Topology of $\mathbb{R}$ Examples

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Topology of $\mathbb{R}$

Example 1. every nonempty open interval is the ϵ neighborhood of some point

Let $a, b \in \mathbb{R}$.
If $a < b$, then $N(\frac{a+b}{2}; \frac{b-a}{2}) = (a, b)$.

Proof. Suppose $a < b$.
Then $b - a > 0$, so $\frac{b-a}{2} > 0$.
Observe that

$$\begin{aligned} N(\frac{a+b}{2}; \frac{b-a}{2}) &= (\frac{a+b}{2} - \frac{b-a}{2}, \frac{a+b}{2} + \frac{b-a}{2}) \\ &= (\frac{a+b-b+a}{2}, \frac{a+b+b-a}{2}) \\ &= (a, b). \end{aligned}$$

□

Example 2. deleted ϵ neighborhood is a subset of the ϵ neighborhood of a point

Let $\epsilon > 0$.
Let $p \in \mathbb{R}$.
Then $N'(p; \epsilon) \subset N(p; \epsilon)$ and $N'(p; \epsilon) = (p - \epsilon, p) \cup (p, p + \epsilon)$.

Proof. We prove $N'(p; \epsilon) \subset N(p; \epsilon)$.
For every set A and B, we have $A - B = A \cap \overline{B} \subset A$, so $A - B \subset A$.
In particular, $N'(p; \epsilon) = N(p; \epsilon) - \{p\} \subset N(p; \epsilon)$. □

Proof. We prove $N'(p; \epsilon) = (p - \epsilon, p) \cup (p, p + \epsilon)$.

We first prove $N'(p; \epsilon) \neq \emptyset$.
Since $\epsilon > 0$, then $\frac{\epsilon}{2} > 0$, so $d(p + \frac{\epsilon}{2}, p) = |(p + \frac{\epsilon}{2}) - p| = |\frac{\epsilon}{2}| = \frac{\epsilon}{2} < \epsilon$.
Thus, $p + \frac{\epsilon}{2} \in N(p; \epsilon)$.
Since $\frac{\epsilon}{2} > 0$, then $p + \frac{\epsilon}{2} > p$, so $p + \frac{\epsilon}{2} \neq p$.
Hence, $p + \frac{\epsilon}{2} \in N'(p; \epsilon)$, so $N'(p; \epsilon) \neq \emptyset$.

We prove $N'(p;\epsilon) \subset (p-\epsilon, p) \cup (p, p+\epsilon)$.
Let $x \in N'(p;\epsilon)$.
Then $x \in N(p;\epsilon)$ and $x \neq p$.
Since $x \in N(p;\epsilon) = (p-\epsilon, p+\epsilon)$, then $p-\epsilon < x < p+\epsilon$, so $p-\epsilon < x$ and $x < p+\epsilon$.
Since $x \neq p$, then either $x < p$ or $x > p$.
We consider these cases separately.
Case 1: Suppose $x < p$.
Since $p-\epsilon < x$ and $x < p$, then $p-\epsilon < x < p$, so $x \in (p-\epsilon, p)$.
Case 2: Suppose $x > p$.
Since $p < x$ and $x < p+\epsilon$, then $p < x < p+\epsilon$, so $x \in (p, p+\epsilon)$.
Thus, either $x \in (p-\epsilon, p)$ or $x \in (p, p+\epsilon)$, so $x \in (p-\epsilon, p) \cup (p, p+\epsilon)$.
Hence, $N'(p;\epsilon) \subset (p-\epsilon, p) \cup (p, p+\epsilon)$.

We prove $(p-\epsilon, p) \cup (p, p+\epsilon) \subset N'(p;\epsilon)$.
Let $y \in (p-\epsilon, p) \cup (p, p+\epsilon)$.
Then either $y \in (p-\epsilon, p)$ or $y \in (p, p+\epsilon)$.
We consider these cases separately.
Case 1: Suppose $y \in (p-\epsilon, p)$.
Then $p-\epsilon < y < p$, so $p-\epsilon < y$ and $y < p$.
Thus, $p-y < \epsilon$ and $0 < p-y$.
Hence, $d(y,p) = d(p,y) = |p-y| = p-y < \epsilon$, so $y \in N(p;\epsilon)$.
Since $y < p$, then $y \neq p$, so $y \in N'(p;\epsilon)$.
Case 2: Suppose $y \in (p, p+\epsilon)$.
Then $p < y < p+\epsilon$, so $p < y$ and $y < p+\epsilon$.
Thus, $0 < y-p$ and $y-p < \epsilon$.
Hence, $d(y,p) = |y-p| = y-p < \epsilon$, so $y \in N(p;\epsilon)$.
Since $y > p$, then $y \neq p$, so $y \in N'(p;\epsilon)$.
Thus, in all cases, $y \in N'(p;\epsilon)$, so $(p-\epsilon, p) \cup (p, p+\epsilon) \subset N'(p;\epsilon)$.

Since $N'(p;\epsilon) \subset (p-\epsilon, p) \cup (p, p+\epsilon)$ and $(p-\epsilon, p) \cup (p, p+\epsilon) \subset N'(p;\epsilon)$, then $N'(p;\epsilon) = (p-\epsilon, p) \cup (p, p+\epsilon)$, as desired. ☐

Example 3. $\emptyset$ has no interior points.
There is no interior point of the empty set.

Proof. Let x be an arbitrary real number.
Let $\epsilon > 0$ be given.
Since $x \in N(x;\epsilon)$ and $x \notin \emptyset$, then x is not an interior point of $\emptyset$.
Thus, every real number is not an interior point of $\emptyset$, so there is no real number that is an interior point of $\emptyset$.
Therefore, there is no interior point of the empty set. ☐

Example 4. A singleton set has no interior points.
Let $p \in \mathbb{R}$.
Then p is not an interior point of the set $\{p\}$.

Proof. Let $\epsilon > 0$ be given.

Let x be the midpoint of p and $p+\epsilon$.

Then $x = \frac{p+p+\epsilon}{2} = p + \frac{\epsilon}{2}$, so $x - p = \frac{\epsilon}{2}$.

Since $\epsilon > 0$, then $\frac{\epsilon}{2} > 0$, so $x - p > 0$.

Since $\frac{1}{2} < 1$ and $\epsilon > 0$, then $\frac{\epsilon}{2} < \epsilon$.

Since $d(x,p) = |x-p| = x - p = \frac{\epsilon}{2} < \epsilon$, then $d(x,p) < \epsilon$, so $x \in N(p;\epsilon)$.

Since $x - p > 0$, then $x > p$, so $x \neq p$.

Thus, $x \notin \{p\}$.

Hence, there exists $x \in N(p;\epsilon)$ such that $x \notin \{p\}$.

Therefore, p is not an interior point of $\{p\}$. □

Example 5. Let $a, b \in \mathbb{R}$ with $a < b$.

Every point in the open interval (a,b) is an interior point of the open interval (a,b).

Proof. Since $a < b$, then $(a,b) \neq \emptyset$.

Let $p \in (a,b)$.

Then $p \in \mathbb{R}$ and $a < p < b$, so $a < p$ and $p < b$.

Hence, $p - a > 0$ and $b - p > 0$.

Let $\epsilon = \min\{d(p,a), d(p,b)\}$.

Then $\epsilon \leq d(p,a)$ and $\epsilon \leq d(p,b)$.

Since $d(p,a) = |p-a| = p - a > 0$, then $d(p,a) > 0$.

Since $d(p,b) = d(b,p) = |b-p| = b - p > 0$, then $d(p,b) > 0$.

Therefore, $\epsilon > 0$.

Let $x \in N(p;\epsilon)$.

Then $x \in (p-\epsilon, p+\epsilon)$, so $p - \epsilon < x < p + \epsilon$.

Hence, $p - \epsilon < x$ and $x < p + \epsilon$.

Since $\epsilon \leq d(p,a)$ and $d(p,a) = p - a$, then $\epsilon \leq p - a$, so $a \leq p - \epsilon$.

Since $\epsilon \leq d(p,b)$ and $d(p,b) = b - p$, then $\epsilon \leq b - p$, so $p + \epsilon \leq b$.

Since $a \leq p - \epsilon$ and $p - \epsilon < x$, then $a < x$.

Since $x < p + \epsilon$ and $p + \epsilon \leq b$, then $x < b$.

Thus, $a < x < b$, so $x \in (a,b)$.

Hence, $N(p;\epsilon) \subset (a,b)$.

Thus, there exists $\epsilon > 0$ such that $N(p;\epsilon) \subset (a,b)$, so p is an interior point of (a,b).

Therefore, every element of (a,b) is an interior point of (a,b). □

Example 6. Let $a, b \in \mathbb{R}$ with $a < b$.

Every point in the closed interval $[a,b]$ except the end points a and b is an interior point of the closed interval $[a,b]$.

Proof. Since $a < b$, then $[a,b] \neq \emptyset$.

Let $p \in [a,b]$.

Since $[a,b] = \{a\} \cup (a,b) \cup \{b\}$, then either $p = a$ or $p \in (a,b)$ or $p = b$.

We consider these cases separately.

Case 1: Suppose $p \in (a,b)$.

Since every element of (a, b) is an interior point of (a, b), then p is an interior point of (a, b).
Since $(a, b) \subset [a, b]$, then p is an interior point of $[a, b]$.
Case 2: Suppose $p = a$.
We prove a is not an interior point of $[a, b]$.
Let $\epsilon > 0$ be given.
Let x be the midpoint of $a - \epsilon$ and a.
Then $x = \frac{a-\epsilon+a}{2} = a - \frac{\epsilon}{2}$.
Thus, $a - x = \frac{\epsilon}{2} > 0$, so $a - x > 0$.
Hence, $x < a$, so $x \notin [a, b]$.
Since $d(a, x) = |a - x| = a - x = \frac{\epsilon}{2} < \epsilon$, then $d(a, x) < \epsilon$, so $x \in N(a; \epsilon)$.
Thus, there exists $x \in N(a; \epsilon)$ such that $x \notin [a, b]$.
Therefore, a is not an interior point of $[a, b]$.
Case 3: Suppose $p = b$.
We prove b is not an interior point of $[a, b]$.
Let $\epsilon > 0$ be given.
Let x be the midpoint of b and $b + \epsilon$.
Then $x = \frac{b+b+\epsilon}{2} = b + \frac{\epsilon}{2}$.
Thus, $x - b = \frac{\epsilon}{2} > 0$, so $x - b > 0$.
Hence, $x > b$, so $x \notin [a, b]$.
Since $d(x, b) = |x - b| = x - b = \frac{\epsilon}{2} < \epsilon$, then $d(x, b) < \epsilon$, so $x \in N(b; \epsilon)$.
Thus, there exists $x \in N(b; \epsilon)$ such that $x \notin [a, b]$.
Therefore, b is not an interior point of $[a, b]$. □

Example 7. No natural number is an interior point of $\mathbb{N}$.

Proof. We prove by contradiction.
Suppose some natural number is an interior point of $\mathbb{N}$.
Then there exists $n \in \mathbb{N}$ such that n is an interior point of $\mathbb{N}$.
Hence, there exists $\epsilon > 0$ such that $N(n; \epsilon) \subset \mathbb{N}$.
Either $\epsilon < 1$ or $\epsilon = 1$ or $\epsilon > 1$.
We consider these cases separately.
Case 1: Suppose $\epsilon < 1$.
Then $0 < \epsilon < 1$, so $0 < \frac{\epsilon}{2} < \frac{1}{2} < 1$.
Hence, $n < n + \frac{\epsilon}{2} < n + \frac{1}{2} < n + 1$.
Let $p = n + \frac{\epsilon}{2}$.
Since $n < n + \frac{\epsilon}{2} < n + 1$ and $n, n + 1 \in \mathbb{N}$, then $n + \frac{\epsilon}{2} \notin \mathbb{N}$, so $p \notin \mathbb{N}$.
Since $n < n + \frac{\epsilon}{2} < n + \epsilon$, then $n + \frac{\epsilon}{2} \in (n, n + \epsilon)$, so $p \in (n, n + \epsilon)$.
Since $(n, n + \epsilon) \subset (n - \epsilon, n + \epsilon)$, then $p \in (n - \epsilon, n + \epsilon)$.
Thus, $p \in N(n; \epsilon)$.
Therefore, there exists p such that $p \in N(n; \epsilon)$ and $p \notin \mathbb{N}$.
This contradicts the fact that $N(n; \epsilon) \subset \mathbb{N}$.
Case 2: Suppose $\epsilon = 1$.
Since $0 < \frac{1}{2} < 1$, then $n < n + \frac{1}{2} < n + 1$.
Since $n, n + 1 \in \mathbb{N}$, then $n + \frac{1}{2} \notin \mathbb{N}$.
Let $p = n + \frac{1}{2}$.

Then $p \notin \mathbb{N}$.
Since $n < n + \frac{1}{2} < n + 1$, then $n + \frac{1}{2} \in (n, n+1)$, so $p \in (n, n+1)$.
Since $(n, n+1) \subset (n-1, n+1)$, then $p \in (n-1, n+1)$, so $p \in N(n; 1)$.
Hence, there exists p such that $p \in N(n; \epsilon)$ and $p \notin \mathbb{N}$.
This contradicts the fact that $N(n; \epsilon) \subset \mathbb{N}$.
Case 3: Suppose $\epsilon > 1$.
Then $n + \epsilon > n + 1$.
Let p be the midpoint of n and $n + 1$.
Then $p = n + \frac{1}{2}$.
Since $0 < \frac{1}{2} < 1$, then $n < n + \frac{1}{2} < n + 1$.
Since $n, n+1 \in \mathbb{N}$, then $n + \frac{1}{2} \notin \mathbb{N}$, so $p \notin \mathbb{N}$.
Since $n < n + \frac{1}{2} < n + 1$, then $n < p < n + 1$, so $n < p$ and $p < n + 1$.
Since $p < n + 1$ and $n + 1 < n + \epsilon$, then $p < n + \epsilon$.
Since $n < p$ and $p < n + \epsilon$, then $p \in (n, n + \epsilon)$.
Since $(n, n + \epsilon) \subset (n - \epsilon, n + \epsilon)$, then $p \in (n - \epsilon, n + \epsilon)$, so $p \in N(n; \epsilon)$.
Hence, there exists p such that $p \in N(n; \epsilon)$ and $p \notin \mathbb{N}$.
This contradicts the fact that $N(n; \epsilon) \subset \mathbb{N}$.
Thus, in all cases, a contradiction is reached.
Therefore, no natural number is an interior point of $\mathbb{N}$. □

Example 8. No rational number is an interior point of $\mathbb{Q}$.

Proof. We prove by contradiction.
Suppose some rational number is an interior point of $\mathbb{Q}$.
Then there exists $q \in \mathbb{Q}$ such that q is an interior point of $\mathbb{Q}$.
Hence, there exists $\delta > 0$ such that $N(q; \delta) \subset \mathbb{Q}$, so $(q - \delta, q + \delta) \subset \mathbb{Q}$.
Since $q \in \mathbb{Q}$ and $\mathbb{Q} \subset \mathbb{R}$, then $q \in \mathbb{R}$, so $q - \delta \in \mathbb{R}$ and $q + \delta \in \mathbb{R}$.
Since $\delta > 0$ and $\delta > 0 \Rightarrow \delta + \delta > 0 \Rightarrow \delta + \delta > q - q \Rightarrow q + \delta > q - \delta$, then $q + \delta > q - \delta$.
Thus, $q - \delta \in \mathbb{R}$ and $q + \delta \in \mathbb{R}$ and $q - \delta < q + \delta$.
Between any two distinct real numbers is an irrational number, so there exists $r \in \mathbb{R}$ such that $r \notin \mathbb{Q}$ and $q - \delta < r < q + \delta$.
Hence, $r \in (q - \delta, q + \delta)$, so $r \in N(q; \delta)$.
Thus, there exists $r \in N(q; \delta)$ such that $r \notin \mathbb{Q}$, which contradicts the fact that $N(q; \delta) \subset \mathbb{Q}$.
Therefore, no rational number is an interior point of $\mathbb{Q}$. □

Example 9. Every real number is an interior point of $\mathbb{R}$.

Proof. Let x be an arbitrary real number.
Let $\delta = 1$.
Since $1 > 0$, then $\delta > 0$.
Since $N(x; \delta) = N(x; 1) = (x - 1, x + 1) \subset \mathbb{R}$, then $N(x; \delta) \subset \mathbb{R}$.
Hence, there exists $\delta > 0$ such that $N(x; \delta) \subset \mathbb{R}$.
Therefore, x is an interior point of $\mathbb{R}$. □

Example 10. accumulation point of a set need not lie in the set

Let $S = (0, 1)$.

Then 1 is an accumulation point of S, but $1 \notin S$.

Proof. Since $1 < 1$ is false, then $1 \notin (0, 1)$.

To prove 1 is an accumulation point of S, let $\delta > 0$ be given.

We must prove there exists $x \in (0, 1)$ such that $x \in N'(1; \delta)$.

Let $M = \max\{1 - \delta, 0\}$.

Then $M \in \mathbb{R}$ and either $M = 1 - \delta$ or $M = 0$, and $1 - \delta \leq M$ and $0 \leq M$.

Since $\delta > 0$, then $\delta > 1 - 1$, so $1 > 1 - \delta$.

Since either $M = 1 - \delta$ or $M = 0$, and $1 - \delta < 1$ and $0 < 1$, then $M < 1$.

Let $x = \frac{M+1}{2}$.

Since $M + 1 > M$ and $M \geq 0$, then $M + 1 > 0$, so $\frac{M+1}{2} > 0$.

Hence, $x > 0$.

Since $M < 1$, then $M + 1 < 2$, so $\frac{M+1}{2} < 1$.

Thus, $x < 1$.

Therefore, $0 < x < 1$, so $x \in (0, 1)$.

Since $\delta > 0$, then $\delta + M > M$.

Since $1 - \delta \leq M$ and $M < \delta + M$, then $1 - \delta < \delta + M$, so $1 - M < 2\delta$.

Since $M < 1$, then $M - 1 < 0$, so $|M - 1| = 1 - M < 2\delta$.

Hence, $|M - 1| < 2\delta$, so $\frac{|M-1|}{2} < \delta$.

Thus, $|\frac{M-1}{2}| < \delta$, so $|\frac{M+1}{2} - 1| < \delta$.

Therefore, $|x - 1| < \delta$, so $x \in N(1; \delta)$.

Since $x < 1$, then $x \neq 1$, so $x \in N'(1; \delta)$, as desired. □

Example 11. point of a set need not be an accumulation point

Let $S = (0, 1) \cup \{2\}$.

Then $2 \in S$, but 2 is not an accumulation point of S.

Proof. Clearly, $2 \in S$.

To prove 2 is not an accumulation point of S, we must prove there exists $\delta > 0$ such that $N'(2; \delta) \cap S = \emptyset$.

Let $\delta = 1$.

We prove $N'(2; 1) \cap S = \emptyset$ by contradiction.

Suppose $N'(2; 1) \cap S \neq \emptyset$.

Then there exists $x \in N'(2; 1) \cap S$, so $x \in N'(2; 1)$ and $x \in S$.

Since $x \in N'(2; 1)$, then $x \in N(2; 1)$ and $x \neq 2$.

Since $x \in N(2; 1)$, then $x \in (1, 3)$, so $1 < x < 3$.

Hence, $1 < x$.

Since $x \in S$, then either $x \in (0, 1)$ or $x = 2$.

Since $x \neq 2$, then $x \in (0, 1)$, so $0 < x < 1$.

Hence, $x < 1$.

Thus, we have $x > 1$ and $x < 1$, a violation of trichotomy.

Therefore, $N'(2; 1) \cap S = \emptyset$, so 2 is not an accumulation point of S, as desired. □

Example 12. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Then 0 is an accumulation point of S and 1 is not an accumulation point of S.

Proof. To prove 0 is an accumulation point of S, let $\delta > 0$ be given.

We must prove there exists $x \in S$ such that $x \in N'(0;\delta)$.

Since $\delta > 0$, then $\frac{1}{\delta} > 0$, so by the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\delta}$.

Since $n > 0$ and $\delta > 0$, then $\delta > \frac{1}{n}$.

Let $x = \frac{1}{n}$.

Since $n \in \mathbb{N}$, then $\frac{1}{n} \in S$, so $x \in S$.

Since $|\frac{1}{n} - 0| = \frac{1}{n} < \delta$, then $\frac{1}{n} \in N(0;\delta)$.

Since $n > 0$, then $\frac{1}{n} > 0$, so $\frac{1}{n} \neq 0$.

Thus, $\frac{1}{n} \in N'(0;\delta)$, as desired. □

Proof. To prove 1 is not an accumulation point of S, we must prove there exists $\delta > 0$ such that $N'(1;\delta) \cap S = \emptyset$.

Let $\delta = \frac{1}{2}$. (Any $\delta \leq \frac{1}{2}$ will work).

We prove $N'(1;\frac{1}{2}) \cap S = \emptyset$ by contradiction.

Suppose $N'(1;\frac{1}{2}) \cap S \neq \emptyset$.

Then there exists $x \in N'(1;\frac{1}{2}) \cap S$, so $x \in N'(1;\frac{1}{2})$ and $x \in S$.

Since $x \in N'(1;\frac{1}{2})$, then $x \in N(1;\frac{1}{2}) - \{1\}$, so $x \in (\frac{1}{2}, \frac{3}{2})$ and $x \neq 1$.

Hence, $\frac{1}{2} < x < \frac{3}{2}$ and $x \neq 1$.

Since $x \in S$, then there exists $n \in \mathbb{N}$ such that $x = \frac{1}{n}$, so $\frac{1}{2} < \frac{1}{n} < \frac{3}{2}$.

Hence, $\frac{1}{2} < \frac{1}{n}$.

Since $n > 0$, then $\frac{n}{2} < 1$, so $n < 2$.

Since $n \in \mathbb{N}$ and $n < 2$, then $n = 1$, so $x = \frac{1}{1} = 1$.

Thus, we have $x = 1$ and $x \neq 1$, a contradiction.

Therefore, $N'(1;\frac{1}{2}) \cap S = \emptyset$, so 1 is not an accumulation point of S, as desired. □

Example 13. $\emptyset$ has no accumulation points.

There is no accumulation point of the empty set.

Proof. To prove there is no accumulation point of $\emptyset$, let p be an arbitrary real number.

To prove p is not an accumulation point of $\emptyset$, we must prove there exists $\delta > 0$ such that $N'(p;\delta) \cap \emptyset = \emptyset$.

Let $\delta = 1$.

Then $\delta > 0$ and $N'(p;1) \cap \emptyset = \emptyset$.

Therefore, p is not an accumulation point of $\emptyset$, as desired. □

Example 14. A singleton set has no accumulation points.

Let $x \in \mathbb{R}$.

There is no accumulation point of the set $\{x\}$.

Proof. To prove there is no accumulation point of $\{x\}$, let p be an arbitrary real number.

To prove p is not an accumulation point of $\{x\}$, we must prove there exists $\delta > 0$ such that $N'(p;\delta) \cap \{x\} = \emptyset$.

Either $p = x$ or $p \neq x$.

We consider these cases separately.

Case 1: Suppose $p = x$.

Let $\delta = 1$.

Then $\delta > 0$.

Suppose $N'(p;1) \cap \{x\} \neq \emptyset$.

Then there exists $t \in N'(p;1) \cap \{x\}$, so $t \in N'(p;1)$ and $t \in \{x\}$.

Hence, $t = x$, so $x \in N'(p;1)$.

Thus, $p \in N'(p;1)$.

But, $p \notin N'(p;1)$, so $N'(p;1) \cap \{x\} = \emptyset$.

Therefore, p is not an accumulation point of $\{x\}$.

Case 2: Suppose $p \neq x$.

Then $x \neq p$, so $d(x,p) > 0$.

Let $\delta = d(x,p)$.

Then $\delta > 0$.

Suppose $N'(p;\delta) \cap \{x\} \neq \emptyset$.

Then there exists $t \in N'(p;\delta) \cap \{x\}$, so $t \in N'(p;\delta)$ and $t \in \{x\}$.

Hence, $t = x$, so $x \in N'(p;\delta)$.

Thus, $x \in N(p;\delta)$, so $d(x,p) < \delta$.

But, this contradicts the fact that $d(x,p) = \delta$.

Hence, $N'(p;\delta) \cap \{x\} = \emptyset$.

Therefore, p is not an accumulation point of $\{x\}$.

Thus, in all cases, p is not an accumulation point of $\{x\}$, as desired. □

Example 15. A finite set has no accumulation points.

There is no accumulation point of a finite set.

Proof. This statement is equivalent to : If S is a finite set, then there is no accumulation point of S.

We prove by contrapositive.

Let S be a set.

Suppose there is an accumulation point of S.

Then there exists p such that p is an accumulation point of S.

Thus, for every $\epsilon > 0$, there exists $x \in S$ such that $x \in N'(p;\epsilon)$.

Let $\epsilon = \frac{1}{n}$ for each $n \in \mathbb{N}$.

Then $\epsilon > 0$ for each $n \in \mathbb{N}$, so there exists $x \in S$ for each $n \in \mathbb{N}$.

Thus, there exists a function $f : \mathbb{N} \to S$ such that $f(n) \in S$.

Hence, there exists an infinite sequence (s_n) such that $s_n \in S$.

Therefore, $\{s_n : n \in \mathbb{N}\} \subset S$.

Since a subset of a finite set is finite, then if $\{s_n : n \in \mathbb{N}\} \subset S$ and S is finite, then $\{s_n : n \in \mathbb{N}\}$ is finite.

Thus, if $\{s_n : n \in \mathbb{N}\} \subset S$ and $\{s_n : n \in \mathbb{N}\}$ is infinite, then S is infinite.

Since $\{s_n : n \in \mathbb{N}\} \subset S$ and $\{s_n : n \in \mathbb{N}\}$ is infinite, then we conclude S is infinite, as desired. ☐

Example 16. Every point in $[a, b]$ is an accumulation point of (a, b).

Let $a, b \in \mathbb{R}$.

If $a < b$, then every point in the closed interval $[a, b]$ is an accumulation point of the open interval (a, b).

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$.

Then $[a, b] \neq \emptyset$, so there is at least one point in the closed interval $[a, b]$.

We prove every point in $[a, b]$ is an accumulation point of (a, b).

Let $x \in [a, b]$.

To prove x is an accumulation point of (a, b), let $\delta > 0$ be given.

We must prove there exists $p \in (a, b)$ such that $p \in N'(x; \delta)$.

Let $M = \max\{a, x - \delta\}$.

Then either $M = a$ or $M = x - \delta$, and $a \leq M$ and $x - \delta \leq M$.

Since $x \in [a, b]$, then $a \leq x \leq b$, so either $a \leq x < b$ or $x = b$.

We consider these cases separately.

Case 1: Suppose $a \leq x < b$.

Then $a \leq x$ and $x < b$.

Let $m = \min\{b, x + \delta\}$.

Then either $m = b$ or $m = x + \delta$, and $m \leq b$ and $m \leq x + \delta$.

Since either $m = b$ or $m = x + \delta$ and $x < b$ and $x < x + \delta$, then $x < m$.

Let p be the midpoint of x and m.

Then $p = \frac{x+m}{2}$ and $x < p < m$, so $x < p$ and $p < m$.

Since either $M = a$ or $M = x - \delta$ and $a \leq x$ and $x - \delta < x$, then $M \leq x$.

Thus, $x - \delta < x < p < m \leq x + \delta$, so $x - \delta < p < x + \delta$.

Hence, $p \in (x - \delta, x + \delta)$, so $p \in N(x; \delta)$.

Since $p > x$, then $p \neq x$, so $p \in N'(x; \delta)$.

Since $a \leq x < p < m \leq b$, then $a < p < b$, so $p \in (a, b)$.

Thus, there exists $p \in (a, b)$ such that $p \in N'(x; \delta)$.

Case 2: Suppose $x = b$.

Since either $M = a$ or $M = x - \delta$, and $a < b = x$ and $x - \delta < x$, then $M < x$.

Let p be the midpoint of M and x.

Then $p = \frac{M+x}{2}$ and $M < p < x$, so $M < p$ and $p < x$.

Since $x - \delta \leq M < p < x < x + \delta$, then $x - \delta < p < x + \delta$, so $p \in (x - \delta, x + \delta)$.

Hence, $p \in N(x; \delta)$.

Since $p < x$, then $p \neq x$, so $p \in N'(x; \delta)$.

Since $a \leq M < p < x = b$, then $a < p < b$, so $p \in (a, b)$.

Thus, there exists $p \in (a, b)$ such that $p \in N'(x; \delta)$.

Hence, in all cases, there exists $p \in (a, b)$ such that $p \in N'(x; \delta)$, so x is an accumulation point of (a, b). ☐

Example 17. Every point in $[a, b]$ is an accumulation point of $[a, b]$.

Let $a, b \in \mathbb{R}$.

If $a < b$, then every point in the closed interval $[a, b]$ is an accumulation point of the closed interval $[a, b]$.

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$.

Then $[a, b] \neq \emptyset$, so there is at least one point in the closed interval $[a, b]$.

Let $x \in [a, b]$.

Since every point in $[a, b]$ is an accumulation point of the open interval (a, b), then x is an accumulation point of (a, b).

Since $(a, b) \subset [a, b]$, then x is an accumulation point of $[a, b]$.

Therefore, every point in $[a, b]$ is an accumulation point of $[a, b]$. □

Example 18. **$\mathbb{N}$ has no accumulation points.**

No integer is an accumulation point of $\mathbb{N}$.

No point in $\mathbb{R}$ is an accumulation point of $\mathbb{N}$.

Proof. We prove the statement 'no integer is an accumulation point of $\mathbb{N}$' by contradiction.

Suppose some integer is an accumulation point of $\mathbb{N}$.

Then there exists $n \in \mathbb{Z}$ such that n is an accumulation point of $\mathbb{N}$.

Thus, for every $\delta > 0$ there exists $k \in \mathbb{N}$ such that $k \in N'(n; \delta)$.

Let $\delta = \frac{1}{2}$.

Then there exists $k \in \mathbb{N}$ such that $k \in N'(n; \frac{1}{2})$.

Since $k \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{Z}$, then $k \in \mathbb{Z}$.

Since $N'(n; \frac{1}{2}) = N(n; \frac{1}{2}) - \{n\} = (n - \frac{1}{2}, n + \frac{1}{2}) - \{n\} = (n - \frac{1}{2}, n) \cup (n, n + \frac{1}{2})$, then $k \in (n - \frac{1}{2}, n) \cup (n, n + \frac{1}{2})$.

Thus, either $k \in (n - \frac{1}{2}, n)$ or $k \in (n, n + \frac{1}{2})$.

We consider these cases separately.

Case 1: Suppose $k \in (n - \frac{1}{2}, n)$.

Then $n - \frac{1}{2} < k < n$.

Since $n - 1 < n - \frac{1}{2} < k < n$, then $n - 1 < k < n$, so k is an integer between two consecutive integers, a contradiction.

(WE NEED TO PROVE THIS!)

Case 2: Suppose $k \in (n, n + \frac{1}{2})$.

Then $n < k < n + \frac{1}{2}$.

Since $n < k < n + \frac{1}{2} < n + 1$, then $n < k < n + 1$, so k is an integer between two consecutive integers, a contradiction.

Thus, in all cases, a contradiction is reached.

Therefore, no integer is an accumulation point of $\mathbb{N}$. □

Proof. We prove the statement 'no point in $\mathbb{R}$ is an accumulation point of $\mathbb{N}$' by contradiction.

Suppose some point in $\mathbb{R}$ is an accumulation point of $\mathbb{N}$.

Then there exists $x \in \mathbb{R}$ such that x is an accumulation point of $\mathbb{N}$.

Since each real number lies between two consecutive integers, then there is a unique integer n such that $n \leq x < n + 1$.

Thus, $n \leq x$ and $x < n + 1$, so either $n < x$ or $n = x$.

We consider these cases separately.

Case 1: Suppose $n = x$.

Then $x \in \mathbb{Z}$.

Since no integer is an accumulation point of $\mathbb{N}$, then every integer is not an accumulation point of $\mathbb{N}$.

In particular, x is not an accumulation point of $\mathbb{N}$.

But, this contradicts the fact that x is an accumulation point of $\mathbb{N}$.

Case 2: Suppose $n < x$.

Then $x - n > 0$.

Since $x < n + 1$, then $n + 1 - x > 0$.

Let $\delta = \min\{d(x, n), d(x, n + 1)\}$.

Then $\delta \leq d(x, n)$ and $\delta \leq d(x, n + 1)$.

Since $d(x, n) = |x - n| = x - n > 0$ and $d(x, n + 1) = |x - (n + 1)| = |(n + 1) - x| = n + 1 - x > 0$, then $\delta > 0$.

Since x is an accumulation point of $\mathbb{N}$ and $\delta > 0$, then there exists $k \in \mathbb{N}$ such that $k \in N'(x; \delta)$.

Since $k \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{Z}$, then $k \in \mathbb{Z}$.

Since $k \in N'(x; \delta)$, then $k \in N(x; \delta)$, so $k \in (x - \delta, x + \delta)$.

Hence, $x - \delta < k < x + \delta$, so $x - \delta < k$ and $k < x + \delta$.

Since $\delta \leq d(x, n) = x - n$, then $\delta \leq x - n$, so $n \leq x - \delta$.

Since $n \leq x - \delta$ and $x - \delta < k$, then $n < k$.

Since $\delta \leq d(x, n + 1) = n + 1 - x$, then $\delta \leq n + 1 - x$, so $x + \delta \leq n + 1$.

Since $k < x + \delta$ and $x + \delta \leq n + 1$, then $k < n + 1$.

Thus, $n < k < n + 1$, so k is an integer between two consecutive integers.

But, this contradicts the fact that there is no integer between two consecutive integers. (WE NEED TO PROVE THIS!)

Therefore, in all cases, a contradiction is reached, so no point in $\mathbb{R}$ is an accumulation point of $\mathbb{N}$. □

Example 19. Every real number is an accumulation point of $\mathbb{Q}$.

Proof. Let x be an arbitrary real number.

Let $\delta > 0$.

Then $\delta > x - x$, so $x + \delta > x$ and $x > x - \delta$.

Since $x < x + \delta$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $q \in \mathbb{Q}$ such that $x < q < x + \delta$.

Since $x - \delta < x < q < x + \delta$, then $x - \delta < q < x + \delta$ and $x < q$.

Since $x - \delta < q < x + \delta$, then $q \in (x - \delta, x + \delta)$, so $q \in N(x; \delta)$.

Since $q > x$, then $q \neq x$, so $q \in N'(x; \delta)$.

Therefore, x is an accumulation point of $\mathbb{Q}$, as desired. □

Example 20. Every real number is an accumulation point of $\mathbb{R}$.

Proof. Let x be an arbitrary real number.

Let $\delta > 0$.

Then $\delta > x - x$, so $x + \delta > x$ and $x > x - \delta$.

Let r be the midpoint of x and $x + \delta$.

Then $r = x + \frac{\delta}{2}$ and $x < r < x + \delta$.

Since $x - \delta < x < r < x + \delta$, then $x - \delta < r < x + \delta$ and $x < r$.

Since $x - \delta < r < x + \delta$, then $r \in (x - \delta, x + \delta)$, so $r \in N(x; \delta)$.

Since $r > x$, then $r \neq x$, so $r \in N'(x;\delta)$.
Therefore, x is an accumulation point of $\mathbb{R}$, as desired. $\square$

Example 21. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.
Then each element of S is an isolated point of S.

Proof. Let $x \in S$.
Then $x = \frac{1}{n}$ for some $n \in \mathbb{N}$.
Since $n \in \mathbb{N}$, then $n \geq 1$, so either $n > 1$ or $n = 1$.
We consider these cases separately.
Case 1: Suppose $n = 1$.
Then $x = \frac{1}{1} = 1$.
Let $\delta = \frac{1}{2}$.
To prove 1 is an isolated point of S, we must prove $N'(1;\frac{1}{2}) \cap S = \emptyset$.
We prove $N'(1;\frac{1}{2}) \cap S = \emptyset$ by contradiction.
Suppose $N'(1;\frac{1}{2}) \cap S \neq \emptyset$.
Then there exists s such that $s \in N'(1;\frac{1}{2}) \cap S$, so $s \in N'(1;\frac{1}{2})$ and $s \in S$.
Since $s \in N'(1;\frac{1}{2})$, then $s \in N(1;\frac{1}{2}) - \{1\}$, so $s \in (\frac{1}{2},\frac{3}{2}) - \{1\}$.
Hence, $\frac{1}{2} < s < \frac{3}{2}$ and $s \neq 1$.
Since $s \in S$, then $s = \frac{1}{m}$ for some $m \in \mathbb{N}$.
Thus, $\frac{1}{2} < \frac{1}{m} < \frac{3}{2}$ and $\frac{1}{m} \neq 1$.
Since $\frac{1}{m} = 1$ iff $m = 1$ and $\frac{1}{m} \neq 1$, then $m \neq 1$.
Since $m \in \mathbb{N}$ and $m \neq 1$, then $m > 1$, so $m \geq 2$.
Since $m > 0$, then $1 \geq \frac{2}{m}$, so $\frac{1}{2} \geq \frac{1}{m}$.
Since $\frac{1}{2} < \frac{1}{m} < \frac{3}{2}$, then $\frac{1}{2} < \frac{1}{m}$.
Thus, we have $\frac{1}{m} \leq \frac{1}{2}$ and $\frac{1}{m} > \frac{1}{2}$, a violation of trichotomy.
Therefore, $N'(1;\frac{1}{2}) \cap S = \emptyset$.
Hence, 1 is an isolated point of S.
Case 2: Suppose $n > 1$.
Then $n - 1 > 0$.
To prove $\frac{1}{n}$ is an isolated point of S, we must prove there exists $\delta > 0$ such that $N'(\frac{1}{n};\delta) \cap S = \emptyset$.
Let $\delta = \min\{d(\frac{1}{n},\frac{1}{n-1}), d(\frac{1}{n},\frac{1}{n+1})\}$.
Then $\delta = \min\{|\frac{1}{n} - \frac{1}{n-1}|, |\frac{1}{n} - \frac{1}{n+1}|\}$, so $\delta = \min\{\frac{1}{n(n-1)}, \frac{1}{n(n+1)}\}$.
Since $n + 1 > n - 1 > 0$ and $n > 0$, then $n(n+1) > n(n-1) > 0$, so $0 < \frac{1}{n(n+1)} < \frac{1}{n(n-1)}$.
Thus, $\delta = \frac{1}{n(n+1)}$ and $\delta > 0$.
We prove $N'(\frac{1}{n};\frac{1}{n(n+1)}) \cap S = \emptyset$ by contradiction.
Suppose $N'(\frac{1}{n};\frac{1}{n(n+1)}) \cap S \neq \emptyset$.
Then there exists $t \in \mathbb{R}$ such that $t \in N'(\frac{1}{n};\frac{1}{n(n+1)}) \cap S$, so $t \in N'(\frac{1}{n};\frac{1}{n(n+1)})$ and $t \in S$.
Since $t \in N'(\frac{1}{n};\frac{1}{n(n+1)})$ and $N'(\frac{1}{n};\frac{1}{n(n+1)}) = N(\frac{1}{n};\frac{1}{n(n+1)}) - \{\frac{1}{n}\} = (\frac{1}{n} - \frac{1}{n(n+1)}, \frac{1}{n} + \frac{1}{n(n+1)}) - \{\frac{1}{n}\}$, then $\frac{1}{n} - \frac{1}{n(n+1)} < t < \frac{1}{n} + \frac{1}{n(n+1)}$ and $t \neq \frac{1}{n}$.
Since $t \in S$, then there exists $m \in \mathbb{N}$ such that $t = \frac{1}{m}$.
Therefore, $\frac{1}{n} - \frac{1}{n(n+1)} < \frac{1}{m} < \frac{1}{n} + \frac{1}{n(n+1)}$ and $\frac{1}{m} \neq \frac{1}{n}$.

Since $\frac{1}{n} - \frac{1}{n(n+1)} < \frac{1}{m} < \frac{1}{n} + \frac{1}{n(n+1)}$, then $\frac{1}{n} - \frac{1}{n(n+1)} - \frac{1}{m} < 0 < \frac{1}{n} + \frac{1}{n(n+1)} - \frac{1}{m}$.

Hence, $\frac{1}{n} - \frac{1}{n(n+1)} - \frac{1}{m} < 0$ and $0 < \frac{1}{n} + \frac{1}{n(n+1)} - \frac{1}{m}$.

Since $\frac{1}{m} = \frac{1}{n}$ iff $m = n$ and $\frac{1}{m} \neq \frac{1}{m}$, then $m \neq n$.

Thus, either $m < n$ or $m > n$.

We consider these cases separately.

Case 2a: Suppose $m < n$.

Then $n - m > 0$, so $n - m \geq 1$.

Since $0 < m < n$, then $0 < \frac{m}{n} < 1$.

Since $n + 1 > n > 0$, then $0 < \frac{1}{n+1} < \frac{1}{n}$.

Since $m > 0$, then $0 < \frac{m}{n+1} < \frac{m}{n}$.

Thus, $0 < \frac{m}{n+1} < \frac{m}{n} < 1 \leq n - m$, so $\frac{m}{n+1} < n - m$.

Observe that

$$\begin{aligned}
\frac{m}{n+1} < n - m \quad &\Rightarrow \quad \frac{1}{n+1} < \frac{n-m}{m} \\
&\Rightarrow \quad \frac{1}{n(n+1)} < \frac{n-m}{mn} \\
&\Rightarrow \quad \frac{1}{n(n+1)} < \frac{1}{m} - \frac{1}{n} \\
&\Rightarrow \quad \frac{1}{n} + \frac{1}{n(n+1)} - \frac{1}{m} < 0.
\end{aligned}$$

Therefore, we have $\frac{1}{n} + \frac{1}{n(n+1)} - \frac{1}{m} < 0$ and $\frac{1}{n} + \frac{1}{n(n+1)} - \frac{1}{m} > 0$, a contradiction.

Case 2b: Suppose $m > n$.

Since $m, n \in \mathbb{N}$ and $m > n$, then $m \geq n + 1$.

Since $m > n > 0$, then $m > 0$, so $1 \geq \frac{n+1}{m}$.

Since $n + 1 > n > 0$, then $n + 1 > 0$, so $\frac{1}{n+1} \geq \frac{1}{m}$.

Observe that

$$\begin{aligned}
\frac{1}{n+1} \geq \frac{1}{m} \quad &\Rightarrow \quad \frac{n}{n(n+1)} \geq \frac{1}{m} \\
&\Rightarrow \quad \frac{(n+1)-1}{n(n+1)} \geq \frac{1}{m} \\
&\Rightarrow \quad \frac{1}{n} - \frac{1}{n(n+1)} \geq \frac{1}{m} \\
&\Rightarrow \quad \frac{1}{n} - \frac{1}{n(n+1)} - \frac{1}{m} \geq 0.
\end{aligned}$$

Therefore, we have $\frac{1}{n} - \frac{1}{n(n+1)} - \frac{1}{m} \geq 0$ and $\frac{1}{n} - \frac{1}{n(n+1)} - \frac{1}{m} < 0$, a contradiction.

Hence, in either case, a contradiction is reached, so $N'(\frac{1}{n}; \frac{1}{n(n+1)}) \cap S = \emptyset$.

Therefore, $\frac{1}{n}$ is an isolated point of S. $\square$

Example 22. No point is an isolated point of $\emptyset$.

Proof. We prove by contradiction.

Suppose some point is an isolated point of $\emptyset$.

Then there exists x such that x is an isolated point of $\emptyset$, so $x \in \emptyset$.

But, this contradicts the fact that $\emptyset$ has no elements.

Therefore, no point is an isolated point of $\emptyset$. $\square$

Example 23. Let $x \in \mathbb{R}$.

Then x is an isolated point of the singleton set $\{x\}$.

Proof. Let $\delta = 1$.

Observe that

$$\begin{aligned} N'(x;1) \cap \{x\} &= (N(x;1) - \{x\}) \cap \{x\} \\ &= (N(x;1) \cap \overline{\{x\}}) \cap \{x\} \\ &= N(x;1) \cap (\overline{\{x\}} \cap \{x\}) \\ &= N(x;1) \cap \emptyset \\ &= \emptyset \end{aligned}$$

and $x \in \{x\}$.

Therefore, x is an isolated point of the set $\{x\}$. $\square$

Example 24. Let $a, b \in \mathbb{R}$.

No point in the open interval (a, b) is an isolated point of (a, b).

Proof. Either $a < b$ or $a = b$ or $a > b$.

We consider these cases separately.

Case 1: Suppose $a \geq b$.

Then $(a, b) = \emptyset$.

Since the empty set has no isolated points, then (a, b) has no isolated points.

Therefore, there is no isolated point in (a, b).

Case 2: Suppose $a < b$.

Then $(a, b) \neq \emptyset$.

Let $x \in (a, b)$.

Since $(a, b) \subset [a, b]$, then $x \in [a, b]$.

Since every point in the closed interval $[a, b]$ is an accumulation point of (a, b), then x is an accumulation point of (a, b).

Hence, x is not an isolated point of (a, b), so every point in (a, b) is not an isolated point of (a, b).

Therefore, there is no point in (a, b) that is an isolated point of (a, b). $\square$

Example 25. Let $a, b \in \mathbb{R}$.

If $a \neq b$, then no point in the closed interval $[a, b]$ is an isolated point of $[a, b]$.

Proof. Suppose $a \neq b$.

Then either $a < b$ or $a > b$.

We consider these cases separately.

Case 1: Suppose $a > b$.

Then $[a, b] = \emptyset$.

Since the empty set has no isolated points, then $[a, b]$ has no isolated points.

Therefore, there is no isolated point in $[a, b]$.

Case 2: Suppose $a < b$.

Then $[a, b] \neq \emptyset$.

Let $x \in [a, b]$.

Since $a < b$, then every point in the closed interval $[a, b]$ is an accumulation point of $[a, b]$.

Thus, x is an accumulation point of $[a, b]$.

Hence, x is not an isolated point of $[a, b]$, so every point in $[a, b]$ is not an isolated point of $[a, b]$.

Therefore, there is no point in $[a, b]$ that is an isolated point of $[a, b]$. ☐

Example 26. Every natural number is an isolated point of $\mathbb{N}$.

Proof. Let $n \in \mathbb{N}$.

Since $\mathbb{N} \subset \mathbb{Z}$, then $n \in \mathbb{Z}$.

Since no integer is an accumulation point of $\mathbb{N}$, then every integer is not an accumulation point of $\mathbb{N}$.

In particular, n is not an accumulation point of $\mathbb{N}$.

Since $n \in \mathbb{N}$ and n is not an accumulation point of $\mathbb{N}$, then n is an isolated point of $\mathbb{N}$, as desired. ☐

Example 27. No rational number is an isolated point of $\mathbb{Q}$.

Proof. Let $q \in \mathbb{Q}$.

Since $\mathbb{Q} \subset \mathbb{R}$, then $q \in \mathbb{R}$.

Since every real number is an accumulation point of $\mathbb{Q}$, then q is an accumulation point of $\mathbb{Q}$.

Hence, q is not an isolated point of $\mathbb{Q}$, so every rational number is not an isolated point of $\mathbb{Q}$.

Therefore, there is no rational number that is an isolated point of $\mathbb{Q}$. ☐

Example 28. No real number is an isolated point of $\mathbb{R}$.

Proof. Let $x \in \mathbb{R}$.

Since every real number is an accumulation point of $\mathbb{R}$, then x is an accumulation point of $\mathbb{R}$.

Hence, x is not an isolated point of $\mathbb{R}$, so every real number is not an isolated point of $\mathbb{R}$.

Therefore, there is no real number that is an isolated point of $\mathbb{R}$. ☐

Example 29. An unbounded open interval is open.

Let $a, b \in \mathbb{R}$.

Then the interval (a, ∞) is open and the interval $(-\infty, b)$ is open.

Proof. We prove the interval (a, ∞) is open.

Let $x \in (a, \infty)$.

Then $x \in \mathbb{R}$ and $x > a$, so $x - a > 0$.

Let $\delta = x - a$.

Then $\delta > 0$.

Let $p \in N(x; \delta)$.

Since $N(x; x-a) = (x-(x-a), x+(x-a)) = (a, 2x-a)$, then $a < p < 2x-a$, so $a < p$.

Since $p > a$, then $p \in (a, \infty)$.

Hence, $N(x; \delta) \subset (a, \infty)$.

Therefore, x is an interior point of (a, ∞), so (a, ∞) is open. ☐

Proof. We prove the interval $(-\infty, b)$ is open.

Let $x \in (-\infty, b)$.

Then $x \in \mathbb{R}$ and $x < b$, so $b - x > 0$.

Let $\delta = b - x$.

Then $\delta > 0$.

Let $p \in N(x; \delta)$.

Since $N(x; b-x) = (x-(b-x), x+(b-x)) = (2x-b, b)$, then $2x-b < p < b$.

Thus, $p < b$, so $p \in (-\infty, b)$.

Hence, $N(x; \delta) \subset (-\infty, b)$.

Therefore, x is an interior point of $(-\infty, b)$, so $(-\infty, b)$ is open. ☐

Example 30. intersection of an infinite collection of open sets is not necessarily open

Let $\{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ be a family of open intervals of $\mathbb{R}$ indexed by $\mathbb{N}$.

Then $\cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n}) = \{0\}$ is not open.

Therefore, at least one collection of open sets in $\mathbb{R}$ is not closed under arbitrary intersection.

Solution. Let $A_n = (\frac{-1}{n}, \frac{1}{n})$ for each $n \in \mathbb{N}$.

We must determine $\cap_{n=1}^{\infty} A_n$.

We sketch out the intervals and intuitively see that the intersection contains zero and conjecture that the intersection contains zero only.

The collection of intervals $\{A_i\}_{i \in \mathbb{N}}$ is a decreasing family of intervals indexed by $\mathbb{N}$.

We formally prove $\cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n}) = \{0\}$. ☐

Proof. We prove $\cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n}) = \{0\}$.

We first prove $\{0\} \subset \cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n})$.

Let $n \in \mathbb{N}$ be given.

Then $n > 0$.

Hence, $\frac{1}{n} > 0$, so $\frac{-1}{n} < 0$.

Since $\frac{-1}{n} < 0$ and $0 < \frac{1}{n}$, then $0 \in (\frac{-1}{n}, \frac{1}{n})$.

Thus, $0 \in (\frac{-1}{n}, \frac{1}{n})$ for all $n \in \mathbb{N}$, so $0 \in \cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n})$.

Therefore, $\{0\} \subset \cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n})$.

We prove $\cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n}) \subset \{0\}$.

Since $0 \in \cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n})$, then $\cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n}) \neq \emptyset$, so there is at least one element in $\cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n})$.

Let $x \in \cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n})$.

Then $x \in (\frac{-1}{n}, \frac{1}{n})$ for all $n \in \mathbb{N}$, so $x \in \mathbb{R}$ and $\frac{-1}{n} < x < \frac{1}{n}$ for every $n \in \mathbb{N}$.

Either $x < 0$ or $x = 0$ or $x > 0$.

We consider these cases separately.

Case 1: Suppose $x > 0$.

Then $x \neq 0$.

Let $n \in \mathbb{N}$ be given.

Then $\frac{-1}{n} < x < \frac{1}{n}$, so $x < \frac{1}{n}$.

Since $n > 0$, then $nx < 1$.

Since $x > 0$, then $n < \frac{1}{x}$.

Thus, $n < \frac{1}{x}$ for every $n \in \mathbb{N}$.

Since $x \in \mathbb{R}$ and $x \neq 0$, then $\frac{1}{x} \in \mathbb{R}$.

Hence, $\frac{1}{x} \in \mathbb{R}$ is an upper bound of $\mathbb{N}$, so $\mathbb{N}$ has an upper bound in $\mathbb{R}$.

But, this contradicts the fact that $\mathbb{N}$ has no upper bound in $\mathbb{R}$, by the Archimedean property of $\mathbb{R}$.

Case 2: Suppose $x < 0$.

Then $x \neq 0$.

Let $n \in \mathbb{N}$ be given.

Then $\frac{-1}{n} < x < \frac{1}{n}$, so $\frac{-1}{n} < x$.

Since $n > 0$, then $-1 < nx$.

Since $x < 0$, then $\frac{-1}{x} > n$, so $n < \frac{-1}{x}$.

Thus, $n < \frac{-1}{x}$ for every $n \in \mathbb{N}$.

Since $x \in \mathbb{R}$ and $x \neq 0$, then $\frac{-1}{x} \in \mathbb{R}$.

Therefore, $\frac{-1}{x} \in \mathbb{R}$ is an upper bound of $\mathbb{N}$, so $\mathbb{N}$ has an upper bound in $\mathbb{R}$.

But, this contradicts the fact that $\mathbb{N}$ has no upper bound in $\mathbb{R}$, by the Archimedean property of $\mathbb{R}$.

Thus, $x = 0$, so $x \in \{0\}$.

Hence, $\cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n}) \subset \{0\}$.

Since $\cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n}) \subset \{0\}$ and $\{0\} \subset \cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n})$, then $\cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n}) = \{0\}$, as desired. ☐

Proof. Since the singleton set $\{a\}$ is not open for every $a \in \mathbb{R}$, then in particular, the singleton set $\{0\}$ is not open.

Since $\cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n}) = \{0\}$, then $\cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n})$ is not open.

Since every bounded open interval is open, then the bounded open interval $(\frac{-1}{n}, \frac{1}{n})$ is open for all $n \in \mathbb{N}$, so the family $\{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ of open intervals is a collection of open sets in $\mathbb{R}$.

Therefore, at least one collection of open sets in $\mathbb{R}$ is not closed under arbitrary intersection. ☐

Example 31. Every bounded closed interval is closed.

Let $a, b \in \mathbb{R}$.

Then the closed interval $[a, b]$ is closed.

Proof. Either $a < b$ or $a = b$ or $a > b$.

We consider these cases separately.

Case 1: Suppose $a > b$.

Then $[a, b] = \emptyset$.

Since the empty set is closed, then $[a, b]$ is closed.

Case 2: Suppose $a = b$.

Then $[a, b] = [a, a] = \{a\}$.

Since the singleton set $\{a\}$ has no accumulation points, then $\{a\}$ is closed.

Therefore, $[a, b]$ is closed.

Case 3: Suppose $a < b$.

Then $[a, b] \neq \emptyset$ and every point in $[a, b]$ is an accumulation point of $[a, b]$.

Let $p \in [a, b]$.

Then p is an accumulation point of $[a, b]$, so there is at least one accumulation point of $[a, b]$.

Let x be an arbitrary accumulation point of $[a, b]$.

To prove $[a, b]$ is closed, we must prove $x \in [a, b]$, so we must prove $a \leq x \leq b$.

We prove $a \leq x$ by contradiction.

Suppose $a > x$.

Then $a - x > 0$.

Since x is an accumulation point of $[a, b]$, then $N'(x; a - x) \cap [a, b] \neq \emptyset$.

Hence, there exists s such that $s \in N'(x; a - x) \cap [a, b]$, so $s \in N'(x; a - x)$ and $s \in [a, b]$.

Since $s \in N'(x; a - x)$, then $s \in N(x; a - x)$, so $s \in (2x - a, a)$.

Thus, $2x - a < s < a$, so $s < a$.

Since $s \in [a, b]$, then $a \leq s \leq b$, so $a \leq s$.

Hence, we have $s < a$ and $s \geq a$, a violation of trichotomy.

Therefore, $a \leq x$.

We prove $x \leq b$ by contradiction.

Suppose $x > b$.

Then $x - b > 0$.

Since x is an accumulation point of $[a, b]$, then $N'(x; x - b) \cap [a, b] \neq \emptyset$.

Hence, there exists t such that $t \in N'(x; x - b) \cap [a, b]$, so $t \in N'(x; x - b)$ and $t \in [a, b]$.

Since $t \in N'(x; x - b)$, then $t \in N(x; x - b)$, so $t \in (b, 2x - b)$.

Thus, $b < t < 2x - b$, so $b < t$.

Since $t \in [a, b]$, then $a \leq t \leq b$, so $t \leq b$.

Hence, we have $t \leq b$ and $t > b$, a violation of trichotomy.

Therefore, $x \leq b$.

Since $a \leq x$ and $x \leq b$, then $a \leq x \leq b$, so $x \in [a, b]$.
Therefore, the closed interval $[a, b]$ is closed. $\square$

Example 32. An unbounded closed interval is closed.
Let $a, b \in \mathbb{R}$.
Then the interval $[a, \infty)$ is closed and the interval $(-\infty, b]$ is closed.

Proof. We prove $[a, \infty)$ is closed.
We first prove a is an accumulation point of $[a, \infty)$.
Let $\delta > 0$ be given.
Let p be the midpoint of a and $a + \delta$.
Then $p = a + \frac{\delta}{2}$, so $p - a = \frac{\delta}{2}$.
Since $\delta > 0$, then $\frac{\delta}{2} > 0$, so $p - a > 0$.
Since $\frac{1}{2} < 1$ and $\delta > 0$, then $\frac{\delta}{2} < \delta$.
Since $d(p, a) = |p - a| = p - a = \frac{\delta}{2} < \delta$, then $d(p, a) < \delta$, so $p \in N(a; \delta)$.
Since $p - a > 0$, then $p > a$, so $p \neq a$.
Hence, $p \in N'(a; \delta)$.
Since $p > a$, then $p \in [a, \infty)$, so $p \in N'(a; \delta) \cap [a, \infty)$.
Therefore, $N'(a; \delta) \cap [a, \infty) \neq \emptyset$, so a is an accumulation point of $[a, \infty)$. $\square$

Proof. We prove if x is an accumulation point of $[a, \infty)$, then $x \in [a, \infty)$.
Since there is at least one accumulation point of $[a, \infty)$, let x be an arbitrary accumulation point of $[a, \infty)$.
To prove $x \in [a, \infty)$, we must prove $x \geq a$.
We prove $x \geq a$ by contradiction.
Suppose $x < a$.
Then $a - x > 0$.
Since x is an accumulation point of $[a, \infty)$, then $N'(x; a - x) \cap [a, \infty) \neq \emptyset$, so there exists p such that $p \in N'(x; a - x) \cap [a, \infty)$.
Hence, $p \in N'(x; a - x)$ and $p \in [a, \infty)$.
Since $p \in N'(x; a - x)$ and $N'(x; a - x) \subset N(x; a - x)$, then $p \in N(x; a - x)$, so $p \in (2x - a, a)$.
Thus, $2x - a < p < a$, so $p < a$.
Since $p \in [a, \infty)$, then $p \geq a$.
Therefore, we have $p < a$ and $p \geq a$, a violation of trichotomy.
Hence $x \geq a$, so $x \in [a, \infty)$.
Therefore, $[a, \infty)$ is closed. $\square$

Proof. We prove $(-\infty, b]$ is closed.
We first prove b is an accumulation point of $(-\infty, b]$.
Let $\delta > 0$ be given.
Let p be the midpoint of b and $b - \delta$.
Then $p = b - \delta + \frac{\delta}{2} = b - \frac{\delta}{2}$, so $\frac{\delta}{2} = b - p$.
Since $\delta > 0$, then $\frac{\delta}{2} > 0$, so $b - p > 0$.
Since $\frac{1}{2} < 1$ and $\delta > 0$, then $\frac{\delta}{2} < \delta$.

Since $d(p,b) = |p-b| = |b-p| = b-p = \frac{\delta}{2} < \delta$, then $d(p,b) < \delta$, so $p \in N(b;\delta)$.

Since $b - p > 0$, then $b > p$, so $p < b$.

Thus, $p \neq b$, so $p \in N'(b;\delta)$.

Since $p < b$, then $p \in (-\infty, b]$, so $p \in N'(b;\delta) \cap (-\infty, b]$.

Therefore, $N'(b;\delta) \cap (-\infty, b] \neq \emptyset$, so b is an accumulation point of $(-\infty, b]$. □

Proof. We prove if x is an accumulation point of $(-\infty, b]$, then $x \in (-\infty, b]$.

Since there is at least one accumulation point of $(-\infty, b]$, let x be an arbitrary accumulation point of $(-\infty, b]$.

To prove $x \in (-\infty, b]$, we must prove $x \leq b$.

We prove $x \leq b$ by contradiction.

Suppose $x > b$.

Then $x - b > 0$.

Since x is an accumulation point of $(-\infty, b]$, then $N'(x; x-b) \cap (-\infty, b] \neq \emptyset$, so there exists p such that $p \in N'(x; x-b) \cap (-\infty, b]$.

Hence, $p \in N'(x; x-b)$ and $p \in (-\infty, b]$.

Since $p \in N'(x; x-b)$ and $N'(x; x-b) \subset N(x; x-b)$, then $p \in N(x; x-b)$, so $p \in (b, 2x-b)$.

Thus, $b < p < 2x - b$, so $b < p$.

Since $p \in (-\infty, b]$, then $p \leq b$.

Therefore, we have $p \leq b$ and $p > b$, a violation of trichotomy.

Hence $x \leq b$, so $x \in (-\infty, b]$.

Therefore, $(-\infty, b]$ is closed. □

Example 33. Let $S = [1, \infty)$.

Let $\mathcal{F}_1 = \{(0,1), (1,\infty)\}$.

Let $\mathcal{F}_2 = \{(0,1], (1,\infty)\}$.

Then $\mathcal{F}_1$ is not a covering of S, but $\mathcal{F}_2$ is a covering of S.

Proof. We prove $\mathcal{F}_1$ is not a covering of S.

Since $1 \in [1,\infty)$, then $1 \in S$.

Since $1 \notin (0,1)$ and $1 \notin (1,\infty)$, then $1 \notin \cup\mathcal{F}_1$.

Thus, $1 \in S$, but $1 \notin \cup\mathcal{F}_1$, so $\mathcal{F}_1$ is not a covering for S. □

Proof. We prove $\mathcal{F}_2$ is a covering of S.

Since $1 \in S$, then $S \neq \emptyset$.

Let $x \in S$.

Then $x \geq 1$, so either $x > 1$ or $x = 1$.

We consider these cases separately.

Case 1: Suppose $x = 1$.

Let $A = (0,1]$.

Since $1 \in (0,1]$ and $(0,1] \in \mathcal{F}_2$, then $1 \in A$ and $A \in \mathcal{F}_2$.

Thus, there exists $A \in \mathcal{F}_2$ such that $x \in A$.

Case 2: Suppose $x > 1$.

Then $x \in (1,\infty)$.

Let $A = (1, \infty)$.
Then $x \in A$ and $A \in \mathcal{F}_2$.
Thus, there exists $A \in \mathcal{F}_2$ such that $x \in A$.
Hence, in all cases, there exists $A \in \mathcal{F}_2$ such that $x \in A$, so $x \in \cup\mathcal{F}_2$.
Therefore, $S \subset \cup\mathcal{F}_2$, so $\mathcal{F}_2$ is a covering for S. ☐

Example 34. a covering of a set is not unique
Let $S = [1, \infty)$.
Let $\mathcal{F}_1 = \{(0, \infty)\}$.
Let $\mathcal{F}_2 = \{(n-1, n+1) : n \in \mathbb{N}\}$.
Then $\mathcal{F}_1$ is a finite covering of S and $\mathcal{F}_2$ is an infinite covering of S.

Proof. We prove $\mathcal{F}_1$ is a covering of S.
Since $1 \in S$, then $S \neq \emptyset$.
Let $x \in S$.
Then $x \geq 1$.
Let $A = (0, \infty)$.
Then $A \in \mathcal{F}_1$.
Since $x \geq 1$ and $1 > 0$, then $x > 0$, so $x \in (0, \infty)$.
Hence, $x \in A$.
Since there exists $A \in \mathcal{F}_1$ such that $x \in A$, then $\mathcal{F}_1$ is a covering of S.
Since the set $\mathcal{F}_1$ contains exactly one element, then $\mathcal{F}_1$ is a finite set.
Therefore, $\mathcal{F}_1$ is a finite covering of S. ☐

Proof. We prove $\mathcal{F}_2$ is a covering of S.
Let $x \in S$.
Then $x \geq 1$.
Since $x \in \mathbb{R}$, then there is a unique integer n such that $n \leq x < n+1$.
Since $n - 1 < n$, then $n - 1 < n \leq x < n + 1$, so $n - 1 < x < n + 1$.
Hence, $x < n + 1$.
Suppose $n \leq 0$.
Then $n + 1 \leq 1$.
Since $x < n + 1$ and $n + 1 \leq 1$, then $x < 1$.
But, this contradicts the fact that $x \geq 1$.
Hence, $n > 0$.
Since $n \in \mathbb{Z}$ and $n > 0$, then $n \in \mathbb{N}$.
Since $n - 1 < x < n + 1$, then $x \in (n - 1, n + 1)$.
Let $A = (n - 1, n + 1)$.
Then $x \in A$.
Since $n \in \mathbb{N}$ and $A = (n - 1, n + 1)$, then $A \in \mathcal{F}_2$.
Hence, there exists $A \in \mathcal{F}_2$ such that $x \in A$, so $x \in \cup\mathcal{F}_2$.
Thus, $S \subset \cup\mathcal{F}_2$, so $\mathcal{F}_2$ is a covering of S.
Since the set $\mathcal{F}_2$ is infinite, then $\mathcal{F}_2$ is an infinite covering of S. ☐

Example 35. Let $S = [1, \infty)$.
Let $\mathcal{F} = \{(0, n) : n \in \mathbb{N}\}$.
Let $\mathcal{G} = \{(0, n) : n \in \mathbb{N}, n \geq 23\}$.

Then $\mathcal{F}$ is an open covering of S and $\mathcal{G}$ is a subcovering of $\mathcal{F}$.

Proof. We prove $\mathcal{F}$ is a covering of S.
Since $1 \in S$, then $S \neq \emptyset$.
Let $x \in S$.
Then $x \geq 1$.
Since $x \geq 1 > 0$, then $x > 0$.
Since $x \in \mathbb{R}$ and $\mathbb{N}$ is unbounded above in $\mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $n > x$.
Thus, $n > x > 0$, so $0 < x < n$.
Hence, $x \in (0, n)$.
Let $A = (0, n)$.
Then $x \in A$.
Since $n \in \mathbb{N}$ and $A = (0, n)$, then $A \in \mathcal{F}$.
Thus, there exists $A \in \mathcal{F}$ such that $x \in A$, so $x \in \cup\mathcal{F}$.
Hence, $S \subset \cup\mathcal{F}$, so $\mathcal{F}$ is a covering of S.
Since the open interval $(0, n)$ is an open set for each $n \in \mathbb{N}$, then each set in $\mathcal{F}$ is an open set.
Therefore, $\mathcal{F}$ is an open covering of S. $\square$

Proof. We prove $\mathcal{G}$ is a subcovering of $\mathcal{F}$.
We prove $\mathcal{G}$ is a covering of S.
Since $1 \in S$, then $S \neq \emptyset$.
Let $x \in S$.
Then $x \geq 1$, so $x \geq 1 > 0$.
Let $M = \max\{x, 23\}$.
Then either $M = x$ or $M = 23$, and $x \leq M$ and $23 \leq M$.
Since either $M = x$ or $M = 23$, then $M \in \mathbb{R}$.
Since $\mathbb{N}$ is unbounded above in $\mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $n > M$.
Since $x \leq M$ and $M < n$, then $x \leq M < n$.
Thus, $0 < 1 \leq x \leq M < n$, so $0 < x < n$.
Hence, $x \in (0, n)$.
Let $A = (0, n)$.
Then $x \in A$.
Since $n > M$ and $M \geq 23$, then $n > 23$.
Since $n \in \mathbb{N}$ and $n > 23$ and $A = (0, n)$, then $A \in \mathcal{G}$.
Thus, there exists $A \in \mathcal{G}$ such that $x \in A$, so $x \in \cup\mathcal{G}$.
Therefore, $S \subset \cup\mathcal{G}$, so $\mathcal{G}$ is a covering of S. $\square$

Proof. We prove $\mathcal{G} \subset \mathcal{F}$.
Since $(0, 23) \in \mathcal{G}$, then $\mathcal{G} \neq \emptyset$.
Let $x \in \mathcal{G}$.
Then there exists $n \in \mathbb{N}$ such that $n \geq 23$ and $x = (0, n)$.
Since $n \in \mathbb{N}$ and $x = (0, n)$, then $x \in \mathcal{F}$.
Therefore, $\mathcal{G} \subset \mathcal{F}$.
Since $\mathcal{G}$ is a covering of S such that $\mathcal{G} \subset \mathcal{F}$, then $\mathcal{G}$ is a subcovering of $\mathcal{F}$. $\square$

Example 36. A finite set is compact.

Proof. Let S be a finite set.
Let $\mathcal{F}$ be an arbitrary open covering of S.
Either $S = \emptyset$ or $S \neq \emptyset$.
We consider these cases separately.
Case 1: Suppose $S = \emptyset$.
Since $\mathcal{F}$ is a covering of S, then $\mathcal{F}$ is a covering of $\emptyset$, so $\emptyset \subset \cup\mathcal{F}$.
Since $\emptyset \subset \emptyset$ and $\cup\emptyset = \emptyset$, then $\emptyset \subset \cup\emptyset$, so $\emptyset$ is a covering of $\emptyset$.
Since the empty set is a subset of every set, then in particular, $\emptyset \subset \mathcal{F}$.
Since $\emptyset \subset \mathcal{F}$ and $\emptyset$ is finite, then $\emptyset$ is a finite subcovering of $\emptyset$.
Therefore, $\emptyset$ is compact, so S is compact.
Case 2: Suppose $S \neq \emptyset$.
Since S is finite and not empty, then there exists a positive integer n such that $S = \{s_1, s_2, ..., s_n\}$.
Since $\mathcal{F}$ is a covering of S, then $S \subset \cup\mathcal{F}$.
Hence, for each $s_k \in S$, there exists $F_k \in \mathcal{F}$ such that $s_k \in F_k$ for each $k \in \{1, 2, ..., n\}$.
Thus, $\{F_1, F_2, ..., F_n\}$ is a subset of $\mathcal{F}$.
Let $\mathcal{G} = \{F_1, F_2, ..., F_n\}$.
Then $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{G}$ is finite.
Since $S \neq \emptyset$, then there is at least one element of S.
Let $x \in S$.
Then there exists $k \in \{1, 2, ..., n\}$ such that $x = s_k$.
Thus, there exists $F_k \in \mathcal{G}$ such that $x \in F_k$.
Since $F_k \in \mathcal{G}$ and $x \in F_k$, then $x \in \cup\mathcal{G}$.
Hence, $S \subset \cup\mathcal{G}$, so $\mathcal{G}$ is a covering of S.
Since $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{G}$ is finite, then $\mathcal{G}$ is a finite subcovering of S.
Therefore, S is compact.
Thus, in all cases, S is compact, as desired. □

Example 37. $\mathbb{N}$ is not compact
Define $I_n = (n - \frac{1}{2}, n + \frac{1}{2})$ for each $n \in \mathbb{N}$.
Let $\mathcal{F} = \{I_n : n \in \mathbb{N}\}$.
Then $\mathcal{F}$ is an open covering of $\mathbb{N}$, but $\mathcal{F}$ contains no finite subcovering of $\mathbb{N}$.
Therefore, $\mathbb{N}$ is not compact.

Proof. Let $n \in \mathbb{N}$ be given.
Since $n - \frac{1}{2} < n < n + \frac{1}{2}$, then $n \in (n - \frac{1}{2}, n + \frac{1}{2})$.
Hence, $n \in I_n$, so $I_n \in \mathcal{F}$.
Thus, there exists $I_n \in \mathcal{F}$ such that $n \in I_n$, so $n \in \cup\mathcal{F}$.
Therefore, $\mathbb{N} \subset \cup\mathcal{F}$, so $\mathcal{F}$ is a covering of $\mathbb{N}$.
For each $n \in \mathbb{N}$, I_n is an open interval, so I_n is an open set.
Thus, each set in $\mathcal{F}$ is an open set, so $\mathcal{F}$ is an open covering of $\mathbb{N}$. □

Proof. We prove $\mathbb{N}$ is not compact by contradiction.

Suppose $\mathbb{N}$ is compact.

Then every open covering of $\mathbb{N}$ contains a finite subcovering of $\mathbb{N}$.

In particular, since $\mathcal{F}$ is an open covering of $\mathbb{N}$, then $\mathcal{F}$ contains a finite subcovering of $\mathbb{N}$.

Thus, there exists $\mathcal{G}$ such that $\mathcal{G}$ is a finite subcover of $\mathcal{F}$.

Since $\mathcal{G}$ is a finite subcover of $\mathcal{F}$, then $\mathcal{G}$ is a covering of $\mathbb{N}$ and $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{G}$ is finite.

Since $\mathcal{G}$ is a covering of $\mathbb{N}$, then $\mathbb{N} \subset \cup\mathcal{G}$.

Since a subset of a finite set is finite, then if $A \subset B$ and B is finite, then A is finite.

Hence, if $A \subset B$ and A is infinite, then B is infinite.

Since $\mathbb{N} \subset \cup\mathcal{G}$ and $\mathbb{N}$ is infinite, then $\cup\mathcal{G}$ is infinite, so $\cup\mathcal{G}$ contains at least one element.

Thus, there exists x such that $x \in X$ for some $X \in \mathcal{G}$.

Since $X \in \mathcal{G}$, then $\mathcal{G} \neq \emptyset$.

Since $\mathcal{G}$ is finite and $\mathcal{G} \neq \emptyset$, then there exists a positive integer k such that $\mathcal{G} = \{A_1, A_2, ..., A_k\}$ and $A_i \in \mathcal{F}$ for each $i = 1, 2, ..., k$.

Let $i \in \{1, 2, ..., k\}$.

Then $A_i \in \mathcal{F}$, so there exists $n \in \mathbb{N}$ such that $A_i = I_n = (n - \frac{1}{2}, n + \frac{1}{2})$.

Since $n - \frac{1}{2} < n < n + \frac{1}{2}$, then $n \in (n - \frac{1}{2}, n + \frac{1}{2})$, so $n \in I_n$.

Hence, $n \in A_i$.

Suppose there exists $m \in \mathbb{N}$ such that $m \in A_i$.

Then $m \in (n - \frac{1}{2}, n + \frac{1}{2})$, so $n - \frac{1}{2} < m < n + \frac{1}{2}$.

Since $n - 1 < n - \frac{1}{2} < m < n + \frac{1}{2} < n + 1$, then $n - 1 < m < n + 1$.

Between any integer $n - 1$ and $n + 1$ is the unique integer n.

Hence, $m = n$, so there exists a unique $n \in \mathbb{N}$ such that $n \in A_i$ for each $i = 1, 2, ..., k$.

Define a function $f : \mathcal{G} \to \mathbb{N}$ by $f(i)$ is the unique natural number such that $f(i) \in A_i$ for each $i = 1, 2, ..., k$.

Let S be the range of f.

Then $S = \{f(i) \in \mathbb{N} : f(i) \in A_i, i = 1, 2, ..., k\}$, so S is a finite set of natural numbers.

Since $A_1 \in \mathcal{G}$, then $f(1) \in A_1$ and $f(1) \in \mathbb{N}$, so $f(1) \in S$.

Hence, S is not empty.

Since $\mathbb{N} \subset \mathbb{R}$, then S is a nonempty finite set of real numbers.

Therefore, $\max S$ exists and $\max S \in \mathbb{N}$.

Let $s = \max S + 1$.

Then $s \in \mathbb{N}$.

Since $\max S + 1 > \max S$, then $s > \max S$, so $s \notin S$.

Since $s \in S$ iff there exists $i \in 1, 2, ..., k$ such that $s \in A_i$, then $s \notin S$ iff for each $i \in 1, 2, ..., k$, $s \notin A_i$.

Thus, $s \notin A_i$ for each $i = 1, 2, ..., k$.

Hence, $s \notin A_1 \cup A_2 \cup ... \cup A_k$, so $s \notin \cup G$.

Therefore, there exists $s \in \mathbb{N}$ such that $s \notin \cup G$.
This contradicts the fact that $\mathbb{N} \subset \cup \mathcal{G}$.
Therefore, $\mathbb{N}$ is not compact. $\square$