = B + 1$.

Then there exists $b > 0$ such that $|x| < b$ for all $x \in S$.

Hence, if a set S is bounded in an ordered field F, then there exists $b > 0$ such that $|x| < b$ for all $x \in S$.

Therefore, if a set S is bounded in an ordered field F, then there exists $b > 0$ such that $-b < x < b$ for all $x \in S$.

Theorem 55. *A subset S of an ordered field F is bounded in F iff S is bounded above and below in F.*

Let $S \subset \mathbb{R}$.

Then S is bounded in $\mathbb{R}$ iff S is bounded above and below in $\mathbb{R}$.

Therefore, S is bounded in $\mathbb{R}$ iff S has an upper and lower bound in $\mathbb{R}$.

Observe that

$$\begin{aligned}
(\exists m \in \mathbb{R})(\forall x \in S)(m \leq x) \wedge (\exists M \in \mathbb{R})(\forall x \in S)(x \leq M) &\Rightarrow \\
(\exists m \in \mathbb{R})(\exists M \in \mathbb{R})[(\forall x \in S)(m \leq x) \wedge (\forall x \in S)(x \leq M)] &\Rightarrow \\
(\exists m \in \mathbb{R})(\exists M \in \mathbb{R})[(\forall x \in S)(m \leq x \wedge x \leq M)] &\Rightarrow \\
(\exists m \in \mathbb{R})(\exists M \in \mathbb{R})[(\forall x \in S)(m \leq x \leq M)].
\end{aligned}$$

Therefore, S is bounded in $\mathbb{R}$ iff there exist $m, M \in \mathbb{R}$ such that $m \leq x \leq M$ for all $x \in S$.

Since S is bounded in $\mathbb{R}$ iff S is bounded above and below in $\mathbb{R}$, then

S is not bounded in $\mathbb{R}$ iff S is not bounded above in $\mathbb{R}$ or S is not bounded below in $\mathbb{R}$.

Therefore, S is unbounded in $\mathbb{R}$ iff either S has no upper bound in $\mathbb{R}$ or S has no lower bound in $\mathbb{R}$.

Proposition 56. *Every element of an ordered field is an upper and lower bound of $\emptyset$.*

Let F be an ordered field.
Let $x \in F$.
Then x is an upper and lower bound of $\emptyset$.
Therefore, $\emptyset$ is bounded above and below in F.
Hence, $\emptyset$ is bounded in F.

Example 57. Every rational number is an upper and lower bound for the empty set.
Therefore, $\emptyset$ is bounded above and below in $\mathbb{Q}$.
Hence, $\emptyset$ is bounded in $\mathbb{Q}$.

Example 58. Every real number is an upper and lower bound for the empty set.
Therefore, $\emptyset$ is bounded above and below in $\mathbb{R}$.
Hence, $\emptyset$ is bounded in $\mathbb{R}$.

Proposition 59. ***A subset of a bounded set is bounded.***
Let A be a bounded subset of an ordered field F.
If $B \subset A$, then B is bounded in F.

Proposition 60. ***A union of bounded sets is bounded.***
Let A and B be subsets of an ordered field F.
If A and B are bounded, then $A \cup B$ is bounded.

Definition 61. least upper bound of a subset of an ordered field
Let F be an ordered field and $S \subset F$.
Then $\beta \in F$ is a **least upper bound for** S **in** F iff β is the least element of the set of all upper bounds of S in F.

Therefore $\beta \in F$ is a **least upper bound of** S iff
1. β is an upper bound for S and
2. $\beta \leq M$ for every upper bound M of S.
$\beta \leq M$ for every upper bound M of S iff
no element of F less than β is an upper bound of S iff
every element of F less than β is not an upper bound of S iff
if $\gamma < \beta$, then γ is not an upper bound of S which means
if $\gamma < \beta$, then there exists $x \in S$ such that $x > \gamma$ which means
for every $\gamma < \beta$, there exists $x \in S$ such that $x > \gamma$ which means
for every $\beta - \gamma > 0$, there exists $x \in S$ such that $x > \beta - (\beta - \gamma)$ which means
for every $\epsilon > 0$, there exists $x \in S$ such that $x > \beta - \epsilon$.
Therefore, $\beta = lub(S)$ iff
1. $(\forall x \in S)(x \leq \beta)$. ($\beta$ is an upper bound of S)
2. $(\forall \epsilon > 0)(\exists x \in S)(x > \beta - \epsilon)$. ($\beta - \epsilon$ is not an upper bound of S).

Theorem 62. ***uniqueness of least upper bound in an ordered field***
A least upper bound of a subset of an ordered field, if it exists, is unique.

Let S be a subset of an ordered field F.

The **least upper bound (lub) of** S is called the **supremum** and is denoted $\sup S$.

Therefore,

1. $(\forall x \in S)(x \leq \sup S)$. ($\sup S$ is an upper bound of S)
2. $(\forall \epsilon > 0)(\exists x \in S)(x > \sup S - \epsilon)$. ($\sup S - \epsilon$ is not an upper bound of S).

Example 63. $\sup(0, 1) = 1$.

Definition 64. greatest lower bound of a subset of an ordered field
Let F be an ordered field and $S \subset F$.

Then $\beta \in F$ is a **greatest lower bound for** S **in** F iff β is the greatest element of the set of all lower bounds of S in F.

Therefore $\beta \in F$ is a **greatest lower bound of** S iff

1. β is a lower bound for S and
2. $M \leq \beta$ for every lower bound M of S.

$M \leq \beta$ for every lower bound M of S iff
no element of F greater than β is a lower bound of S iff
every element of F greater than β is not a lower bound of S iff
if $\gamma > \beta$, then γ is not a lower bound of S which means
if $\gamma > \beta$, then there exists $x \in S$ such that $x < \gamma$ which means
for every $\gamma > \beta$, there exists $x \in S$ such that $x < \gamma$ which means
for every $\gamma - \beta > 0$, there exists $x \in S$ such that $x < \beta + (\gamma - \beta)$ which means
for every $\epsilon > 0$, there exists $x \in S$ such that $x < \beta + \epsilon$.

Therefore, $\beta = glb(S)$ iff

1. $(\forall x \in S)(\beta \leq x)$. ($\beta$ is a lower bound of S)
2. $(\forall \epsilon > 0)(\exists x \in S)(x < \beta + \epsilon)$. ($\beta + \epsilon$ is not a lower bound of S).

Theorem 65. ***uniqueness of greatest lower bound in an ordered field***
A greatest lower bound of a subset of an ordered field, if it exists, is unique.

Let S be a subset of an ordered field F.

The **greatest lower bound (glb) of** S is called the **infimum** and is denoted $\inf S$.

Therefore,

1. $(\forall x \in S)(\inf S \leq x)$. ($\inf S$ is a lower bound of S)
2. $(\forall \epsilon > 0)(\exists x \in S)(x < \inf S + \epsilon)$. ($\inf S + \epsilon$ is not a lower bound of S)

Example 66. $\inf(0, 1) = 0$.

Proposition 67. *1. There is no least upper bound of $\emptyset$ in an ordered field.*
2. There is no greatest lower bound of $\emptyset$ in an ordered field.

Let F be an ordered field.

Then $\sup \emptyset$ does not exist in F and $\inf \emptyset$ does not exist in F.

Therefore, $\sup \emptyset$ does not exist in $\mathbb{Q}$ and $\inf \emptyset$ does not exist in $\mathbb{Q}$ and $\sup \emptyset$ does not exist in $\mathbb{R}$ and $\inf \emptyset$ does not exist in $\mathbb{R}$.

Let $S \subset F$.

If $S = \emptyset$, then $\sup S$ does not exist, so if $\sup S$ exists, then $S \neq \emptyset$.

If $S = \emptyset$, then $\inf S$ does not exist, so if $\inf S$ exists, then $S \neq \emptyset$.

Theorem 68. ***approximation property of suprema and infima***

Let S be a subset of an ordered field F.

1. If $\sup S$ exists, then $(\forall \epsilon > 0)(\exists x \in S)(\sup S - \epsilon < x \leq \sup S)$.

2. If $\inf S$ exists, then $(\forall \epsilon > 0)(\exists x \in S)(\inf S \leq x < \inf S + \epsilon)$.

If $\sup S$ exists, then there is some element of S arbitrarily close to $\sup S$.

If $\inf S$ exists, then there is some element of S arbitrarily close to $\inf S$.

Proposition 69. *Let S be a subset of an ordered field F.*

If $\sup S$ and $\inf S$ exist, then $\inf S \leq \sup S$.

Proposition 70. *Let S be a subset of an ordered field F.*

Let $-S = \{-s : s \in S\}$.

1. If $\inf S$ exists, then $\sup(-S) = -\inf S$.

2. If $\sup S$ exists, then $\inf(-S) = -\sup S$.

Lemma 71. *Let S be a subset of an ordered field F.*

Let $k \in F$.

Let $K = \{k\}$.

Let $k + S = \{k + s : s \in S\}$.

Let $K + S = \{k + s : k \in K, s \in S\}$. Then

1. $\sup K = k$.

2. $\inf K = k$.

3. $k + S = K + S$.

Proposition 72. ***additive property of suprema and infima***

Let A and B be subsets of an ordered field F.

Let $A + B = \{a + b : a \in A, b \in B\}$.

1. If $\sup A$ and $\sup B$ exist, then $\sup(A + B) = \sup A + \sup B$.

2. If $\inf A$ and $\inf B$ exist, then $\inf(A + B) = \inf A + \inf B$.

Corollary 73. *Let S be a subset of an ordered field F.*

Let $k \in F$.

Let $k + S = \{k + s : s \in S\}$.

1. If $\sup S$ exists, then $\sup(k + S) = k + \sup S$.

2. If $\inf S$ exists, then $\inf(k + S) = k + \inf S$.

Corollary 74. *Let A and B be subsets of an ordered field F.*

Let $A - B = \{a - b : a \in A, b \in B\}$.

If $\sup A$ and $\inf B$ exist, then $\sup(A - B) = \sup A - \inf B$.

Proposition 75. ***comparison property of suprema and infima***

Let A and B be subsets of an ordered field F such that $A \subset B$.

1. If $\sup A$ and $\sup B$ exist, then $\sup A \leq \sup B$.

2. If $\inf A$ and $\inf B$ exist, then $\inf B \leq \inf A$.

Proposition 76. ***scalar multiple property of suprema and infima***
Let S be a subset of an ordered field F.
Let $k \in F$.
Let $kS = \{ks : s \in S\}$.
1. If $k > 0$ and $\sup S$ exists, then $\sup(kS) = k \sup S$.
2. If $k > 0$ and $\inf S$ exists, then $\inf(kS) = k \inf S$.
3. If $k < 0$ and $\inf S$ exists, then $\sup(kS) = k \inf S$.
4. If $k < 0$ and $\sup S$ exists, then $\inf(kS) = k \sup S$.

Proposition 77. ***sufficient conditions for existence of supremum and infimum in an ordered field***
Let S be a subset of an ordered field F.
1. If $\max S$ exists, then $\sup S = \max S$.
2. If $\min S$ exists, then $\inf S = \min S$.

Proposition 78. *Let S be a subset of an ordered field F.*
Let $-S = \{-s : s \in S\}$.
1. If $\min S$ exists, then $\max(-S) = -\min S$.
2. If $\max S$ exists, then $\min(-S) = -\max S$.

Lemma 79. *Let A and B be nonempty subsets of an ordered field F.*
Then $u \in F$ is an upper bound of $A \cup B$ iff u is an upper bound of A and B.

Proposition 80. *Let A and B be subsets of an ordered field F.*
If $\sup A$ and $\sup B$ exist, then $\sup(A \cup B) = \max \{\sup A, \sup B\}$.

Let A and B be subsets of an ordered field F.
If $\max A$ and $\max B$ exist in F, then $\sup A = \max A$ and $\sup B = \max B$.
Thus, $\sup(A \cup B) = \max\{\max A, \max B\}$.

Lemma 81. *Let A and B be subsets of an ordered field F.*
If $\max A$ and $\max B$ exist in F, then $\max(A \cup B) = \max \{\max A, \max B\}$.

Theorem 82. *Every nonempty finite subset of an ordered field has a maximum.*

Let S be a nonempty finite subset of an ordered field F.
Then $\max S$ exists.
Since $S \subset F$ and $\max S$ exists, then $\sup S = \max S$.

Example 83. Every nonempty finite subset of $\mathbb{R}$ has a maximum.

Complete ordered fields

Definition 84. complete ordered field
An ordered field F is **complete** iff every nonempty subset of F that is bounded above in F has a least upper bound in F. Otherwise, F is said to be **incomplete**.

Axiom 85. $\mathbb{R}$ ***is Dedekind complete.***

Every nonempty subset of $\mathbb{R}$ *that is bounded above in* $\mathbb{R}$ *has a least upper bound in* $\mathbb{R}$.

Let S be a nonempty subset of $\mathbb{R}$ that is bounded above in $\mathbb{R}$.
Then S has a least upper bound in $\mathbb{R}$.
Therefore $\sup S$ is the least upper bound of S in $\mathbb{R}$.
Hence $\sup S \in \mathbb{R}$ and
1. $(\forall x \in S)(x \leq \sup S)$.
2. If b is any upper bound of S, then $\sup S \leq b$.

Equivalently,
1. $(\forall x \in S)(x \leq \sup S)$.
2. $(\forall \epsilon > 0)(\exists x \in S)(x > \sup S - \epsilon)$.

Theorem 86. ***greatest lower bound property in a complete ordered field***

Every nonempty subset of a complete ordered field F that is bounded below in F has a greatest lower bound in F.

Example 87. Every nonempty set of real numbers that is bounded below in $\mathbb{R}$ has a greatest lower bound in $\mathbb{R}$.

Let S be a nonempty subset of $\mathbb{R}$ that is bounded below in $\mathbb{R}$.
Then S has a greatest lower bound in $\mathbb{R}$.
Therefore $\inf S$ is the greatest lower bound of S in $\mathbb{R}$.
Hence $\inf S \in \mathbb{R}$ and
1. $(\forall x \in S)(\inf S \leq x)$.
2. If b is any lower bound of S, then $b \leq \inf S$.

Equivalently,
1. $(\forall x \in S)(\inf S \leq x)$.
2. $(\forall \epsilon > 0)(\exists x \in S)(x < \inf S + \epsilon)$.

Proposition 88. *There is no rational number x such that $x^2 = 2$.*

Example 89. $\mathbb{Q}$ **is not a complete ordered field.**

The set $\{q \in \mathbb{Q} : q^2 < 2\}$ is bounded above in $\mathbb{Q}$, but does not have a least upper bound in $\mathbb{Q}$.

Therefore, $\mathbb{Q}$ is not a complete ordered field.
Since $\mathbb{Q}$ is not complete, then the Hasse diagram of $\mathbb{Q}$ is linear with 'holes'.
Thus, $\mathbb{Q}$ is incomplete and the number line for $\mathbb{Q}$ has holes, while $\mathbb{R}$ is complete and the number line for $\mathbb{R}$ does not have any holes.
Rework this section.

Proposition 90. *Let A and B be subsets of* $\mathbb{R}$ *such that* $\sup A$ *and* $\sup B$ *exist in* $\mathbb{R}$.

If $A \cap B \neq \emptyset$, then $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$.

Moreover, if A and B are bounded intervals such that $A \cap B \neq \emptyset$, then $\sup(A \cap B) = \min\{\sup A, \sup B\}$.

Archimedean ordered fields

Definition 91. Archimedean ordered field
An ordered field F is **Archimedean ordered** iff $(\forall a \in F, b > 0)(\exists n \in \mathbb{N})(nb > a)$.

Let F be an Archimedean ordered field.
Then regardless of how small b is and how large a is, a sufficient number of repeated additions of b to itself will exceed a.
Equivalently, an ordered field F is Archimedean ordered iff $(\forall a \in F, b > 0)(\exists n \in \mathbb{N})(n > \frac{a}{b})$.

Theorem 92. ***Archimedean property of*** $\mathbb{Q}$
The field $(\mathbb{Q}, +, \cdot, \leq)$ *is Archimedean ordered.*

Therefore, for all $q \in \mathbb{Q}, \epsilon > 0$ there exists $n \in \mathbb{N}$ such that $n\epsilon > q$.

Theorem 93. ***Archimedean property of*** $\mathbb{R}$
A complete ordered field is necessarily Archimedean ordered.

Since $(\mathbb{R}, +, \cdot, \leq)$ is a complete ordered field, then $(\mathbb{R}, +, \cdot, \leq)$ is Archimedean ordered.
Therefore, for all $x \in \mathbb{R}, \epsilon > 0$ there exists $n \in \mathbb{N}$ such that $n\epsilon > x$.

Theorem 94. $\mathbb{N}$ ***is unbounded in an Archimedean ordered field.***
Let F *be an Archimedean ordered field.*
Then for every $x \in F$*, there exists* $n \in \mathbb{N}$ *such that* $n > x$*.*

Since $(\mathbb{R}, +, \cdot, \leq)$ is Archimedean ordered, then for every real number x, there exists a natural number n such that $n > x$.
In symbols, $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(n > x)$.
Therefore, $\mathbb{N}$ is unbounded in $\mathbb{R}$.

Proposition 95. *Let* F *be an Archimedean ordered field.*
For every positive $\epsilon \in F$*, there exists* $n \in \mathbb{N}$ *such that* $\frac{1}{n} < \epsilon$*.*

Since $\mathbb{R}$ is Archimedean ordered, then for every positive real ϵ, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.
In symbols, $(\forall \epsilon > 0)(\exists n \in \mathbb{N})(\frac{1}{n} < \epsilon)$.

Example 96. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.
Then $\max S = \sup S = 1$ and $\min S$ does not exist and $\inf S = 0$.

Lemma 97. ***Each real number lies between two consecutive integers***
For each real number x *there is a unique integer* n *such that* $n \leq x < n + 1$*.*

In symbols, $(\forall x \in \mathbb{R})(\exists! n \in \mathbb{Z})(n \leq x < n + 1)$.
Let $x \in \mathbb{R}$.
Then there is a unique integer n such that $n \leq x < n + 1$.

Theorem 98. $\mathbb{Q}$ ***is dense in*** $\mathbb{R}$

For every $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Therefore, between any two distinct real numbers is a rational number.
Hence, if $a < b$, then there exists $q \in \mathbb{Q}$ in the open interval (a, b).
Therefore, there is a rational number in every nonempty open interval.

Corollary 99. ***between any two distinct real numbers is a nonzero rational number***

For every $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $q \neq 0$ and $a < q < b$.

Existence of square roots in $\mathbb{R}$

Definition 100. square root of a real number

Let $r \in \mathbb{R}$.
A square root of r is a real number x such that $x^2 = r$.

Proposition 101. *A square root of a negative real number does not exist in $\mathbb{R}$.*

Proposition 102. *Zero is the unique square root of 0.*

Lemma 103. *Let F be an ordered field.*
Let $a, b \in F$.
If $0 < a < b$, then $0 < a^2 < ab < b^2$.

Lemma 104. *Let F be an ordered field.*
Let $a \in F$.
If $|a| < \epsilon$ for all $\epsilon > 0$, then $a = 0$.

Theorem 105. ***existence and uniqueness of positive square roots***

Let $r \in \mathbb{R}$.
A unique positive square root of r exists in $\mathbb{R}$ iff $r > 0$.

Definition 106. nonnegative square root of a real number

Let $x \in \mathbb{R}$ such that $x \geq 0$.
The nonnegative square root of x is denoted $\sqrt{x}$.

Therefore, $\sqrt{x} \geq 0$ and $(\sqrt{x})^2 = x$.

Let $x \in \mathbb{R}$.
Then $\sqrt{x} > 0$ iff $x > 0$.

Proposition 107. *Let $x \in \mathbb{R}$.*
Then $\sqrt{x} \in \mathbb{R}$ iff $x \geq 0$.

Proposition 108. *Let $x \in \mathbb{R}$.*
Then $\sqrt{x} \geq 0$ iff $x \geq 0$.

Let $x \in \mathbb{R}$.
If $x > 0$, then $\sqrt{x} > 0$ and $-\sqrt{x} < 0$ and $(\sqrt{x})^2 = x$ and $(-\sqrt{x})^2 = x$.

Proposition 109. *Let $a, b \in \mathbb{R}$ with $a \geq 0$ and $b \geq 0$.*
Then $\sqrt{a} = \sqrt{b}$ iff $a = b$.

Proposition 110. *Let $a, b \in \mathbb{R}$.*
If $a \geq 0$ and $b \geq 0$, then $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Proposition 111. *Let $x \in \mathbb{R}$. Then*
1. $\sqrt{x} = 0$ iff $x = 0$.
2. $\sqrt{x^2} = |x|$.

Let $x \in \mathbb{R}$.
Since $\sqrt{x} = 0$ iff $x = 0$, then $\sqrt{0} = 0$.
Since $\sqrt{x^2} = |x|$, then either $\sqrt{x^2} = x$ or $\sqrt{x^2} = -x$.

Lemma 112. *Let $x \in \mathbb{R}$.*
If $x > 0$, then $\sqrt{\frac{1}{x}} = \frac{1}{\sqrt{x}}$.

Proposition 113. *Let $a, b \in \mathbb{R}$.*
If $a \geq 0$ and $b > 0$, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

Lemma 114. *Let $a, b \in \mathbb{R}$.*
If $0 < a \leq b$, then $0 < a^2 \leq b^2$.

Proposition 115. *Let $a, b \in \mathbb{R}$.*
Then $0 < a < b$ iff $0 < \sqrt{a} < \sqrt{b}$.

Corollary 116. *Let $x \in \mathbb{R}$.*
1. If $0 < x < 1$, then $0 < x^2 < x < \sqrt{x} < 1$.
2. If $x > 1$, then $1 < \sqrt{x} < x < x^2$.

Definition 117. irrational number
A real number that is not rational is said to be **irrational**.

Let $r \in \mathbb{R}$ such that r is not rational.
Then $r \notin \mathbb{Q}$, so r is irrational.

Example 118. If $S = \{q \in \mathbb{Q} : q^2 < 2\}$, then $\sup S = \sqrt{2}$ in $\mathbb{R}$.
Thus, there exists $x \in \mathbb{R}$ such that $x^2 = 2$ and $x = \sqrt{2}$.
Since there is no rational x such that $x^2 = 2$, then $x \notin \mathbb{Q}$, so $\sqrt{2} \notin \mathbb{Q}$.
Therefore, $\sqrt{2}$ is irrational.

The set $\mathbb{R} - \mathbb{Q}$ is called the set of **irrational numbers**.

Proposition 119. ***the additive inverse of an irrational number is irrational***
Let $a \in \mathbb{R}$.
If a is irrational, then $-a$ is irrational.

Proposition 120. ***the sum of a rational and irrational number is irrational***

Let $a, b \in \mathbb{R}$.

If a is rational and b is irrational, then $a + b$ is irrational.

Proposition 121. ***the reciprocal of an irrational number is irrational***

Let $a \in \mathbb{R}$.

If a is irrational, then $\frac{1}{a}$ is irrational.

Proposition 122. ***the product of a nonzero rational and irrational number is irrational***

Let $a, b \in \mathbb{R}$.

If a is a nonzero rational and b is irrational, then ab is irrational.

Corollary 123. ***the quotient of a nonzero rational and irrational number is irrational***

Let $a, b \in \mathbb{R}$.

If a is a nonzero rational and b is irrational, then $\frac{a}{b}$ is irrational.

Proposition 124. $\mathbb{R} - \mathbb{Q}$ ***is dense in*** $\mathbb{Q}$

For every $a, b \in \mathbb{Q}$ with $a < b$, there exists $r \in \mathbb{R} - \mathbb{Q}$ such that $a < r < b$.

Therefore, between any two distinct rational numbers is an irrational number.

If $a, b \in \mathbb{Q}$ and $a < b$, then $a < a + \frac{b-a}{2}\sqrt{2} < b$.

Proposition 125. $\mathbb{R} - \mathbb{Q}$ ***is dense in*** $\mathbb{R}$

For every $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{R} - \mathbb{Q}$ such that $a < r < b$.

Therefore, between any two distinct real numbers is an irrational number.