Topology of $\mathbb{R}$ Theory

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Topology of $\mathbb{R}$

Theorem 1. ***properties of the distance function***

Let F be an ordered field. For all $x, y, z \in F$

D1. $d(x, y) \geq 0$.

D2. $d(x, y) = 0$ *iff* $x = y$.

D3. $d(x, y) = d(y, x)$.

D4. $d(x, y) \leq d(x, z) + d(z, y)$.

Proof. We prove 1.

Let $x, y \in F$.

To prove $d(x, y) \geq 0$, we must prove $|x - y| \geq 0$.

Since $x, y \in F$, then $x - y \in F$, by closure of F under subtraction.

Since $|a| \geq 0$ for all $a \in F$, then in particular, $|x - y| \geq 0$. □

Proof. We prove 2.

Let $x, y \in F$.

To prove $d(x, y) = 0$ iff $x = y$, we must prove $|x - y| = 0$ iff $x = y$.

Since $x, y \in F$, then $x - y \in F$.

Since $|a| = 0$ iff $a = 0$ for all $a \in F$, then in particular, $|x - y| = 0$ iff $x - y = 0$.

Since $x - y = 0$ iff $x = y$, then this implies $|x - y| = 0$ iff $x = y$, as desired. □

Proof. We prove 3.

Let $x, y \in F$.

To prove $d(x, y) = d(y, x)$, we must prove $|x - y| = |y - x|$.

Since $x, y \in F$, then $x - y \in F$.

Observe that

$$\begin{aligned} |x - y| &= |-(x - y)| \\ &= |-x + y| \\ &= |y - x|. \end{aligned}$$

□

Proof. We prove 4.

Let $x, y, z \in F$.

To prove $d(x, y) \leq d(x, z) + d(z, y)$, we must prove $|x - y| \leq |x - z| + |z - y|$.

Observe that

$$\begin{aligned} |x - y| &= |x + 0 - y| \\ &= |x + (-z + z) - y| \\ &= |(x - z) + (z - y)| \\ &\leq |x - z| + |z - y|. \end{aligned}$$

Therefore, $|x - y| \leq |x - z| + |z - y|$, as desired. □

Proposition 2. *Let $a, b \in \mathbb{R}$.*

If $a \geq b$, then $(a, b) = \emptyset$.

If $a < b$, then $(a, b) \neq \emptyset$.

Proof. Let $a, b \in \mathbb{R}$.

Either $a < b$ or $a = b$ or $a > b$.

We consider these cases separately.

Case 1: Suppose $a \geq b$.

We prove $(a, b) = \emptyset$ by contradiction.

Suppose $(a, b) \neq \emptyset$.

Then there is at least one element in (a, b).

Let $x \in (a, b)$.

Then $x \in \mathbb{R}$ and $a < x < b$, so $a < b$.

Thus, we have $a < b$ and $a \geq b$, a violation of trichotomy.

Therefore, $(a, b) = \emptyset$, as desired.

Case 2: Suppose $a < b$.

We prove $(a, b) \neq \emptyset$.

By the density of $\mathbb{R}$, between any two real numbers is another real number.

Since $a, b \in \mathbb{R}$, then there exists $c \in \mathbb{R}$ such that $a < c < b$.

Therefore, $c \in (a, b)$, so $(a, b) \neq \emptyset$, as desired. □

Proposition 3. *Let $a, b \in \mathbb{R}$.*

Then $(a, b) \subset [a, b]$.

Proof. Either $a < b$ or $a = b$ or $a > b$.

We consider these cases separately.

Case 1: Suppose $a \geq b$.

Then $(a, b) = \emptyset$.

Since the empty set is a subset of every set, then in particular, $\emptyset \subset [a, b]$.

Therefore, $(a, b) \subset [a, b]$, as desired.

Case 2: Suppose $a < b$.

Then $(a, b) \neq \emptyset$, so there is at least one element in (a, b).

Let $x \in (a, b)$.

Then $x \in \mathbb{R}$ and $a < x < b$, so $a < x$ and $x < b$.

Since $a < x$, then either $a < x$ or $a = x$, so $a \leq x$.

Since $x < b$, then either $x < b$ or $x = b$, so $x \leq b$.
Thus, $a \leq x$ and $x \leq b$, so $a \leq x \leq b$.
Therefore, $x \in [a, b]$, so $(a, b) \subset [a, b]$, as desired. □

Proposition 4. *Let $a, b \in \mathbb{R}$.*
If $a > b$, then $[a, b] = \emptyset$.
If $a = b$, then $[a, b] = \{a\}$.
If $a < b$, then $[a, b] \neq \emptyset$.

Proof. Either $a < b$ or $a = b$ or $a > b$.
We consider these cases separately.
Case 1: Suppose $a > b$.
We prove $[a, b] = \emptyset$ by contradiction.
Suppose $[a, b] \neq \emptyset$.
Then there is at least one element in $[a, b]$.
Let $x \in [a, b]$.
Then $x \in \mathbb{R}$ and $a \leq x \leq b$, so $a \leq b$.
Thus, we have $a \leq b$ and $a > b$, a violation of trichotomy.
Therefore, $[a, b] = \emptyset$, as desired.
Case 2: Suppose $a = b$.
Since $a = a$, then $a \leq a$, so $a \leq a$ and $a \leq a$.
Thus, $a \leq a \leq a$, so $a \leq a \leq b$.
Hence, $a \in [a, b]$, so $\{a\} \subset [a, b]$ and $[a, b] \neq \emptyset$.
Therefore, there is at least one element in $[a, b]$.
Let $x \in [a, b]$.
Then $x \in \mathbb{R}$ and $a \leq x \leq b$, so $a \leq x \leq a$.
Thus, $a \leq x$ and $x \leq a$, so $a = x$.
Therefore, $x \in \{a\}$, so $[a, b] \subset \{a\}$.
Since $[a, b] \subset \{a\}$ and $\{a\} \subset [a, b]$, then $[a, b] = \{a\}$, as desired.
Case 3: Suppose $a < b$.
Then $(a, b) \neq \emptyset$.
Hence, there is at least one element x such that $x \in (a, b)$.
Since $x \in (a, b)$ and $(a, b) \subset [a, b]$, then $x \in [a, b]$.
Therefore, $[a, b] \neq \emptyset$, as desired. □

Proposition 5. ***The distance between any two points in the interval (a, b) is less than $b - a$.***
Let $a, b \in \mathbb{R}$.
Let x, y be any real numbers such that $a < x < b$ and $a < y < b$.
Then $|x - y| < b - a$.

Proof. We must prove $|x - y| < b - a$.
Since $a < x < b$ and $a < y < b$, then $a < x$ and $x < b$ and $a < y$ and $y < b$.
Since $x, y \in \mathbb{R}$, then $x - y \in \mathbb{R}$.
Either $x - y \geq 0$ or $x - y < 0$.
We consider these cases separately.
Case 1: Suppose $x - y \geq 0$.

Since $a < y$ and $x < b$, then $a + x < y + b$, so $x - y < b - a$.
Therefore, $|x - y| = x - y < b - a$.
Case 2: Suppose $x - y < 0$.
Since $a < x$ and $y < b$, then $a + y < x + b$, so $y - x < b - a$.
Therefore, $|x - y| = -(x - y) = -x + y = y - x < b - a$.
Hence, in all cases, $|x - y| < b - a$, as desired. □

Corollary 6. *Let $a, b \in \mathbb{R}$ with $a < b$.*
Let x, y be any real numbers such that $x \in [a, b]$ and $y \in [a, b]$.
Then $|x - y| \leq b - a$.

Proof. Since $a < b$, then $b - a > 0$.
Since $x \in [a, b]$ and $y \in [a, b]$, then $a \leq x \leq b$ and $a \leq y \leq b$.
Thus, $x = a$ or $a < x < b$ or $x = b$ and $y = a$ or $a < y < b$ or $y = b$.
Hence, either $x = a$ and $y = a$ or $x = a$ and $a < y < b$ or $x = a$ and $y = b$ or $a < x < b$ and $y = a$ or $a < x < b$ and $a < y < b$ or $a < x < b$ and $y = b$ or $x = b$ and $y = a$ or $x = b$ and $a < y < b$ or $x = b$ and $y = b$.
We consider these cases separately.
Case 1: Suppose $x = a$ and $y = a$.
Then $|x - y| = |a - a| = 0 < b - a$.
Case 2: Suppose $x = a$ and $a < y < b$.
Since $a < y < b$, then $a < y$ and $y < b$.
Since $a < y$, then $y - a > 0$.
Thus, $|x - y| = |a - y| = |y - a| = y - a < b - a$.
Case 3: Suppose $x = a$ and $y = b$.
Then $|x - y| = |a - b| = |b - a| = b - a$.
Case 4: Suppose $a < x < b$ and $y = a$.
Since $a < x < b$, then $a < x$ and $x < b$.
Since $a < x$, then $x - a > 0$.
Thus, $|x - y| = |x - a| = x - a < b - a$.
Case 5: Suppose $a < x < b$ and $a < y < b$.
Then $|x - y| < b - a$.
Case 6: Suppose $a < x < b$ and $y = b$.
Since $a < x < b$, then $a < x$ and $x < b$.
Since $a < x$, then $-a > -x$, so $-x < -a$.
Since $x < b$, then $x - b < 0$.
Thus, $|x - y| = |x - b| = -(x - b) = -x + b = b - x < b - a$.
Case 7: Suppose $x = b$ and $y = a$.
Then $|x - y| = |b - a| = b - a$.
Case 8: Suppose $x = b$ and $a < y < b$.
Since $a < y < b$, then $a < y$ and $y < b$.
Since $a < y$, then $-a > -y$, so $-y < -a$.
Since $y < b$, then $y - b < 0$.
Thus, $|x - y| = |b - y| = |y - b| = -(y - b) = -y + b = b - y < b - a$.
Case 9: Suppose $x = b$ and $y = b$.
Then $|x - y| = |b - b| = 0 < b - a$.

Therefore, in all cases, either $|x - y| < b - a$ or $|x - y| = b - a$, so $|x - y| \leq b - a$. ☐

Proposition 7. *Let $I \subset \mathbb{R}$ be an interval.*
If $a \in I$ and $b \in I$ and $a < b$, then $[a, b] \subset I$.

Proof. Suppose $a \in I$ and $b \in I$ and $a < b$.
Since $a \in I$ and $I \subset \mathbb{R}$, then $a \in \mathbb{R}$.
Since $b \in I$ and $I \subset \mathbb{R}$, then $b \in \mathbb{R}$.
Since $a < b$, then by the density of $\mathbb{R}$, there exists $c \in \mathbb{R}$ such that $a < c < b$, so there exists $c \in \mathbb{R}$ such that $c \in (a, b)$.
Since $(a, b) \subset [a, b]$, then $c \in [a, b]$.
To prove $[a, b] \subset I$, let $x \in [a, b]$ be arbitrary.
Since $[a, b] = \{a\} \cup (a, b) \cup \{b\}$, then either $x \in \{a\}$ or $x \in (a, b)$ or $x \in \{b\}$.
We consider these cases separately.
Case 1: Suppose $x \in \{a\}$.
Then $x = a$.
Since $a \in I$, then $x \in I$.
Case 2: Suppose $x \in \{b\}$.
Then $x = b$.
Since $b \in I$, then $x \in I$.
Case 3: Suppose $x \in (a, b)$.
Then $a < x < b$.
Since $a \in I$ and $b \in I$ and $a < x < b$ and I is an interval, then $x \in I$.
Therefore, in all cases $x \in I$, so $[a, b] \subset I$. ☐

Proposition 8. ***intersection of any two intervals is an interval***
If I_1 and I_2 are intervals, then $I_1 \cap I_2$ is an interval.

Proof. Let a, b, and c be arbitrary real numbers.
Suppose $a \in I_1 \cap I_2$ and $b \in I_1 \cap I_2$ and $a < c < b$.
Since $a \in I_1 \cap I_2$, then $a \in I_1$ and $a \in I_2$.
Since $b \in I_1 \cap I_2$, then $b \in I_1$ and $b \in I_2$.
Since I_1 is an interval and $a \in I_1$ and $b \in I_1$ and $a < c < b$, then $c \in I_1$.
Since I_2 is an interval and $a \in I_2$ and $b \in I_2$ and $a < c < b$, then $c \in I_2$.
Thus, $c \in I_1$ and $c \in I_2$, so $c \in I_1 \cap I_2$.
Therefore, $I_1 \cap I_2$ is an interval. ☐

Proposition 9. ***intersection of a countable collection of intervals is an interval***
If $\{I_n : n \in \mathbb{Z}^+\}$ is a collection of intervals, then $\cap_{n=1}^{\infty} I_n$ is an interval.

Solution. Let a, b, c be arbitrary real numbers.
To prove $a, b \in \cap_{n=1}^{\infty} I_n \wedge a < c < b \rightarrow c \in \cap_{n=1}^{\infty} I_n$, we assume $a, b \in \cap_{n=1}^{\infty} I_n \wedge a < c < b$.
To prove $c \in \cap_{n=1}^{\infty} I_n$, we must prove that $(\forall n \in \mathbb{Z}^+)(c \in I_n)$. ☐

Proof. Let $\{I_n : n \in \mathbb{Z}^+\}$ be a collection of intervals.

To prove $\cap_{n=1}^{\infty} I_n$ is an interval, let a, b, and c be arbitrary real numbers such that $a \in \cap_{n=1}^{\infty} I_n$ and $b \in \cap_{n=1}^{\infty} I_n$ and $a < c < b$.

To prove $c \in \cap_{n=1}^{\infty} I_n$, let k be an arbitrary positive integer.

We must prove $c \in I_k$.

Since $a \in \cap_{n=1}^{\infty} I_n$, then $a \in I_n$ for every $n \in \mathbb{Z}^+$.

In particular, $a \in I_k$.

Since $b \in \cap_{n=1}^{\infty} I_n$, then $b \in I_n$ for every $n \in \mathbb{Z}^+$.

In particular, $b \in I_k$.

Since I_k is an interval and $a \in I_k$ and $b \in I_k$ and $a < c < b$, then $c \in I_k$, as desired. □

Proposition 10. *Let $p \in \mathbb{R}$.*

Let $\delta, \epsilon \in \mathbb{R}$.

If $0 < \delta \leq \epsilon$, then $N(p; \delta) \subset N(p; \epsilon)$.

Proof. Suppose $0 < \delta \leq \epsilon$.

Then $0 < \delta$ and $\delta \leq \epsilon$.

Since $\delta > 0$, then $p \in N(p; \delta)$, so $N(p; \delta) \neq \emptyset$.

Let $x \in N(p; \delta)$.

Then $x \in \mathbb{R}$ and $d(x, p) < \delta$.

Since $d(x, p) < \delta$ and $\delta \leq \epsilon$, then $d(x, p) < \epsilon$, so $x \in N(p; \epsilon)$.

Therefore, $N(p; \delta) \subset N(p; \epsilon)$. □

Proposition 11. ***Every ϵ neighborhood of a point is a neighborhood of the point.***

Let $p \in \mathbb{R}$.

Then $N(p; \epsilon)$ is a neighborhood of p for every $\epsilon > 0$.

Proof. Let $\epsilon > 0$ be given.

Since $\epsilon > 0$ and $N(p; \epsilon) \subset N(p; \epsilon)$, then $N(p; \epsilon)$ is a neighborhood of p. □

Types of points in $\mathbb{R}$

Proposition 12. *Let A and B be sets.*

If p is an interior point of A and $A \subset B$, then p is an interior point of B.

Proof. Suppose p is an interior point of A and $A \subset B$.

Since p is an interior point of A, then there exists $\epsilon > 0$ such that $N(p; \epsilon) \subset A$.

Since $A \subset B$, then $N(p; \epsilon) \subset B$.

Hence, there exists $\epsilon > 0$ such that $N(p; \epsilon) \subset B$.

Therefore, p is an interior point of B. □

Lemma 13. *Let A and B be sets.*

If p is an accumulation point of A and $A \subset B$, then p is an accumulation point of B.

Proof. Suppose p is an accumulation point of A and $A \subset B$.

Let $\epsilon > 0$ be given.

Since p is an accumulation point of A, then there exists $x \in A$ such that $x \in N'(p;\epsilon)$.

Since $x \in A$ and $A \subset B$, then $x \in B$.

Thus, there exists $x \in B$ such that $x \in N'(p;\epsilon)$.

Therefore, p is an accumulation point of B. □

Proposition 14. ***Every point in an interval of at least two elements is an accumulation point of the interval.***

Let $I \subset \mathbb{R}$ be an interval with at least two elements.

If $a \in I$, then a is an accumulation point of I.

Proof. Let $a \in I$ be arbitrary.

Since I has at least two elements, then there exists an element of I that is distinct from a, so there exists $b \in I$ such that $b \neq a$.

Thus, either $b < a$ or $b > a$.

We consider these cases separately.

Case 1: Suppose $b > a$.

Then $a < b$.

To prove a is an accumulation point of I, let $\delta > 0$ be given.

We must prove there exists $p \in I$ such that $p \in N'(a;\delta)$.

Let $m = \min\{b, a+\delta\}$.

Then either $m = b$ or $m = a+\delta$, and $m \leq b$ and $m \leq a+\delta$.

Since $a < b$ and $a < a+\delta$ and either $m = b$ or $m = a+\delta$, then $a < m$.

Let p be the midpoint of a and m.

Then $p = \frac{a+m}{2}$.

Since $a < \frac{a+m}{2} < m$, then $a < p < m$, so $a - \delta < a < p < m \leq a+\delta$ and $a < p < m \leq b$.

Since $a < p < m \leq b$, then $a < p < b$.

Since $a \in I$ and $b \in I$ and $a < p < b$ and I is an interval, then $p \in I$.

Since $a - \delta < a < p < m \leq a+\delta$, then $a < p$ and $a - \delta < p < a+\delta$.

Since $a - \delta < p < a+\delta$, then $p \in (a-\delta, a+\delta) = N(a;\delta)$, so $p \in N(a;\delta)$.

Since $p > a$, then $p \neq a$, so $p \in N'(a;\delta)$.

Thus, there exists $p \in I$ such that $p \in N'(a;\delta)$, so a is an accumulation point of I.

Case 2: Suppose $b < a$.

To prove a is an accumulation point of I, let $\delta > 0$ be given.

We must prove there exists $p \in I$ such that $p \in N'(a;\delta)$.

Let $m = \max\{b, a-\delta\}$.

Then either $m = b$ or $m = a-\delta$, and $b \leq m$ and $a-\delta \leq m$.

Since $b < a$ and $a - \delta < a$ and either $m = b$ or $m = a-\delta$, then $m < a$.

Let p be the midpoint of m and a.

Then $p = \frac{m+a}{2}$.

Since $m < \frac{m+a}{2} < a$, then $m < p < a$, so $b \leq m < p < a$ and $a - \delta \leq m < p < a < a+\delta$.

Since $b \leq m < p < a$, then $b < p < a$.
Since $b \in I$ and $a \in I$ and $b < p < a$ and I is an interval, then $p \in I$.
Since $a - \delta \leq m < p < a < a + \delta$, then $a - \delta < p < a + \delta$ and $p < a$.
Since $a - \delta < p < a + \delta$, then $p \in (a - \delta, a + \delta) = N(a; \delta)$, so $p \in N(a; \delta)$.
Since $p < a$, then $p \neq a$, so $p \in N'(a; \delta)$.
Thus, there exists $p \in I$ such that $p \in N'(a; \delta)$, so a is an accumulation point of I.
Therefore, in all cases, a is an accumulation point of I, as desired. □

Proposition 15. ***Every interior point of a set S is an accumulation point of S.***
Let S be a set.
If p is an interior point of S, then p is an accumulation point of S.

Proof. Suppose p is an interior point of S.
Let $\epsilon > 0$ be given.
Since p is an interior point of S, then there exists $\epsilon_1 > 0$ such that $N(p; \epsilon_1) \subset S$.
Let $m = \min\{\epsilon, \epsilon_1\}$.
Since $\epsilon > 0$ and $\epsilon_1 > 0$, then $m > 0$.
Let x be the midpoint of p and $p + m$.
Then $x = \frac{p+p+m}{2} = p + \frac{m}{2}$.
Since $d(x, p) = |x - p| = |\frac{m}{2}| = \frac{m}{2} < m \leq \epsilon_1$, then $d(x, p) < \epsilon_1$, so $x \in N(p; \epsilon_1)$.
Since $N(p; \epsilon_1) \subset S$, then $x \in S$.
Since $d(x, p) = |x - p| = |\frac{m}{2}| = \frac{m}{2} < m \leq \epsilon$, then $d(x, p) < \epsilon$, so $x \in N(p; \epsilon)$.
Since $d(x, p) = |x - p| = |\frac{m}{2}| = \frac{m}{2} > 0$, then $d(x, p) > 0$, so $x \neq p$.
Hence, there exists $x \in S$ such that $x \in N(p; \epsilon)$ and $x \neq p$.
Therefore, p is an accumulation point of S. □

Proposition 16. *Every element of a nonempty set is either an accumulation point or an isolated point.*

Proof. Let S be a nonempty set.
Then there is at least one element of S.
Let $p \in S$.
To prove either p is an accumulation point or p is an isolated point of S, we prove by contrapositive.
Suppose p is not an accumulation point of S.
Since $p \in S$ and p is not an accumulation point of S, then p is an isolated point of S, as desired. □

Sets in $\mathbb{R}$

Proposition 17. *Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.*
If $A \subset B$, then $A^\circ \subset B^\circ$.

Proof. Suppose $A \subset B$.

Let A° be the interior of A.

Let B° be the interior of B.

Either $A^\circ = \emptyset$ or $A^\circ \neq \emptyset$.

We consider these cases separately.

Case 1: Suppose $A^\circ = \emptyset$.

Since the empty set is a subset of every set, then in particular, $\emptyset \subset B^\circ$.

Therefore, $A^\circ \subset B^\circ$.

Case 2: Suppose $A^\circ \neq \emptyset$.

Then there is at least one element in A°.

Let $x \in A^\circ$.

Then x is an interior point of A, so there exists $\delta > 0$ such that $N(x;\delta) \subset A$.

Since $N(x;\delta) \subset A$ and $A \subset B$, then $N(x;\delta) \subset B$.

Hence, there exists $\delta > 0$ such that $N(x;\delta) \subset B$.

Thus, x is an interior point of B, so $x \in B^\circ$.

Therefore, $A^\circ \subset B^\circ$.

Thus, in all cases, $A^\circ \subset B^\circ$, as desired. ☐

Proposition 18. *Let $a, b \in \mathbb{R}$. Then*

1. $(a,b)^\circ = (a,b)$.
2. $[a,b]^\circ = (a,b)$.

Proof. We prove 1.

Let $(a,b)^\circ$ be the interior of the open interval (a,b).

We must prove $(a,b)^\circ = (a,b)$.

Either $a < b$ or $a = b$ or $a > b$.

We consider these cases separately.

Case 1: Suppose $a \geq b$.

Then $(a,b) = \emptyset$.

Therefore, $(a,b)^\circ = \emptyset^\circ = \emptyset = (a,b)$.

Case 2: Suppose $a < b$.

Since $a < b$, then $(a,b) \neq \emptyset$.

Since $S^\circ \subset S$ for every set S, then in particular, $(a,b)^\circ \subset (a,b)$.

We prove $(a,b) \subset (a,b)^\circ$.

Let $x \in (a,b)$.

Then $x \in \mathbb{R}$ and $a < x < b$, so $a < x$ and $x < b$.

Hence, $x - a > 0$ and $b - x > 0$.

Let $\delta = \min\{d(a,x), d(x,b)\}$.

Then $\delta \leq d(a,x)$ and $\delta \leq d(x,b)$.

Since $d(a,x) = |a-x| = |x-a| = x-a > 0$, then $d(a,x) > 0$.

Since $d(x,b) = |x-b| = |b-x| = b-x > 0$, then $d(x,b) > 0$.

Therefore, $\delta > 0$.

Let $p \in N(x;\delta)$.
Then $p \in (x-\delta, x+\delta)$, so $x - \delta < p < x + \delta$.
Hence, $x - \delta < p$ and $p < x + \delta$.
Since $\delta \leq d(a,x)$ and $d(a,x) = x - a$, then $\delta \leq x - a$, so $a \leq x - \delta$.
Since $\delta \leq d(x,b)$ and $d(x,b) = b - x$, then $\delta \leq b - x$, so $x + \delta \leq b$.
Since $a \leq x - \delta$ and $x - \delta < p$, then $a < p$.
Since $p < x + \delta$ and $x + \delta \leq b$, then $p < b$.
Thus, $a < p < b$, so $p \in (a,b)$.
Hence, $N(x;\delta) \subset (a,b)$.
Since there exists $\delta > 0$ such that $N(x;\delta) \subset (a,b)$, then x is an interior point of (a,b).
Therefore, $x \in (a,b)^\circ$, so $(a,b) \subset (a,b)^\circ$.

Since $(a,b)^\circ \subset (a,b)$ and $(a,b) \subset (a,b)^\circ$, then $(a,b)^\circ = (a,b)$, as desired. □

Proof. We prove 2.
Let $[a,b]^\circ$ be the interior of the closed interval $[a,b]$.
We must prove $[a,b]^\circ = (a,b)$.
Either $a < b$ or $a = b$ or $a > b$.
We consider these cases separately.
Case 1: Suppose $a > b$.
Then $(a,b) = \emptyset = [a,b]$.
Therefore, $[a,b]^\circ = \emptyset^\circ = \emptyset = (a,b)$.
Case 2: Suppose $a = b$.
Then $(a,b) = \emptyset$ and $[a,b] = \{a\}$.
Therefore, $[a,b]^\circ = \{a\}^\circ = \emptyset = (a,b)$.
Case 3: Suppose $a < b$.

We prove $(a,b) \subset [a,b]^\circ$.
Since $(a,b) \subset [a,b]$, then $(a,b)^\circ \subset [a,b]^\circ$.
Therefore, $(a,b) \subset [a,b]^\circ$.

We prove $[a,b]^\circ \subset (a,b)$.
Either $[a,b]^\circ = \emptyset$ or $[a,b]^\circ \neq \emptyset$.
We consider these cases separately.
Case 3a: Suppose $[a,b]^\circ = \emptyset$.
Since the empty set is a subset of every set, then in particular, $\emptyset \subset (a,b)$.
Therefore, $[a,b]^\circ \subset (a,b)$.
Case 3b: Suppose $[a,b]^\circ \neq \emptyset$.
Then there is at least one element in $[a,b]^\circ$.
Let $x \in [a,b]^\circ$.
Then x is an interior point of $[a,b]$.
For every nonempty set S, $S^\circ \subset S$.
Since $[a,b] \neq \emptyset$, then $[a,b]^\circ \subset [a,b]$.
Since $x \in [a,b]^\circ$ and $[a,b]^\circ \subset [a,b]$, then $x \in [a,b]$, so $a \leq x \leq b$.
Thus, $a \leq x$ and $x \leq b$.

We prove $x \neq a$ by contradiction.
Suppose $x = a$.
Then a is an interior point of $[a, b]$, so there exists $\delta > 0$ such that $N(a; \delta) \subset [a, b]$.
Let p be the midpoint of $a - \delta$ and a.
Then $p = \frac{(a-\delta)+a}{2} = a - \frac{\delta}{2}$.
Since $\delta > 0$, then $\frac{\delta}{2} > 0$, so $\delta - \frac{\delta}{2} > a - a$.
Hence, $a - \frac{\delta}{2} > a - \delta$, so $p > a - \delta$.
Since $\delta > 0$, then $\frac{3\delta}{2} > 0$, so $\delta + \frac{\delta}{2} > a - a$.
Hence, $a + \delta > a - \frac{\delta}{2}$, so $a + \delta > p$.
Thus, $a - \delta < p < a + \delta$, so $p \in N(a; \delta)$.
Since $\frac{\delta}{2} > 0$, then $\frac{\delta}{2} > a - a$, so $a > a - \frac{\delta}{2}$.
Hence, $a > p$, so $p < a$.
Thus, $p \notin [a, b]$.
Since $p \in N(a; \delta)$ and $p \notin [a, b]$, then a is not an interior point of $[a, b]$.
This contradicts the assumption that a is an interior point of $[a, b]$.
Therefore, $x \neq a$.

We prove $x \neq b$ by contradiction.
Suppose $x = b$.
Then b is an interior point of $[a, b]$, so there exists $\delta > 0$ such that $N(b; \delta) \subset [a, b]$.
Let p be the midpoint of b and $b + \delta$.
Then $p = \frac{b+(b+\delta)}{2} = b + \frac{\delta}{2}$.
Since $\delta > 0$, then $\frac{3\delta}{2} > 0$, so $\delta + \frac{\delta}{2} > b - b$.
Hence, $b + \frac{\delta}{2} > b - \delta$, so $p > b - \delta$.
Since $\delta > 0$, then $\frac{\delta}{2} > 0$, so $\delta - \frac{\delta}{2} > b - b$.
Hence, $b + \delta > b + \frac{\delta}{2}$, so $b + \delta > p$.
Thus, $b - \delta < p < b + \delta$, so $p \in N(b; \delta)$.
Since $\frac{\delta}{2} > 0$, then $\frac{\delta}{2} > b - b$, so $b + \frac{\delta}{2} > b$.
Hence, $p > b$, so $p \notin [a, b]$.
Since $p \in N(b; \delta)$ and $p \notin [a, b]$, then b is not an interior point of $[a, b]$.
This contradicts the assumption that b is an interior point of $[a, b]$.
Therefore, $x \neq b$.

Since $a \leq x \leq b$ and $x \neq a$ and $x \neq b$, then $a < x < b$, so $x \in (a, b)$.
Therefore, $[a, b]^\circ \subset (a, b)$.
Thus, in all cases, $[a, b]^\circ \subset (a, b)$.

Since $[a, b]^\circ \subset (a, b)$ and $(a, b) \subset [a, b]^\circ$, then $[a, b]^\circ = (a, b)$, as desired. ☐

Proposition 19. *Let $A \subset \mathbb{R}$.*
Then $A^{\circ\circ} = A^\circ$.

Proof. Let A° be the interior of A.

Let $A^{\circ\circ}$ be the interior of A°.

We must prove $A^{\circ\circ} = A^\circ$.

Either $A^\circ = \emptyset$ or $A^\circ \neq \emptyset$.

We consider these cases separately.

Case 1: Suppose $A^\circ = \emptyset$.

Then $A^{\circ\circ} = \emptyset^\circ = \emptyset = A^\circ$, as desired.

Case 2: Suppose $A^\circ \neq \emptyset$.

For every nonempty set S, $S^\circ \subset S$.

Since $A^\circ \neq \emptyset$, then $A^{\circ\circ} \subset A^\circ$.

We prove $A^\circ \subset A^{\circ\circ}$.

Since $A^\circ \neq \emptyset$, then there is at least one element in A°.

Let $x \in A^\circ$.

Then x is an interior point of A, so there exists $\delta > 0$ such that $N(x;\delta) \subset A$.

Let $p \in N(x;\delta)$.

Then $N(x;\delta)^\circ = (x-\delta, x+\delta)^\circ = (x-\delta, x+\delta) = N(x;\delta)$, so $p \in N(x;\delta)^\circ$.

Since $N(x;\delta) \subset A$, then $N(x;\delta)^\circ \subset A^\circ$.

Hence, $p \in A^\circ$, so $N(x;\delta) \subset A^\circ$.

Therefore, there exists $\delta > 0$ such that $N(x;\delta) \subset A^\circ$, so x is an interior point of A°.

Thus, $x \in A^{\circ\circ}$, so $A^\circ \subset A^{\circ\circ}$.

Since $A^{\circ\circ} \subset A^\circ$ and $A^\circ \subset A^{\circ\circ}$, then $A^{\circ\circ} = A^\circ$, as desired. □

Proposition 20. *Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.*

Then $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Proof. Let A° be the interior of A.

Let B° be the interior of B.

Let $(A \cap B)^\circ$ be the interior of $A \cap B$.

We must prove $(A \cap B)^\circ = A^\circ \cap B^\circ$.

We prove $(A \cap B)^\circ \subset A^\circ \cap B^\circ$.

Let $x \in (A \cap B)^\circ$.

Since $A \cap B \subset A$, then $(A \cap B)^\circ \subset A^\circ$, so $x \in A^\circ$.

Since $A \cap B \subset B$, then $(A \cap B)^\circ \subset B^\circ$, so $x \in B^\circ$.

Therefore, $x \in A^\circ \cap B^\circ$, so $(A \cap B)^\circ \subset A^\circ \cap B^\circ$.

We prove $A^\circ \cap B^\circ \subset (A \cap B)^\circ$.

Let $x \in A^\circ \cap B^\circ$.

Then $x \in A^\circ$ and $x \in B^\circ$.

Since $x \in A^\circ$, then x is an interior point of A, so there exists $\delta_1 > 0$ such that $N(x;\delta_1) \subset A$.

Since $x \in B^\circ$, then x is an interior point of B, so there exists $\delta_2 > 0$ such that $N(x;\delta_2) \subset B$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Since $\delta_1 > 0$ and $\delta_2 > 0$, then $\delta > 0$.
Let $p \in N(x;\delta)$.
Since $0 < \delta \leq \delta_1$, then $N(x;\delta) \subset N(x;\delta_1)$.
Since $p \in N(x;\delta)$ and $N(x;\delta) \subset N(x;\delta_1) \subset A$, then $p \in A$.
Since $0 < \delta \leq \delta_2$, then $N(x;\delta) \subset N(x;\delta_2)$.
Since $p \in N(x;\delta)$ and $N(x;\delta) \subset N(x;\delta_2) \subset B$, then $p \in B$.
Thus, $p \in A \cap B$, so $N(x;\delta) \subset A \cap B$.
Since there exists $\delta > 0$ such that $N(x;\delta) \subset A \cap B$, then x is an interior point of $A \cap B$, so $x \in (A \cap B)^\circ$.
Therefore, $A^\circ \cap B^\circ \subset (A \cap B)^\circ$.

Since $(A \cap B)^\circ \subset A^\circ \cap B^\circ$ and $A^\circ \cap B^\circ \subset (A \cap B)^\circ$, then $(A \cap B)^\circ = A^\circ \cap B^\circ$, as desired. □

Proposition 21. *Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.*
Then $(A \cup B)^\circ \supset A^\circ \cup B^\circ$.

Proof. Let A° be the interior of A.
Let B° be the interior of B.
Let $(A \cup B)^\circ$ be the interior of $A \cup B$.
We must prove $A^\circ \cup B^\circ \subset (A \cup B)^\circ$.

Let $x \in A^\circ \cup B^\circ$.
Then either $x \in A^\circ$ or $x \in B^\circ$.
We consider these cases separately.
Case 1: Suppose $x \in A^\circ$.
Then x is an interior point of A, so there exists $\delta > 0$ such that $N(x;\delta) \subset A$.
Since $N(x;\delta) \subset A$ and $A \subset A \cup B$, then $N(x;\delta) \subset A \cup B$.
Therefore, x is an interior point of $A \cup B$, so $x \in (A \cup B)^\circ$.
Case 2: Suppose $x \in B^\circ$.
Then x is an interior point of B, so there exists $\delta > 0$ such that $N(x;\delta) \subset B$.
Since $N(x;\delta) \subset B$ and $B \subset A \cup B$, then $N(x;\delta) \subset A \cup B$.
Therefore, x is an interior point of $A \cup B$, so $x \in (A \cup B)^\circ$.
Hence, in all cases, $x \in (A \cup B)^\circ$, so $A^\circ \cup B^\circ \subset (A \cup B)^\circ$, as desired. □

Proposition 22. *A set S is open iff $S^\circ = S$.*

Proof. Let S be a set.
Let S° be the interior of S.
Then $S^\circ = \{x : x \text{ is an interior point of } S\}$.
We prove if S is open, then $S^\circ = S$.
Suppose S is open.
Then if $x \in S$, then x is an interior point of S.
Thus, if $x \in S$, then $x \in S^\circ$.
Hence, $S \subset S^\circ$.
Since the interior of a set is a subset of the set, then $S^\circ \subset S$.
Thus, $S^\circ \subset S$ and $S \subset S^\circ$, so $S^\circ = S$. □

Proof. Conversely, we prove if $S^\circ = S$, then S is open.
Suppose $S^\circ = S$.
Then $S^\circ \subset S$ and $S \subset S^\circ$, so $S \subset S^\circ$.
Hence, if $x \in S$, then $x \in S^\circ$.
Thus, if $x \in S$, then x is an interior point of S.
Therefore, S is open. □

Proposition 23. *Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.*
1. If A is open and B is open, then $A \cup B$ is open.
2. If A is open and B is open, then $A \cap B$ is open.

Proof. We prove 1.
Suppose A is open and B is open.
Either $A \cup B = \emptyset$ or $A \cup B \neq \emptyset$.
We consider these cases separately.
Case 1: Suppose $A \cup B = \emptyset$.
Since the empty set is open, then $A \cup B$ is open.
Case 2: Suppose $A \cup B \neq \emptyset$.
Then there is at least one element of $A \cup B$.
Let $x \in A \cup B$.
Then either $x \in A$ or $x \in B$.
We consider these cases separately.
Case 2a: Suppose $x \in A$.
Since A is open, then x is an interior point of A, so there exists $\delta_1 > 0$ such that $N(x; \delta_1) \subset A$.
Since $A \subset A \cup B$, then $N(x; \delta_1) \subset A \cup B$.
Thus, x is an interior point of $A \cup B$.
Case 2b: Suppose $x \in B$.
Since B is open, then x is an interior point of B, so there exists $\delta_2 > 0$ such that $N(x; \delta_2) \subset B$.
Since $B \subset A \cup B$, then $N(x; \delta_2) \subset A \cup B$.
Thus, x is an interior point of $A \cup B$.
Therefore, in either case, x is an interior point of $A \cup B$, so $A \cup B$ is open. □

Proof. We prove 2.
Suppose A is open and B is open.
Either $A \cap B = \emptyset$ or $A \cap B \neq \emptyset$.
We consider these cases separately.
Case 1: Suppose $A \cap B = \emptyset$.
Since the empty set is open, then $A \cap B$ is open.
Case 2: Suppose $A \cap B \neq \emptyset$.
Then there is at least one element of $A \cap B$.
Let $x \in A \cap B$.
Then $x \in A$ and $x \in B$.
Since $x \in A$ and A is open, then x is an interior point of A, so there exists $\delta_1 > 0$ such that $N(x; \delta_1) \subset A$.

Since $x \in B$ and B is open, then x is an interior point of B, so there exists $\delta_2 > 0$ such that $N(x;\delta_2) \subset B$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then either $\delta = \delta_1$ or $\delta = \delta_2$, and $\delta \leq \delta_1$ and $\delta \leq \delta_2$.

Since $\delta_1 > 0$ and $\delta_2 > 0$ and either $\delta = \delta_1$ or $\delta = \delta_2$, then $\delta > 0$.

Since $N(x;\delta) \neq \emptyset$, let $p \in N(x;\delta)$.

Then $d(p,x) < \delta$.

Since $d(p,x) < \delta$ and $\delta \leq \delta_1$, then $d(p,x) < \delta_1$, so $p \in N(x;\delta_1)$.

Since $N(x;\delta_1) \subset A$, then $p \in A$.

Since $d(p,x) < \delta$ and $\delta \leq \delta_2$, then $d(p,x) < \delta_2$, so $p \in N(x;\delta_2)$.

Since $N(x;\delta_2) \subset B$, then $p \in B$.

Hence, $p \in A \cap B$, so $N(x;\delta) \subset A \cap B$.

Thus, there exists $\delta > 0$ such that $N(x;\delta) \subset A \cap B$, so x is an interior point of $A \cap B$.

Therefore, $A \cap B$ is open. □

Theorem 24. ***topological properties of open sets in*** $\mathbb{R}$

1. *The union of any collection of open sets in* $\mathbb{R}$ *is open.*
2. *The intersection of any finite collection of open sets in* $\mathbb{R}$ *is open.*

Proof. We prove 1.

Let $\mathcal{F}$ be an arbitrary collection of open sets in $\mathbb{R}$.

We must prove $\cup\mathcal{F}$ is open.

Either $\cup\mathcal{F} = \emptyset$ or $\cup\mathcal{F} \neq \emptyset$.

We consider these cases separately.

Case 1: Suppose $\cup\mathcal{F} = \emptyset$.

Since the empty set is open, then $\cup\mathcal{F}$ is open.

Case 2: Suppose $\cup\mathcal{F} \neq \emptyset$.

Then there is at least one element of $\cup\mathcal{F}$.

Let $x \in \cup\mathcal{F}$.

Then there exists $S \in \mathcal{F}$ such that $x \in S$.

Since $S \in \mathcal{F}$, then S is an open set in $\mathbb{R}$.

Since S is open and $x \in S$, then x is an interior point of S.

Hence, there exists $\delta > 0$ such that $N(x;\delta) \subset S$.

Since $S \in \mathcal{F}$, then $S \subset \cup\mathcal{F}$.

Since $N(x;\delta) \subset S$ and $S \subset \cup\mathcal{F}$, then $N(x;\delta) \subset \cup\mathcal{F}$.

Hence, there exists $\delta > 0$ such that $N(x;\delta) \subset \cup\mathcal{F}$, so x is an interior point of $\cup\mathcal{F}$.

Therefore, $\cup\mathcal{F}$ is open.

Thus, in all cases, $\cup\mathcal{F}$ is open, as desired. □

Proof. We prove 2.

Let $\mathcal{F}$ be an arbitrary finite collection of open sets in $\mathbb{R}$.

We must prove $\cap\mathcal{F}$ is open.

Since $\mathcal{F}$ is a finite set, then $\mathcal{F}$ contains exactly n elements for some nonnegative integer n.

Thus, either $n = 0$ or $n = 1$ or $n > 1$.

We consider these cases separately.

Case 1: Suppose $n = 0$.

Then $\mathcal{F}$ contains zero elements, so $\mathcal{F} = \emptyset$.

Thus, $\cap\mathcal{F} = \cap\emptyset = \emptyset$.

Since the empty set is open, then $\cap\mathcal{F}$ is open.

Case 2: Suppose $n = 1$.

Then $\mathcal{F}$ contains exactly one element.

Thus, there exists F_1 such that $\mathcal{F} = \{F_1\}$.

Since $F_1 \in \mathcal{F}$, then F_1 is an open set in $\mathbb{R}$.

Hence, $\cap\mathcal{F} = F_1$ is open.

Case 3: Suppose $n > 1$.

Then $n \geq 2$, so $\mathcal{F} = \{F_1, F_2, ..., F_n\}$ and F_i is an open set in $\mathbb{R}$ for each $i = 1, 2, ..., n$.

We must prove for each natural number $n \geq 2$, if $F_1, F_2, ..., F_n$ are all open sets in $\mathbb{R}$, then $F_1 \cap F_2 \cap ... \cap F_n$ is open.

We prove by induction on n.

Basis: Let $n = 2$.

Suppose F_1 and F_2 are open sets.

Since the intersection of any two open sets is open, then $F_1 \cap F_2$ is open.

Induction:

Let $k \in \mathbb{N}$ such that $k \geq 2$.

Suppose the statement 'if the sets $S_1, S_2, ..., S_k$ are open sets in $\mathbb{R}$, then $S_1 \cap S_2 \cap ... \cap S_k$ is open' is true.

We must prove 'if the sets $S_1, S_2, ..., S_k, S_{k+1}$ are open sets in $\mathbb{R}$, then $S_1 \cap S_2 \cap ... \cap S_{k+1}$ is open'.

Assume the sets $S_1, S_2, ..., S_k, S_{k+1}$ are open sets in $\mathbb{R}$.

Since the sets $S_1, S_2, ..., S_k$ are open, then by the induction hypothesis, the set $S_1 \cap S_2 \cap ... \cap S_k$ is open.

Since the intersection of any two open sets is open and $S_1 \cap S_2 \cap ... \cap S_k$ is open and S_{k+1} is open, then the set $(S_1 \cap S_2 \cap ... \cap S_k) \cap S_{k+1}$ is open.

Hence, $S_1 \cap S_2 \cap ... \cap S_k \cap S_{k+1}$ is open.

Therefore, by PMI, for all natural numbers $n \geq 2$, if the sets $S_1, S_2, ..., S_n$ are open sets in $\mathbb{R}$, then the intersection $S_1 \cap S_2 \cap ... \cap S_n$ is open, as desired. □

Corollary 25. $\mathbb{R}$ ***is a topological space***

Let τ be the set of all open subsets of $\mathbb{R}$. Then

T1. $\emptyset \in \tau$ and $\mathbb{R} \in \tau$.

T2. The union of any collection of sets in τ is in τ.

T3. The intersection of any finite collection of sets in τ is in τ.

Proof. Let τ be the set of all open subsets of $\mathbb{R}$.

Then $\tau = \{S : S \text{ is an open subset of } \mathbb{R}\}$.

We prove T1.

Since $\emptyset \subset \mathbb{R}$ and $\emptyset$ is open, then $\emptyset \in \tau$.

Since $\mathbb{R} \subset \mathbb{R}$ and $\mathbb{R}$ is open, then $\mathbb{R} \in \tau$. □

Proof. We prove T2.

Let S be a collection of sets in τ.

Then S is a collection of open subsets of $\mathbb{R}$, so $S \subset \tau$.

Let $\cup S$ be the union of the elements of S.

Either $\cup S = \emptyset$ or $\cup S \neq \emptyset$.

We consider these cases separately.

Case 1: Suppose $\cup S = \emptyset$.

Since $\emptyset \in \tau$, then $\cup S \in \tau$.

Case 2: Suppose $\cup S \neq \emptyset$.

Then there is at least one element of $\cup S$.

Let $x \in \cup S$.

Then there exists $X \in S$ such that $x \in X$.

Since $X \in S$ and $S \subset \tau$, then $X \in \tau$, so X is an open subset of $\mathbb{R}$.

Thus, $X \subset \mathbb{R}$.

Since $x \in X$ and $X \subset \mathbb{R}$, then $x \in \mathbb{R}$.

Thus, $\cup S \subset \mathbb{R}$.

Since the union of any collection of open sets in $\mathbb{R}$ is open and S is a collection of open sets in $\mathbb{R}$, then $\cup S$ is open.

Thus, $\cup S$ is an open subset of $\mathbb{R}$, so $\cup S \in \tau$.

Therefore, in all cases, $\cup S \in \tau$, as desired.

Proof. □

We prove T3.

Let S be a finite collection of sets in τ.

Then S is a finite collection of open subsets of $\mathbb{R}$.

Since S is a collection of open subsets of $\mathbb{R}$, then $S \subset \tau$.

Let $\cap S$ be the intersection of the elements of S.

Either $\cap S = \emptyset$ or $\cap S \neq \emptyset$.

We consider these cases separately.

Case 1: Suppose $\cap S = \emptyset$.

Since $\emptyset \in \tau$, then $\cap S \in \tau$.

Case 2: Suppose $\cap S \neq \emptyset$.

Then there is at least one element of $\cap S$.

Let $x \in \cap S$.

Then $x \in X$ for each $X \in S$.

Let $X \in S$.

Then $x \in X$.

Since $X \in S$ and $S \subset \tau$, then $X \in \tau$, so X is an open subset of $\mathbb{R}$.

Thus, $X \subset \mathbb{R}$.

Since $x \in X$ and $X \subset \mathbb{R}$, then $x \in \mathbb{R}$.

Thus, $\cap S \subset \mathbb{R}$.

Since the intersection of any finite collection of open sets in $\mathbb{R}$ is open and S is a finite collection of open sets in $\mathbb{R}$, then $\cap S$ is open.

Thus, $\cap S$ is an open subset of $\mathbb{R}$, so $\cap S \in \tau$.

Therefore, in all cases, $\cap S \in \tau$, as desired. □

Theorem 26. ***characterization of open sets in*** $\mathbb{R}$

A nonempty subset of $\mathbb{R}$ *is open iff it is a union of bounded open intervals.*

Proof. Let S be a nonempty subset of $\mathbb{R}$.

We prove if S is a union of bounded open intervals, then S is open.

Let $\mathcal{F}$ be a family of bounded open intervals in $\mathbb{R}$.

Suppose $S = \cup\mathcal{F}$.

Since $\mathcal{F}$ is a collection of bounded open intervals and every bounded open interval is open, then $\mathcal{F}$ is a collection of open sets in $\mathbb{R}$.

Therefore, $\cup\mathcal{F}$ is open, so S is open, as desired.

Conversely, we prove if S is open, then S is a union of bounded open intervals.

Suppose S is open.

Then every point in S is an interior point of S.

Hence, for every $x \in S$, there exists $\delta > 0$ such that $N(x;\delta) \subset S$.

Let $\mathcal{F}$ be a family of δ neighborhoods centered at x of radius δ for each $x \in S$.

Since $N(x;\delta) = (x-\delta, x+\delta)$ for each $x \in S$, then every δ neighborhood is a bounded open interval.

Thus, $\mathcal{F}$ is a collection $\{I_x : x \in S\}$ of bounded open intervals and the bounded open interval I_x is a subset of S for each $x \in S$.

Hence, $\mathcal{F} = \{I_x : x \in S\}$ and $I_x \subset S$ for each $x \in S$.

By a previous proposition, if I is an index set and B is a set and $A_i \subset B$ for all $i \in I$, then $\cup_{i\in I} A_i \subset B$.

In particular, since S is an index set and $I_x \subset S$ for all $x \in S$, then $\cup_{x\in S} I_x \subset S$.

Therefore, $\cup\mathcal{F} \subset S$.

Since S is not empty, then there is at least one point in S.

Let $y \in S$.

Then $I_y \subset S$.

Since I_y is an δ neighborhood of y, then $y \in I_y$.

Let $s = y$.

Since $y \in S$, then $s \in S$.

Since $y \in I_y$, then $y \in I_s$.

Hence, there exists $s \in S$ such that $y \in I_s$, so $y \in \cup_{x\in S} I_x$.

Thus, $y \in \cup\mathcal{F}$, so $S \subset \cup\mathcal{F}$.

Since $S \subset \cup\mathcal{F}$ and $\cup\mathcal{F} \subset S$, then $S = \cup\mathcal{F}$.

Therefore, S is a union of bounded open intervals, as desired. □

Proposition 27. *A set with no accumulation points is closed.*

Proof. Let S be a set with no accumulation points.

Then no point is an accumulation point of S, so there is no x such that x is an accumulation point of S.

Hence, for every x, x is not an accumulation point of S.

Thus, for every x, if $x \notin S$, then x is not an accumulation point of S.

Therefore, for every x, if x is an accumulation point of S, then $x \in S$.

Hence, S is closed. $\square$

Proposition 28. *Let $S \subset \mathbb{R}$.*

1. *If S is non empty, closed, and bounded above in $\mathbb{R}$, then $\max S$ exists.*
2. *If S is non empty, closed, and bounded below in $\mathbb{R}$, then $\min S$ exists.*

Proof. We prove 1.

Suppose S is non empty, closed, and bounded above in $\mathbb{R}$.

Since $S \neq \emptyset$ and S is bounded above in $\mathbb{R}$, then by the completeness of $\mathbb{R}$, $\sup S$ exists.

Let $\epsilon > 0$ be given.

Since $\sup S$ is the least upper bound of S, then there exists $x \in S$ such that $x > \sup S - \epsilon$.

Since $x \in S$ and $\sup S$ is an upper bound of S, then $x \leq \sup S$, so either $x < \sup S$ or $x = \sup S$.

We consider these cases separately.

Case 1: Suppose $x = \sup S$.

Since $x \in S$, then $\sup S \in S$.

Case 2: Suppose $x < \sup S$.

Then $x \neq \sup S$.

Since $\sup S - \epsilon < x < \sup S < \sup S + \epsilon$, then $\sup S - \epsilon < x < \sup S + \epsilon$, so $x \in (\sup S - \epsilon, \sup S + \epsilon)$.

Hence, $x \in N(\sup S; \epsilon)$.

Since $x \neq \sup S$, then $x \in N'(\sup S; \epsilon)$.

Thus, $x \in N'(\sup S; \epsilon) \cap S$, so $N'(\sup S; \epsilon) \cap S \neq \emptyset$.

Therefore, $\sup S$ is an accumulation point of S.

Since S is closed, then every accumulation point of S is an element of S, so $\sup S \in S$.

Thus, in all cases, $\sup S \in S$.

Since $\sup S$ is an upper bound of S and $\sup S \in S$, then $\sup S = \max S$, so $\max S$ exists. $\square$

Proof. We prove 2.

Suppose S is non empty, closed, and bounded below in $\mathbb{R}$.

Since $S \neq \emptyset$ and S is bounded below in $\mathbb{R}$, then by the completeness of $\mathbb{R}$, $\inf S$ exists.

Let $\epsilon > 0$ be given.

Since $\inf S$ is the greatest lower bound of S, then there exists $x \in S$ such that $x < \inf S + \epsilon$.

Since $x \in S$ and $\inf S$ is a lower bound of S, then $\inf S \leq x$, so either $\inf S < x$ or $\inf S = x$.

We consider these cases separately.

Case 1: Suppose $\inf S =$ x.

Since $x \in S$, then $\inf S \in S$.

Case 2: Suppose $\inf S < x$.

Then $x > \inf S$, so $x \neq \inf S$.

Since $\inf S - \epsilon < \inf S < x < \inf S + \epsilon$, then $\inf S - \epsilon < x < \inf S + \epsilon$, so $x \in (\inf S - \epsilon, \inf S + \epsilon)$.

Hence, $x \in N(\inf S; \epsilon)$.

Since $x \neq \inf S$, then $x \in N'(\inf S; \epsilon)$.

Thus, $x \in N'(\inf S; \epsilon) \cap S$, so $N'(\inf S; \epsilon) \cap S \neq \emptyset$.

Therefore, $\inf S$ is an accumulation point of S.

Since S is closed, then every accumulation point of S is an element of S, so $\inf S \in S$.

Thus, in all cases, $\inf S \in S$.

Since $\inf S$ is a lower bound of S and $\inf S \in S$, then $\inf S = \min S$, so $\min S$ exists. ☐

Theorem 29. *Let $S \subset \mathbb{R}$.*

Then S is open iff $\mathbb{R} - S$ is closed.

Proof. We prove if S is open, then $\mathbb{R} - S$ is closed.

Either there is some accumulation point of the set $\mathbb{R} - S$ or there is not.

We consider these cases separately.

Case 1: Suppose there is no accumulation point of $\mathbb{R} - S$.

Then $\mathbb{R} - S$ is a set with no accumulation points.

Since a set with no accumulation points is closed, then $\mathbb{R} - S$ is closed.

Thus, the conditional 'if S is open, then $\mathbb{R} - S$ is closed' is trivially true.

Case 2: Suppose there is some accumulation point of $\mathbb{R} - S$.

Suppose S is open.

Since there is some accumulation point of $\mathbb{R} - S$, then there is at least one accumulation point of $\mathbb{R} - S$.

Let x be an arbitrary accumulation point of $\mathbb{R} - S$.

Suppose for the sake of contradiction $x \in S$.

Since $S \subset \mathbb{R}$, then $x \in \mathbb{R}$.

Since S is open and $x \in S$, then x is an interior point of S, so there exists $\delta > 0$ such that $N(x; \delta) \subset S$.

Since x is an accumulation point of $\mathbb{R} - S$ and $\delta > 0$, then there exists $p \in (\mathbb{R} - S)$ such that $p \in N'(x; \delta)$.

Since $p \in N'(x; \delta)$ and $N'(x; \delta) \subset N(x; \delta)$, then $p \in N(x; \delta)$.

Since $N(x; \delta) \subset S$, then $p \in S$.

Since $p \in \mathbb{R} - S$, then $p \in \mathbb{R}$ and $p \notin S$.

Thus, we have $p \in S$ and $p \notin S$, a contradiction.

Therefore, $x \notin S$.

Since $x \in \mathbb{R}$ and $x \notin S$, then $x \in (\mathbb{R} - S)$, so $\mathbb{R} - S$ is closed.

Hence, if S is open, then $\mathbb{R} - S$ is closed.

Thus, in either case, the implication 'if S is open, then $\mathbb{R} - S$ is closed' is true. ☐

Proof. Conversely, we prove if $\mathbb{R} - S$ is closed, then S is open.

Either $S = \emptyset$ or $S \neq \emptyset$.

We consider these cases separately.

Case 1: Suppose $S = \emptyset$.

Since the empty set is open, then S is open.

Therefore, the conditional 'if $\mathbb{R} - S$ is closed, then S is open' is trivially true.

Case 2: Suppose $S \neq \emptyset$.

Suppose $\mathbb{R} - S$ is closed.

Since $S \neq \emptyset$, then there is at least one element of S.

Let $x \in S$.

Since $\mathbb{R} - S$ is closed, then if x is an accumulation point of $\mathbb{R} - S$, then $x \in (\mathbb{R} - S)$.

Hence, if $x \notin (\mathbb{R} - S)$, then x is not an accumulation point of $\mathbb{R} - S$.

Since $x \in S$, then $x \notin (\mathbb{R} - S)$, so x is not an accumulation point of $\mathbb{R} - S$.

Thus, there exists $\delta > 0$ such that $N'(x; \delta) \cap (\mathbb{R} - S) = \emptyset$.

Since $N(x; \delta) \neq \emptyset$, let $p \in N(x; \delta)$.

Since $N(x; \delta) \subset \mathbb{R}$, then $p \in \mathbb{R}$.

Either $p = x$ or $p \neq x$.

We consider these cases separately.

Case 2a: Suppose $p = x$.

Since $x \in S$, then $p \in S$.

Case 2a: Suppose $p \neq x$.

Then $p \in N'(x; \delta)$.

Suppose $p \notin S$.

Since $p \in \mathbb{R}$ and $p \notin S$, then $p \in (\mathbb{R} - S)$.

Thus, $p \in N'(x; \delta) \cap (\mathbb{R} - S)$.

This contradicts the fact that $N'(x; \delta) \cap (\mathbb{R} - S) = \emptyset$.

Thus, $p \in S$.

Hence, in all cases, $p \in S$.

Consequently, if $p \in N(x; \delta)$, then $p \in S$, so $N(x; \delta) \subset S$.

Thus, there exists $\delta > 0$ such that $N(x; \delta) \subset S$, so x is an interior point of S.

Therefore, if $x \in S$, then x is an interior point of S, so S is open.

Hence, if $\mathbb{R} - S$ is closed, then S is open.

Thus, in either case, the implication 'if $\mathbb{R} - S$ is closed, then S is open' is true. □

Theorem 30. ***Heine-Borel covering theorem***

Every open covering of a closed and bounded set S in $\mathbb{R}$ contains a finite subcovering of S.

Proof. Let S be a closed and bounded set of real numbers.

Either $S = \emptyset$ or $S \neq \emptyset$.

We consider these cases separately.

Case 1: Suppose $S = \emptyset$.

Since the empty set is compact, then every open covering of $\emptyset$ contains a finite subcovering of $\emptyset$.

Case 2: Suppose $S \neq \emptyset$.

Let $\mathcal{F}$ be an arbitrary open covering of S.

Then $S \subset \cup\mathcal{F}$ and each set in $\mathcal{F}$ is open.

To prove $\mathcal{F}$ contains a finite subcovering of S, we prove there exists a finite subset of $\mathcal{F}$ that covers S.

Define the set $S_x = \{s \in S : s \leq x\}$ for each $x \in \mathbb{R}$.

Let $B = \{x \in \mathbb{R} : \text{some finite subset of } \mathcal{F} \text{ covers } S_x\}$.

Since S is bounded, then S is bounded above and below in $\mathbb{R}$.

Since $S \subset \mathbb{R}$ and $S \neq \emptyset$ and S is closed and bounded above in $\mathbb{R}$, then $\max S$ exists.

Since $S \subset \mathbb{R}$ and $S \neq \emptyset$ and S is closed and bounded below in $\mathbb{R}$, then $\min S$ exists.

Thus, $S_{\min S} = \{s \in S : s \leq \min S\} = \{\min S\}$.

Since $\min S \in S$ and $S \subset \cup\mathcal{F}$, then $\min S \in \cup\mathcal{F}$, so there exists $A \in \mathcal{F}$ such that $\min S \in A$.

Since $\min S \in A$, then $\{\min S\} \subset A$.

Let $G = \{A\}$.

Then $S_{\min S} = \{\min S\} \subset A = \cup G$, so $S_{\min S} \subset \cup G$.

Thus, G covers $S_{\min S}$.

Since $G \subset \mathcal{F}$ and G is a finite set, then G is a finite subset of $\mathcal{F}$ that covers $S_{\min S}$.

Therefore, $\min S \in B$, so $B \neq \emptyset$.

We prove the set B is not bounded above in $\mathbb{R}$ by contradiction.

Suppose B is bounded above in $\mathbb{R}$.

Since $B \subset \mathbb{R}$ and $B \neq \emptyset$ and B is bounded above in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\sup B$ exists.

Either $\sup B \in S$ or $\sup B \notin S$.

We consider these cases separately.

Case 2a: Suppose $\sup B \in S$.

Since $S \subset \cup\mathcal{F}$, then $\sup B \in \cup\mathcal{F}$, so there exists $F_0 \in \mathcal{F}$ such that $\sup B \in F_0$.

Since $F_0 \in \mathcal{F}$, then F_0 is an open set.

Since $\sup B \in F_0$, then $\sup B$ is an interior point of F_0, so there exists $\delta > 0$ such that $N(\sup B; \delta) \subset F_0$.

Since $\delta > 0$ and $\sup B$ is the least upper bound of B, then there exists $b \in B$ such that $\sup B - \delta < b < \sup B$.

Why can we assume $b < \sup B$ versus $b \leq \sup B$? We need to prove this!!!!

Since $b \in B$, then some finite subset of $\mathcal{F}$ covers S_b.

Hence, there exists $G = \{F_1, F_2, ..., F_k\}$ such that $F_i \in F$ for each i and $S_b \subset \cup G$.

Let $t = \sup B + \frac{\delta}{2}$.

Then $t > \sup B$ and $S_t = \{s \in S : s \leq t\}$.

We prove $G' = \{F_0, F_1, ..., F_k\}$ covers S_t.
Since $\sup B \in S$ and $\sup B < t$, then $\sup B \in S_t$, so $S_t \neq \emptyset$.
Let $x \in S_t$.
Then $x \in S$ and $x \leq t$.
Either $x \leq \sup B - \delta$ or $x > \sup B - \delta$.
We consider these cases separately.
Case 2a1: Suppose $x \leq \sup B - \delta$.
Since $x \leq \sup B - \delta$ and $\sup B - \delta < b$, then $x < b$.
Since $x \in S$ and $x < b$, then $x \in S_b$.
Since $S_b \subset \cup G$, then $x \in \cup G$.
Case 2a2: Suppose $x > \sup B - \delta$.
Since $x \leq t$ and $t < \sup B + \delta$, then $x < \sup B + \delta$.
Thus, $\sup B - \delta < x < \sup B + \delta$, so $x \in (\sup B - \delta, \sup B + \delta)$.
Hence, $x \in N(\sup B; \delta)$.
Since $N(\sup B; \delta) \subset F_0$, then $x \in F_0$.
Hence, either $x \in F_0$ or $x \in \cup G$, so $x \in F_0 \cup (\cup G)$.
Thus, $x \in F_0 \cup (F_1 \cup F_2 \cup ... \cup F_k)$, so $x \in \cup G'$.
Therefore, $S_t \subset \cup G'$, so G' is a covering for S_t.
Since $G' \subset \mathcal{F}$ and G' is a finite set, then G' is a finite subset of $\mathcal{F}$ that covers S_t, so $t \in B$.
Since $\sup B$ is an upper bound of B and $t \in B$, then $t \leq \sup B$.
Thus, we have $t \leq \sup B$ and $t > \sup B$, a contradiction.
Therefore, $\sup B \in S$ is false.
Case 2b: Suppose $\sup B \notin S$.
Since S is closed, then $\sup B$ is not an accumulation point of S.
Hence, there exists $\delta > 0$ such that $N'(\sup B; \delta) \cap S = \emptyset$.

$\square$