Topology of $\mathbb{R}$ Exercises

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Topology of $\mathbb{R}$

Exercise 1. There are exactly two real numbers whose distance from the number 3 is 7.

Proof. Let S be the set of all real numbers whose distance from 3 is 7.
Then $S = \{x \in \mathbb{R} : d(x, 3) = 7\}$.
We prove $S = \{-4, 10\}$.
Let $x \in S$.
Then $x \in \mathbb{R}$ and $d(x, 3) = 7$, so $|x - 3| = 7$.
Either $x - 3 \geq 0$ or $x - 3 < 0$.
We consider these cases separately.
Case 1: Suppose $x - 3 \geq 0$.
Then $7 = |x - 3| = x - 3$, so $7 = x - 3$.
Thus, $x = 10$.
Case 2: Suppose $x - 3 < 0$.
Then $7 = |x - 3| = -(x - 3) = -x + 3 = 3 - x$, so $7 = 3 - x$.
Thus, $x = -4$.
Hence, either $x = 10$ or $x = -4$.
Therefore, $x \in \{-4, 10\}$, so $S \subset \{-4, 10\}$.

Let $y \in \{-4, 10\}$.
Then either $y = -4$ or $y = 10$.
We consider these cases separately.
Case 1: Suppose $y = -4$.
Since $-4 \in \mathbb{R}$ and $d(-4, 3) = |-4 - 3| = |-7| = 7$, then $y \in S$.
Case 2: Suppose $y = 10$.
Since $10 \in \mathbb{R}$ and $d(10, 3) = |10 - 3| = |7| = 7$, then $y \in S$.
Hence, in all cases, $y \in S$, so $\{-4, 10\} \subset S$.
Since $S \subset \{-4, 10\}$ and $\{-4, 10\} \subset S$, then $S = \{-4, 10\}$.
Since -4 and 10 are the only real numbers whose distance from the number 3 is 7, then there are exactly two real numbers whose distance from the number 3 is 7. $\square$

Exercise 2. Describe the set of all points in $\mathbb{R}$ which are within 5 units of the number -2.

Proof. Let S be the set of all points in $\mathbb{R}$ which are within 5 units of the number -2.

Then $S = \{x \in \mathbb{R} : d(x, -2) \leq 5\}$.
We prove $S = [-7, 3]$.
Let $x \in S$.
Then $x \in \mathbb{R}$ and $d(x, -2) \leq 5$, so $5 \geq d(x, -2) = |x - (-2)| = |x + 2|$.
Hence, $5 \geq |x + 2|$, so $|x + 2| \leq 5$.
Thus, $-5 \leq x + 2 \leq 5$, so $-7 \leq x \leq 3$.
Therefore, $x \in [-7, 3]$, so $S \subset [-7, 3]$.

Let $y \in [-7, 3]$.
Then $-7 \leq y \leq 3$, so $-5 \leq y + 2 \leq 5$.
Hence, $|y + 2| \leq 5$, so $|y - (-2)| \leq 5$.
Thus, $d(y, -2) \leq 5$.
Since $y \in \mathbb{R}$ and $d(y, -2) \leq 5$, then $y \in S$, so $[-7, 3] \subset S$.
Since $S \subset [-7, 3]$ and $[-7, 3] \subset S$, then $S = [-7, 3]$. □

Exercise 3. Describe the set of all real numbers whose distance from 4 is greater than 15.

Proof. Let S be the set of all real numbers whose distance from 4 is greater than 15.

Then $S = \{x \in \mathbb{R} : d(x, 4) > 15\}$.
We prove $S = (-\infty, -11) \cup (19, \infty)$.
Let $x \in S$.
Then $x \in \mathbb{R}$ and $d(x, 4) > 15$, so $|x - 4| > 15$.
Hence, either $x - 4 > 15$ or $x - 4 < -15$, so either $x > 19$ or $x < -11$.
Thus, either $x \in (19, \infty)$ or $x \in (-\infty, -11)$, so $x \in (19, \infty) \cup (-\infty, -11)$.
Therefore, $x \in (-\infty, -11) \cup (19, \infty)$, so $S \subset (-\infty, -11) \cup (19, \infty)$.

Let $y \in (-\infty, -11) \cup (19, \infty)$.
Then either $y \in (-\infty, -11)$ or $y \in (19, \infty)$, so either $y < -11$ or $y > 19$.
Hence, either $y - 4 < -15$ or $y - 4 > 15$, so either $y - 4 > 15$ or $y - 4 < -15$.
Thus, $|y - 4| > 15$, so $d(y, 4) > 15$.
Since $y \in \mathbb{R}$ and $d(y, 4) > 15$, then $y \in S$, so $(-\infty, -11) \cup (19, \infty) \subset S$.
Since $S \subset (-\infty, -11) \cup (19, \infty)$ and $(-\infty, -11) \cup (19, \infty) \subset S$, then $S = (-\infty, -11) \cup (19, \infty)$. □

Exercise 4. If I_1 and I_2 are intervals such that $I_1 \cap I_2 \neq \emptyset$, then $I_1 \cup I_2$ is an interval.

Solution.
Our hypothesis is I_1 is an interval and I_2 is an interval and $I_1 \cap I_2 \neq \emptyset$.

To prove our conclusion $I_1 \cup I_2$ is an interval, we must prove $(\forall a, b, c \in \mathbb{R})[a, b \in I_1 \cup I_2 \wedge a < c < b \rightarrow c \in I_1 \cup I_2]$.

We let a, b, c be arbitrary real numbers.

To prove $c \in I_1 \cup I_2$, we assume $a \in I_1 \cup I_2$ and $b \in I_1 \cup I_2$ and $a < c < b$.

We must prove either $c \in I_1$ or $c \in I_2$.

To prove $c \in I_1$ we must prove $a, b \in I_1$ and $a < c < b$.

To prove $c \in I_2$ we must prove $a, b \in I_2$ and $a < c < b$.

Since $a \in I_1 \cup I_2$, then either $a \in I_1$ or $a \in I_2$.

Since $b \in I_1 \cup I_2$, then either $b \in I_1$ or $b \in I_2$.

Hence, we have 4 cases to consider:

1. $a \in I_1, b \in I_1$.
2. $a \in I_1, b \in I_2$.
3. $a \in I_2, b \in I_1$.
4. $a \in I_2, b \in I_2$. □

Proof. Let I_1 and I_2 be intervals such that $I_1 \cap I_2 \neq \emptyset$.

Let a, b, and c be arbitrary real numbers.

To prove $I_1 \cup I_2$ is an interval, we assume $a \in I_1 \cup I_2$ and $b \in I_1 \cup I_2$ and $a < c < b$.

To prove $c \in I_1 \cup I_2$, we must prove either $c \in I_1$ or $c \in I_2$.

Since $a \in I_1 \cup I_2$, then either $a \in I_1$ or $a \in I_2$.

Since $b \in I_1 \cup I_2$, then either $b \in I_1$ or $b \in I_2$.

There are 4 cases to consider.

Case 1: Suppose $a \in I_1$ and $b \in I_1$.

Since $a \in I_1$ and $b \in I_1$ and c is between a and b, and since I_1 is an interval, then we conclude $c \in I_1$.

Case 2: Suppose $a \in I_1$ and $b \in I_2$.

Since $I_1 \cap I_2$ is not empty, then there exists an element in $I_1 \cap I_2$.

Let x be an arbitrary element of $I_1 \cap I_2$.

Then $x \in I_1$ and $x \in I_2$.

By trichotomy, either $x < c$ or $x = c$ or $x > c$.

We consider these cases separately.

Case 2a: Suppose $x > c$.

Then $c < x$.

Since $a < c < b$, then $a < c$ and $c < b$.

Since $a < c$ and $c < x$, then $a < c < x$.

Since $a \in I_1$ and $x \in I_1$ and c is between a and x, and since I_1 is an interval, then we conclude $c \in I_1$.

Case 2b: Suppose $x < c$.

Since $a < c < b$, then $a < c$ and $c < b$.

Since $x < c$ and $c < b$, then $x < c < b$.

Since $b \in I_2$ and $x \in I_2$ and c is between x and b, and since I_2 is an interval, then we conclude $c \in I_2$.

Case 2c: Suppose $x = c$.

Since $x \in I_1 \cap I_2$ and $I_1 \cap I_2 \subset I_1 \cup I_2$, then $x \in I_1 \cup I_2$.

Since $x = c$, then $c \in I_1 \cup I_2$.

Hence, in all cases, either $c \in I_1$ or $c \in I_2$, so $c \in I_1 \cup I_2$.

Case 3: Suppose $a \in I_2$ and $b \in I_1$.

The argument is the same as case 2, with a and b reversed.

Case 4: Suppose $a \in I_2$ and $b \in I_2$.

The argument is the same as case 1, with I_2 instead of I_1.

Hence, in all 4 cases, c is contained in $I_1 \cup I_2$, so $I_1 \cup I_2$ is an interval, as desired. □

Exercise 5. Let $a, b, a', b' \in \mathbb{R}$ with $a < b$ and $a' < b'$.

Let $[a, b]$ and $[a', b']$ be closed intervals.

Then $[a, b] \subset [a', b']$ iff $a' \leq a$ and $b \leq b'$.

Proof. We first prove if $a' \leq a$ and $b \leq b'$, then $[a, b] \subset [a', b']$.

Suppose $a' \leq a$ and $b \leq b'$.

Since $a < b$, then $[a, b] \neq \emptyset$.

Let $x \in [a, b]$.

Then $a \leq x \leq b$, so $a \leq x$ and $x \leq b$.

Since $a' \leq a$ and $a \leq x$, then $a' \leq x$.

Since $x \leq b$ and $b \leq b'$, then $x \leq b'$.

Thus, $a' \leq x$ and $x \leq b'$, so $a' \leq x \leq b'$.

Therefore, $x \in [a', b']$, so $[a, b] \subset [a', b']$. □

Proof. Conversely, we prove if $[a, b] \subset [a', b']$, then $a' \leq a$ and $b \leq b'$.

Suppose $[a, b] \subset [a', b']$.

Since $a \in [a, b]$ and $[a, b] \subset [a', b']$, then $a \in [a', b']$, so $a' \leq a \leq b'$.

Hence, $a' \leq a$ and $a \leq b'$, so $a' \leq a$.

Since $b \in [a, b]$ and $[a, b] \subset [a', b']$, then $b \in [a', b']$, so $a' \leq b \leq b'$.

Hence, $a' \leq b$ and $b \leq b'$, so $b \leq b'$.

Therefore, $a' \leq a$ and $b \leq b'$, as desired. □

Exercise 6. Let S be a nonempty subset of $\mathbb{R}$.

Then S is bounded iff there is a closed bounded interval I such that $S \subset I$.

Proof. Suppose S is bounded.

Then S is bounded above and below in $\mathbb{R}$, so there exist real numbers a and b such that $a \leq x \leq b$ for all $x \in S$.

Let $I = [a, b]$.

Then I is a closed bounded interval.

Since S is nonempty, let $x \in S$.

Then $a \leq x \leq b$.

Since $x \in S$ and $S \subset \mathbb{R}$, then $x \in \mathbb{R}$.

Since $x \in \mathbb{R}$ and $a \leq x \leq b$, then $x \in I$, so $S \subset I$.

Therefore, there is a closed bounded interval I such that $S \subset I$.

Conversely, suppose there is a closed bounded interval I such that $S \subset I$.

Then there are real numbers a and b such that $I = [a, b]$.

Since S is not empty, let $x \in S$.

Since $S \subset I$, then $x \in I$, so $a \leq x \leq b$.

Hence, $a \leq x \leq b$ for all $x \in S$.

Thus, there exist real numbers a and b such that $a \leq x \leq b$ for all $x \in S$.

Therefore, S is bounded. □

Exercise 7. Let S be a nonempty bounded subset of $\mathbb{R}$.

Let $I = [\inf S, \sup S]$.

Then $S \subset I$.

Proof. Since S is a bounded subset of $\mathbb{R}$, then S is bounded above and below in $\mathbb{R}$.

Since S is nonempty and bounded above in $\mathbb{R}$, then $\sup S$ exists.

Since S is nonempty and bounded below in $\mathbb{R}$, then $\inf S$ exists.

Let $I = [\inf S, \sup S]$.

Since S is not empty, let $x \in S$.

Since $\sup S$ is an upper bound of S, then $x \leq \sup S$.

Since $\inf S$ is a lower bound of S, then $\inf S \leq x$.

Thus, $\inf S \leq x \leq \sup S$, so $x \in [\inf S, \sup S]$.

Therefore, $x \in I$, so $S \subset I$. □

Exercise 8. Let S be a nonempty bounded subset of $\mathbb{R}$.

Let $I = [\inf S, \sup S]$.

Let J be a closed bounded interval such that $S \subset J$.

Then $I \subset J$.

Proof. Since J is a closed bounded interval, then there exist real numbers a and b with $a < b$ such that $J = [a, b]$.

Since $S \neq \emptyset$, let $x \in S$.

Since $S \subset J$, then $x \in J$, so $x \in [a, b]$.

Hence, $a \leq x \leq b$, so $a \leq x$ and $x \leq b$.

Thus, $a \leq x$ and $x \leq b$ for all $x \in S$, so $a \leq x$ for all $x \in S$ and $x \leq b$ for all $x \in S$.

Consequently, a is a lower bound of S and b is an upper bound of S.

Since b is an upper bound of S and $\sup S$ is the least upper bound of S, then $\sup S \leq b$.

Since a is a lower bound of S and $\inf S$ is the greatest lower bound of S, then $a \leq \inf S$.

Let $y \in I$.

Then $\inf S \leq y \leq \sup S$, so $\inf S \leq y$ and $y \leq \sup S$.

Since $a \leq \inf S$ and $\inf S \leq y$ and $y \leq \sup S$ and $\sup S \leq b$, then $a \leq y \leq b$, so $y \in [a, b]$.

Therefore, $y \in J$, so $I \subset J$, as desired. □

Exercise 9. Let $S_n = [0, \frac{1}{n}]$ for each $n \in \mathbb{N}$.
Then $\cap_{n=1}^{\infty} S_n = \{0\}$.

Solution. We can draw several of these intervals and observe that each of the intervals gets smaller, so the collection $\{S_n : n \in \mathbb{N}\}$ is a decreasing family of sets indexed by $\mathbb{N}$. It appears that the intersection of all these intervals is the singleton set $\{0\}$. □

Proof. To prove $\{0\} \subset \cap_{n=1}^{\infty} S_n$, let $n \in \mathbb{N}$ be arbitrary.
Then $S_n = [0, \frac{1}{n}]$.
Since $n \in \mathbb{N}$, then $n > 0$, so $\frac{1}{n} > 0$.
Thus, $S_n \neq \emptyset$ and $0 \in S_n$.
Since n is arbitrary, then $0 \in S_n$ for each $n \in \mathbb{N}$.
Hence, $0 \in \cap_{n=1}^{\infty} S_n$, so $\{0\} \subset \cap_{n=1}^{\infty} S_n$.

To prove $\cap_{n=1}^{\infty} S_n \subset \{0\}$, let $x \in \cap_{n=1}^{\infty} S_n$.
Then $x \in S_n$ for each $n \in \mathbb{N}$.
Thus, $x \in S_1 = [0, 1]$, so $0 \leq x \leq 1$.
Hence, $0 \leq x$, so $x \geq 0$.
Therefore, either $x > 0$ or $x = 0$.
Suppose $x > 0$.
Then by the Archimedean property of $\mathbb{R}$, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < x$.
Thus, $x \notin [0, \frac{1}{m}]$.
Hence, there exists $m \in \mathbb{N}$ such that $x \notin S_m$.
But, this contradicts the fact that $x \in S_n$ for every $n \in \mathbb{N}$.
Therefore, $x = 0$, so $x \in \{0\}$.
Thus, $\cap_{n=1}^{\infty} S_n \subset \{0\}$.

Since $\cap_{n=1}^{\infty} S_n \subset \{0\}$ and $\{0\} \subset \cap_{n=1}^{\infty} S_n$, then $\cap_{n=1}^{\infty} S_n = \{0\}$. □

Exercise 10. Let $S_n = (\frac{1}{n}, 1)$ for each $n \in \mathbb{N}$.
Then $\cup_{n=1}^{\infty} S_n = (0, 1)$.

Solution. We can draw several of these intervals and observe that each of the intervals gets larger, so the collection $\{S_n : n \in \mathbb{N}\}$ is a increasing family of sets indexed by $\mathbb{N}$. It appears that the union of all these intervals is $(0, 1)$. □

Proof. To prove $\cup_{n=1}^{\infty} S_n = (0, 1)$, we prove $\cup_{n=1}^{\infty} S_n \subset (0, 1)$ and $(0, 1) \subset \cup_{n=1}^{\infty} S_n$.
We prove $\cup_{n=1}^{\infty} S_n \subset (0, 1)$.
Let $x \in \cup_{n=1}^{\infty} S_n$.
Then there exists $n \in \mathbb{N}$ such that $x \in S_n$.
Thus, there exists $n \in \mathbb{N}$ such that $x \in \mathbb{R}$ and $\frac{1}{n} < x < 1$.
Since $\frac{1}{n} < x < 1$, then $\frac{1}{n} < x$ and $x < 1$.
Since $n \in \mathbb{N}$, then $n > 0$, so $n \neq 0$.
Since $n \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{R}$, then $n \in \mathbb{R}$.

Since $n \in \mathbb{R}$ and $n \neq 0$, then $\frac{1}{n} \in \mathbb{R}$.
Since $n > 0$, then $\frac{1}{n} > 0$, so $0 < \frac{1}{n}$.
Since $0 < \frac{1}{n}$ and $\frac{1}{n} < x$, then $0 < x$.
Thus, $0 < x$ and $x < 1$, so $0 < x < 1$.
Since $x \in \mathbb{R}$ and $0 < x < 1$, then $x \in (0,1)$.
Hence, $x \in \cup_{n=1}^{\infty} S_n$ implies $x \in (0,1)$, so $\cup_{n=1}^{\infty} S_n \subset (0,1)$.
We prove $(0,1) \subset \cup_{n=1}^{\infty} S_n$.
Let $y \in (0,1)$.
Then $y \in \mathbb{R}$ and $0 < y < 1$, so $0 < y$ and $y < 1$.
To prove $y \in \cup_{n=1}^{\infty} S_n$, we must show there exists $k \in \mathbb{N}$ such that $y \in S_k$.
Since $0 < y$, then $y > 0$, so $y \neq 0$.
Since $y \in \mathbb{R}$ and $y \neq 0$, then $\frac{1}{y} \in \mathbb{R}$.
By the Archimedean property of $\mathbb{R}$, for every real number there corresponds a larger natural number.
Hence, there exists a natural number that is larger than the real number $\frac{1}{y}$.
Choose $k \in \mathbb{N}$ such that $k > \frac{1}{y}$.
Since $y > 0$, we multiply by y to get $ky > 1$.
Since $k \in \mathbb{N}$, then $k > 0$, so we divide by k to get $y > \frac{1}{k}$.
Hence, $\frac{1}{k} < y$.
Since $\frac{1}{k} < y$ and $y < 1$, then $\frac{1}{k} < y < 1$.
Since $y \in \mathbb{R}$ and $\frac{1}{k} < y < 1$, then $y \in (\frac{1}{k}, 1)$, so $y \in S_k$.
Thus, there exists $k \in \mathbb{N}$ such that $y \in S_k$, so $y \in \cup_{n=1}^{\infty} S_n$.
Therefore, $y \in (0,1)$ implies $y \in \cup_{n=1}^{\infty} S_n$, so $(0,1) \subset \cup_{n=1}^{\infty} S_n$.
Since $\cup_{n=1}^{\infty} S_n \subset (0,1)$ and $(0,1) \subset \cup_{n=1}^{\infty} S_n$, then $\cup_{n=1}^{\infty} S_n = (0,1)$. ☐

Exercise 11. Let $S_n = (0, \frac{1}{n})$ for each $n \in \mathbb{N}$.
Then $\cap_{n=1}^{\infty} S_n = \emptyset$.

Solution. We can draw several of these intervals and note that each of the intervals gets smaller, so $S = \{S_n : n \in \mathbb{N}\}$ is a decreasing family of sets indexed by $\mathbb{N}$. It appears that the intersection of all these intervals is empty. ☐

Proof. We prove by contradiction.
Suppose $\cap_{n=1}^{\infty} S_n \neq \emptyset$.
Then there is an element in $\cap_{n=1}^{\infty} S_n$.
Let x be an element of $\cap_{n=1}^{\infty} S_n$.
Then $x \in S_n$ for every $n \in \mathbb{N}$.
Hence, $x \in \mathbb{R}$ and $0 < x < \frac{1}{n}$ for every $n \in \mathbb{N}$, so $0 < x$ and $x < \frac{1}{n}$ for every $n \in \mathbb{N}$.
Since $x \in \mathbb{R}$ and $x > 0$, then $x \neq 0$.
Since $x \in \mathbb{R}$ and $x \neq 0$, then $\frac{1}{x} \in \mathbb{R}$.
By the Archimedean property of $\mathbb{R}$, there exists a natural number that is larger than the real number $\frac{1}{x}$.
Thus, there exists $k \in \mathbb{N}$ such that $k > \frac{1}{x}$.
Since $k \in \mathbb{N}$, then $x < \frac{1}{k}$ and $k > 0$.
Consequently, $kx < 1$.

Since $x > 0$, then $k < \frac{1}{x}$.
Hence, we have $k > \frac{1}{x}$ and $k < \frac{1}{x}$, a contradiction.
Therefore, $\cap_{n=1}^{\infty} S_n = \emptyset$, as desired. □

Exercise 12. What is $\cup_{n=1}^{\infty}(\mathbb{R} - (0, \frac{1}{n}))$?

Solution. Observe that

$$\begin{aligned} \cup_{n=1}^{\infty}(\mathbb{R} - (0, \frac{1}{n})) &= \mathbb{R} - \cap_{n=1}^{\infty}(0, \frac{1}{n}) \\ &= \mathbb{R} - \emptyset \\ &= \mathbb{R}. \end{aligned}$$

□

Exercise 13. Compute $\cap_{n=1}^{\infty}(\frac{-1}{n}, 1 + \frac{1}{n})$.

Solution. Let $S = \{S_n : n \in \mathbb{N}\}$ be a family of intervals $S_n = (\frac{-1}{n}, 1 + \frac{1}{n})$ indexed by $\mathbb{N}$.

We must compute $\cap_{n=1}^{\infty} S_n$.

We sketch S_1, S_2, S_3, S_4 intervals and observe that $S_1 \supset S_2 \supset S_3 \supset S_4 \supset ...$, so S is a decreasing family of nested intervals.

Intuitively, we see that the interval $(0, 1)$ is contained in the intersection of this family of intervals.

We need to check the endpoints 0 and 1 to determine if they are in this intersection.

We show that $0 \in \cap_{n=1}^{\infty} S_n$.
Let $n \in \mathbb{N}$.
Then $n > 0$, so $\frac{1}{n} > 0$.
Hence, $\frac{-1}{n} < 0$.
Since $-1 < 0$ and $0 < \frac{1}{n}$, then $-1 < \frac{1}{n}$.
Thus, $0 < 1 + \frac{1}{n}$.
Therefore, $\frac{-1}{n} < 0$ and $0 < 1 + \frac{1}{n}$, so $\frac{-1}{n} < 0 < 1 + \frac{1}{n}$.
Hence, $0 \in (\frac{-1}{n}, 1 + \frac{1}{n})$, so $0 \in S_n$.
Since n is arbitrary, then $0 \in S_n$ for all $n \in \mathbb{N}$.
Therefore, $0 \in \cap_{n=1}^{\infty} S_n$.
Thus, the interval $[0, 1)$ is contained in $\cap_{n=1}^{\infty} S_n$.
We show that $1 \in \cap_{n=1}^{\infty} S_n$.
Let $n \in \mathbb{N}$.
Then $n > 0$, so $\frac{1}{n} > 0$.
Hence, $\frac{-1}{n} < 0$.
Since $\frac{-1}{n} < 0$ and $0 < 1$, then $\frac{-1}{n} < 1$.
Since $0 < \frac{1}{n}$, then $1 < 1 + \frac{1}{n}$.
Thus, $\frac{-1}{n} < 1$ and $1 < 1 + \frac{1}{n}$, so $\frac{-1}{n} < 1 < 1 + \frac{1}{n}$.
Hence, $1 \in (\frac{-1}{n}, 1 + \frac{1}{n})$, so $1 \in S_n$.
Since n is arbitrary, then $1 \in S_n$ for all $n \in \mathbb{N}$.
Therefore, $1 \in \cap_{n=1}^{\infty} S_n$.
Thus, the interval $[0, 1]$ is contained in $\cap_{n=1}^{\infty} S_n$.
We see that $\cap_{n=1}^{\infty} S_n = [0, 1]$. □

Proof. To prove $\cap_{n=1}^{\infty} S_n = [0,1]$, we prove $\cap_{n=1}^{\infty} S_n \subset [0,1]$ and $[0,1] \subset \cap_{n=1}^{\infty} S_n$.

We first prove $\cap_{n=1}^{\infty} S_n \subset [0,1]$.

Let $x \in \cap_{n=1}^{\infty} S_n$.

Then $x \in S_n$ for all $n \in \mathbb{N}$, so $x \in (\frac{-1}{n}, 1 + \frac{1}{n})$ for all $n \in \mathbb{N}$.

To prove $x \in [0,1]$, we must prove $x \in \mathbb{R}$ and $0 \leq x \leq 1$.

Let $n \in \mathbb{N}$.

Then $x \in (\frac{-1}{n}, 1 + \frac{1}{n})$, so $x \in \mathbb{R}$ and $\frac{-1}{n} < x < 1 + \frac{1}{n}$.

Thus, $\frac{-1}{n} < x$ and $x < 1 + \frac{1}{n}$.

Since $n \in \mathbb{N}$, then $n > 0$.

Since $\frac{-1}{n} < x$, we multiply by n to get $-1 < nx$.

Suppose $x < 0$.

Then we divide by x to get $\frac{-1}{x} > n$.

Thus, $n < \frac{-1}{x}$, so $n \leq \frac{-1}{x}$.

Since $x < 0$, then $x \neq 0$, so $\frac{1}{x} \in \mathbb{R}$.

Hence, $\frac{-1}{x} \in \mathbb{R}$.

Since n is arbitrary then $n \leq \frac{-1}{x}$ for all $n \in \mathbb{N}$.

Thus, the real number $\frac{-1}{x}$ is an upper bound for $\mathbb{N}$ in $\mathbb{R}$, so $\mathbb{N}$ has an upper bound in $\mathbb{R}$.

By the Archimedean property of $\mathbb{R}$, $\mathbb{N}$ has no upper bound in $\mathbb{R}$.

Hence, we have $\mathbb{N}$ has an upper bound in $\mathbb{R}$ and $\mathbb{N}$ has no upper bound in $\mathbb{R}$, a contradiction.

Therefore, x cannot be negative.

Hence, $x \geq 0$, so $0 \leq x$.

Suppose $x > 1$.

Then $x - 1 > 0$.

Since $x < 1 + \frac{1}{n}$, then $x - 1 < \frac{1}{n}$.

Since $n > 0$, we multiply by n to get $n(x-1) < 1$.

Since $x - 1 > 0$, we divide by $x - 1$ to get $n < \frac{1}{x-1}$, so $n \leq \frac{1}{x-1}$.

Since $x - 1 > 0$, then $x - 1 \neq 0$, so $\frac{1}{x-1} \in \mathbb{R}$.

Since n is arbitrary, then $n \leq \frac{1}{x-1}$ for all $n \in \mathbb{N}$.

Thus, the real number $\frac{1}{x-1}$ is an upper bound for $\mathbb{N}$ in $\mathbb{R}$, so $\mathbb{N}$ has an upper bound in $\mathbb{R}$.

By the Archimedean property of $\mathbb{R}$, $\mathbb{N}$ has no upper bound in $\mathbb{R}$.

Hence, we have $\mathbb{N}$ has an upper bound in $\mathbb{R}$ and $\mathbb{N}$ has no upper bound in $\mathbb{R}$, a contradiction.

Therefore, x cannot be greater than 1.

Thus, $x \leq 1$.

Since $0 \leq x$ and $x \leq 1$, then $0 \leq x \leq 1$, so $x \in [0,1]$.

Therefore, $x \in \cap_{n=1}^{\infty} S_n$ implies $x \in [0,1]$, so $\cap_{n=1}^{\infty} S_n \subset [0,1]$.

We prove $[0,1] \subset \cap_{n=1}^{\infty} S_n$.

Let $y \in [0,1]$.

Then $y \in \mathbb{R}$ and $0 \leq y \leq 1$.

To prove $y \in \cap_{n=1}^{\infty} S_n$, we must prove $y \in S_n$ for all $n \in \mathbb{N}$.

Thus, we must prove $y \in (\frac{-1}{n}, 1 + \frac{1}{n})$ for all $n \in \mathbb{N}$, so we must prove $y \in \mathbb{R}$ and $\frac{-1}{n} < y < 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.
Then $n > 0$, so $\frac{1}{n} > 0$, so $\frac{-1}{n} < 0$.
Since $0 \leq y \leq 1$, then either $y = 0$ or $y = 1$ or $0 < y < 1$.
We consider these cases separately.
Case 1: Suppose $0 < y < 1$.
Then $0 < y$ and $y < 1$.
Since $\frac{-1}{n} < 0$ and $0 < y$, then $\frac{-1}{n} < y$.
Since $\frac{1}{n} > 0$, then $0 < \frac{1}{n}$, so $1 < 1 + \frac{1}{n}$.
Since $y < 1$ and $1 < 1 + \frac{1}{n}$, then $y < 1 + \frac{1}{n}$.
Thus, $\frac{-1}{n} < y$ and $y < 1 + \frac{1}{n}$, so $\frac{-1}{n} < y < 1 + \frac{1}{n}$.
Case 2: Suppose $y = 0$.
Since $\frac{-1}{n} < 0$, then $\frac{-1}{n} < y$.
Since $-1 < 0$ and $0 < \frac{1}{n}$, then $-1 < \frac{1}{n}$.
Thus, $0 < 1 + \frac{1}{n}$, so $y < 1 + \frac{1}{n}$.
Hence, $\frac{-1}{n} < y$ and $y < 1 + \frac{1}{n}$, so $\frac{-1}{n} < y < 1 + \frac{1}{n}$.
Case 3: Suppose $y = 1$.
Since $\frac{-1}{n} < 0$ and $0 < 1$, then $\frac{-1}{n} < 1$, so $\frac{-1}{n} < y$.
Since $0 < \frac{1}{n}$, then $1 < 1 + \frac{1}{n}$, so $y < 1 + \frac{1}{n}$.
Hence, $\frac{-1}{n} < y$ and $y < 1 + \frac{1}{n}$, so $\frac{-1}{n} < y < 1 + \frac{1}{n}$.
Therefore, in all cases, $\frac{-1}{n} < y < 1 + \frac{1}{n}$.
Since $y \in \mathbb{R}$ and $\frac{-1}{n} < y < 1 + \frac{1}{n}$, then $y \in (\frac{-1}{n}, 1 + \frac{1}{n})$.
Since n is arbitrary, then $y \in (\frac{-1}{n}, 1 + \frac{1}{n})$ for all $n \in \mathbb{N}$, so $y \in S_n$ for all $n \in \mathbb{N}$.
Therefore, $y \in \cap_{n=1}^{\infty} S_n$.
Thus, $y \in [0, 1]$ implies $y \in \cap_{n=1}^{\infty} S_n$, so $[0, 1] \subset \cap_{n=1}^{\infty} S_n$.
Since $\cap_{n=1}^{\infty} S_n \subset [0, 1]$ and $[0, 1] \subset \cap_{n=1}^{\infty} S_n$, then $\cap_{n=1}^{\infty} S_n = [0, 1]$. □

Lemma 14. *Let $r \in \mathbb{R}$.*
If $r \leq \frac{1}{n}$ for all $n \in \mathbb{N}$, then $r \leq 0$.

Proof. We prove by contrapositive.
Suppose $r > 0$.
Then, by the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < r$.
Therefore, there exists $n \in \mathbb{N}$ such that $r > \frac{1}{n}$, as desired. □

Exercise 15. Let $S_n = (n, \infty)$ for each $n \in \mathbb{N}$.
Then $\cap_{n=1}^{\infty} S_n = \emptyset$.

Proof. We prove by contradiction.
Suppose $\cap_{n=1}^{\infty} S_n \neq \emptyset$.
Then there exists $x \in \cap_{n=1}^{\infty} S_n$, so $x \in S_n$ for all $n \in \mathbb{N}$.
Hence, $x \in (n, \infty)$ for each $n \in \mathbb{N}$, so $x > n$ for each $n \in \mathbb{N}$.
Thus, there exists $x \in \mathbb{R}$ such that $n < x$ for each $n \in \mathbb{N}$, so x is an upper bound of $\mathbb{N}$.
Consequently, $\mathbb{N}$ is bounded above in $\mathbb{R}$.
Since $0 < n$ for all $n \in \mathbb{N}$, then 0 is a lower bound of $\mathbb{N}$, so $\mathbb{N}$ is bounded below in $\mathbb{R}$.

Since $\mathbb{N}$ is bounded above and below in $\mathbb{R}$, then $\mathbb{N}$ is bounded in $\mathbb{R}$.
But, this contradict the Archimedean property that $\mathbb{N}$ is unbounded in $\mathbb{R}$.
Therefore, $\cap_{n=1}^{\infty} S_n = \emptyset$. □

Exercise 16. Compute $\bigcup_{n=1}^{\infty}(\mathbb{R} - [\frac{1}{n}, 2 + \frac{1}{n}])$.

Solution. We prove $\bigcup_{n=1}^{\infty}(\mathbb{R} - [\frac{1}{n}, 2 + \frac{1}{n}]) = (-\infty, 1) \cup (2, \infty)$. □

Proof. We first prove $\cap_{n=1}^{\infty}[\frac{1}{n}, 2 + \frac{1}{n}] = [1, 2]$.

Let $x \in \cap_{n=1}^{\infty}[\frac{1}{n}, 2 + \frac{1}{n}]$.
Then $x \in [\frac{1}{n}, 2 + \frac{1}{n}]$ for each $n \in \mathbb{N}$, so $\frac{1}{n} \leq x \leq 2 + \frac{1}{n}$ for each $n \in \mathbb{N}$.
Hence, $\frac{1}{n} \leq x$ and $x \leq 2 + \frac{1}{n}$ for each $n \in \mathbb{N}$, so $\frac{1}{n} \leq x$ for each $n \in \mathbb{N}$ and $x \leq 2 + \frac{1}{n}$ for each $n \in \mathbb{N}$.
Since $\frac{1}{n} \leq x$ for each $n \in \mathbb{N}$, then for $n = 1$, we have $\frac{1}{1} \leq x$, so $1 \leq x$.
Since $x \leq 2 + \frac{1}{n}$ for each $n \in \mathbb{N}$, then $x - 2 \leq \frac{1}{n}$ for each $n \in \mathbb{N}$.
By a previous lemma, if $r \leq \frac{1}{n}$ for each $n \in \mathbb{N}$, then $r \leq 0$.
Thus, $x - 2 \leq 0$, so $x \leq 2$.
Since $1 \leq x$ and $x \leq 2$, then $1 \leq x \leq 2$, so $x \in [1, 2]$.
Therefore, $\cap_{n=1}^{\infty}[\frac{1}{n}, 2 + \frac{1}{n}] \subset [1, 2]$.

Let $y \in [1, 2]$.
Then $1 \leq y \leq 2$, so $1 \leq y$ and $y \leq 2$.
Since $y \leq 2$, then $y - 2 \leq 0$.

Let $n \in \mathbb{N}$.
Then $n \geq 1 > 0$, so $n \geq 1$ and $n > 0$.
Hence, $1 \geq \frac{1}{n} > 0$, so $1 \geq \frac{1}{n}$ and $\frac{1}{n} > 0$.
Since $\frac{1}{n} \leq 1$ and $1 \leq y$, then $\frac{1}{n} \leq y$.
Since $y - 2 \leq 0$ and $0 < \frac{1}{n}$, then $y - 2 < \frac{1}{n}$, so $y < 2 + \frac{1}{n}$.
Thus, $\frac{1}{n} \leq y$ and $y < 2 + \frac{1}{n}$, so $\frac{1}{n} \leq y < 2 + \frac{1}{n}$.
Hence, $y \in [\frac{1}{n}, 2 + \frac{1}{n}]$.
Therefore, $y \in [\frac{1}{n}, 2 + \frac{1}{n}]$ for all $n \in \mathbb{N}$, so $y \in \cap_{n=1}^{\infty}[\frac{1}{n}, 2 + \frac{1}{n}]$.
Consequently, if $y \in [1, 2]$, then $y \in \cap_{n=1}^{\infty}[\frac{1}{n}, 2 + \frac{1}{n}]$, so $[1, 2] \subset \cap_{n=1}^{\infty}[\frac{1}{n}, 2 + \frac{1}{n}]$.

Since $\cap_{n=1}^{\infty}[\frac{1}{n}, 2 + \frac{1}{n}] \subset [1, 2]$ and $[1, 2] \subset \cap_{n=1}^{\infty}[\frac{1}{n}, 2 + \frac{1}{n}]$, then $\cap_{n=1}^{\infty}[\frac{1}{n}, 2 + \frac{1}{n}] = [1, 2]$.

Observe that

$$\begin{aligned}
\cup_{n=1}^{\infty}(\mathbb{R} - [\frac{1}{n}, 2 + \frac{1}{n}]) &= \mathbb{R} - \cap_{n=1}^{\infty}[\frac{1}{n}, 2 + \frac{1}{n}] \\
&= \mathbb{R} - [1, 2] \\
&= (-\infty, 1) \cup (2, \infty).
\end{aligned}$$

□

Exercise 17. Does there exist $\epsilon > 0$ such that the ϵ neighborhood of $\frac{1}{3}$ contains both $\frac{1}{4}$ and $\frac{1}{2}$, but does not contain $\frac{17}{30}$?

Solution. We must find a positive real ϵ, if one exists, such that $\frac{1}{4} \in N(\frac{1}{3}; \epsilon)$ and $\frac{1}{2} \in N(\frac{1}{3}; \epsilon)$ and $\frac{17}{30} \notin N(\frac{1}{3}; \epsilon)$.

We see that any $\epsilon \in (\frac{1}{6}, \frac{7}{30}]$ will work.

Let $\epsilon = \frac{1}{5}$.

Since $d(\frac{1}{4}, \frac{1}{3}) = |\frac{1}{4} - \frac{1}{3}| = \frac{1}{12} < \frac{1}{5}$, then $\frac{1}{4} \in N(\frac{1}{3}; \frac{1}{5})$.

Since $d(\frac{1}{2}, \frac{1}{3}) = |\frac{1}{2} - \frac{1}{3}| = \frac{1}{6} < \frac{1}{5}$, then $\frac{1}{2} \in N(\frac{1}{3}; \frac{1}{5})$.

Since $d(\frac{17}{30}, \frac{1}{3}) = |\frac{17}{30} - \frac{1}{3}| = \frac{7}{30} > \frac{1}{5}$, then $\frac{17}{30} \notin N(\frac{1}{3}; \frac{1}{5})$. ☐

Exercise 18. The interval $[0, 1]$ is a neighborhood of $\frac{2}{3}$.

Proof. To prove $[0, 1]$ is a neighborhood of $\frac{2}{3}$, let $\epsilon = \frac{1}{6}$.

Then $N(\frac{2}{3}; \frac{1}{6}) = (\frac{2}{3} - \frac{1}{6}, \frac{2}{3} + \frac{1}{6}) = (\frac{3}{6}, \frac{5}{6}) \subset [0, 1]$, as desired. ☐

Exercise 19. Does there exist $\epsilon > 0$ such that the ϵ neighborhood of $\frac{1}{3}$ contains $\frac{11}{12}$, but does not contain either $\frac{1}{2}$ or $\frac{5}{8}$?

Solution. ☐

Exercise 20. Let $a, b \in \mathbb{R}$.

If $a < b$, then there exists a bijective function from the interval (a, b) onto the interval $(0, 1)$.

Proof. Suppose $a < b$.

Then $b - a > 0$, so $b - a \neq 0$.

Let $f : (a, b) \to (0, 1)$ be a function defined by $f(x) = \frac{x-a}{b-a}$ for all $x \in (a, b)$.

We prove f is bijective.

Let $x_1, x_2 \in (a, b)$ such that $f(x_1) = f(x_2)$.

Then $\frac{x_1-a}{b-a} = \frac{x_2-a}{b-a}$, so $x_1 - a = x_2 - a$.

Therefore, $x_1 = x_2$, so f is injective.

Let $t \in (0, 1)$.

Then $0 < t < 1$.

Since $b - a > 0$, then $0 < t(b - a) < b - a$.

Hence, $a < a + t(b - a) < b$.

Let $s = a + t(b - a)$.

Then $a < s < b$, so $s \in (a, b)$.

Observe that

$$\begin{aligned} f(s) &= f(a + t(b - a)) \\ &= \frac{[a + t(b - a)] - a}{b - a} \\ &= \frac{t(b - a)}{b - a} \\ &= t. \end{aligned}$$

Therefore, there exists $s \in (a,b)$ such that $f(s) = t$, so f is surjective.

Since f is injective and surjective, then f is bijective, as desired. $\square$

Exercise 21. Let x and y be distinct real numbers.

Then there is a neighborhood P of x and a neighborhood Q of y such that $P \cap Q = \emptyset$.

Proof. Since x and y are distinct real numbers, then $x, y \in \mathbb{R}$ and $x \neq y$, so either $x < y$ or $x > y$.

Without loss of generality, assume $x < y$.

Let $\delta = \frac{y-x}{2}$.

Since $x < y$, then $y - x > 0$, so $\frac{y-x}{2} > 0$.

Hence, $\delta > 0$.

Let $P = (x - \delta, x + \delta)$ and $Q = (y - \delta, y + \delta)$.

Since $x - \delta < x + \delta$ and $y - \delta < y + \delta$, then P and Q are open intervals, so P is a δ neighborhood of x and Q is a δ neighborhood of y.

Thus, P is a neighborhood of x and Q is a neighborhood of y.

We prove $P \cap Q = \emptyset$ by contradiction.

Suppose $P \cap Q \neq \emptyset$.

Then there exists $p \in P \cap Q$, so $p \in P$ and $p \in Q$.

Since $p \in P$, then $p \in (x - \delta, x + \delta)$, so $x - \delta < p < x + \delta$.

Thus, $p < x + \delta$, so $p < x + \frac{y-x}{2}$.

Hence, $p < \frac{x+y}{2}$.

Since $p \in Q$, then $p \in (y - \delta, y + \delta)$, so $y - \delta < p < y + \delta$.

Hence, $y - \delta < p$, so $y - \frac{y-x}{2} < p$.

Thus, $y + \frac{x-y}{2} < p$, so $\frac{x+y}{2} < p$.

Hence, we have $p < \frac{x+y}{2}$ and $p > \frac{x+y}{2}$, a contradiction.

Therefore, $P \cap Q = \emptyset$, as desired. $\square$

Exercise 22. Let $x \in \mathbb{R}$ and $\epsilon > 0$.

Then $N(x;\epsilon)$ is a neighborhood of each of its members.

Proof. Let $y \in N(x;\epsilon)$.

To prove $N(x;\epsilon)$ is a neighborhood of y, we must prove there exists $\delta > 0$ such that $N(y;\delta) \subset N(x;\epsilon)$.

Let $\delta = \min\{x + \epsilon - y, y - x + \epsilon\}$.

Then $\delta \leq x + \epsilon - y$ and $\delta \leq y - x + \epsilon$, and either $\delta = x + \epsilon - y$ and $\delta = y - x + \epsilon$.

Since $y \in N(x;\epsilon) = (x - \epsilon, x + \epsilon)$, then $x - \epsilon < y < x + \epsilon$, so $x - \epsilon < y$ and $y < x + \epsilon$.

Thus, $0 < y - x + \epsilon$ and $0 < x + \epsilon - y$, so $\delta > 0$.

To prove $N(y;\delta) \subset N(x;\epsilon)$, let $p \in N(y;\delta)$.

Then $|p - y| < \delta$, so $-\delta < p - y < \delta$.

Hence, $y - \delta < p < y + \delta$, so $y - \delta < p$ and $p < y + \delta$.

Since $\delta \leq x + \epsilon - y$, then $y + \delta \leq x + \epsilon$.

Since $p < y + \delta$ and $y + \delta \leq x + \epsilon$, then $p < x + \epsilon$.

Since $\delta \leq y - x + \epsilon$, then $x - \epsilon \leq y - \delta$.
Since $x - \epsilon \leq y - \delta$ and $y - \delta < p$, then $x - \epsilon < p$.
Since $x - \epsilon < p$ and $p < x + \epsilon$, then $x - \epsilon < p < x + \epsilon$, so $p \in (x - \epsilon, x + \epsilon) = N(x; \epsilon)$, as desired. ☐

Exercise 23. The interval $(0, \infty)$ is an open subset of $\mathbb{R}$.

Proof. Since $(0, \infty) = \{x \in \mathbb{R} : x > 0\}$, then $(0, \infty) \subset \mathbb{R}$.
Let $x \in (0, \infty)$.
Then $x > 0$.
Let $\epsilon = x$.
Then $\epsilon > 0$ and $N(x; \epsilon) = N(x; x) = (x - x, x + x) = (0, 2x)$.
Let $p \in N(x; \epsilon)$.
Then $p \in (0, 2x)$, so $0 < p < 2x$.
Hence, $0 < p$, so $p \in (0, \infty)$.
Thus, $N(x; \epsilon) \subset (0, \infty)$.
Therefore, there exists $\epsilon > 0$ such that $N(x; \epsilon) \subset (0, \infty)$, so x is an interior point of $(0, \infty)$.
Hence, every point in $(0, \infty)$ is an interior point of $(0, \infty)$, so $(0, \infty)$ is open. ☐

Exercise 24. Let A be a set and $B = \mathbb{R} - A$.
Then every interior point of A is not an accumulation point of B.

Proof. Let x be an arbitrary interior point of A.
Then there exists $\epsilon > 0$ such that $N(x; \epsilon) \subset A$.
We prove x is not an accumulation point of B by contradiction.
Suppose x is an accumulation point of B.
Since $\epsilon > 0$, then $N'(x; \epsilon) \cap B \neq \emptyset$.
Hence, there exists p such that $p \in N'(x; \epsilon) \cap B$, so $p \in N'(x; \epsilon)$ and $p \in B$.
Since $p \in N'(x; \epsilon)$ and $N'(x; \epsilon) \subset N(x; \epsilon) \subset A$, then $p \in A$.
Since $p \in B$, then $p \in \mathbb{R}$ and $p \notin A$.
Thus, we have $p \in A$ and $p \notin A$, a contradiction.
Therefore, x is not an accumulation point of B, as desired. ☐

Exercise 25. Every real number is an accumulation point of irrational numbers.

Proof. Let $\mathbb{R} - \mathbb{Q}$ be the set of all irrational numbers.
Let x be an arbitrary real number.
We must prove x is an accumulation point of $\mathbb{R} - \mathbb{Q}$.
Let $\epsilon > 0$.
Then $\epsilon > x - x$, so $x + \epsilon > x$ and $x > x - \epsilon$.
Between any two distinct real numbers is an irrational number.
Since $x, x + \epsilon \in \mathbb{R}$ and $x < x + \epsilon$, then there exists $r \in \mathbb{R} - \mathbb{Q}$ such that $x < r < x + \epsilon$.
Hence, $x - \epsilon < x < r < x + \epsilon$, so $x - \epsilon < r < x + \epsilon$.
Thus, $r \in (x - \epsilon, x + \epsilon)$, so $r \in N(x; \epsilon)$.
Since $r > x$, then $r \neq x$, so $r \in N'(x; \epsilon)$.

Thus, $r \in N'(x;\epsilon) \cap (\mathbb{R} - \mathbb{Q})$, so x is an accumulation point of $\mathbb{R} - \mathbb{Q}$, as desired. ☐

Exercise 26. Let S be a set of real numbers.

If there exists $\delta > 0$ such that the distance between every distinct pair of elements of S is greater than δ, then S has no accumulation points.

Proof. Suppose there exists $\delta > 0$ such that the distance between every distinct pair of elements of S is greater than δ.

Then for every $a, b \in S$ with $a \neq b$, then $d(a,b) = |a-b| > \delta$.

We prove S has no accumulation points by contradiction.

Suppose S has an accumulation point.

Then there exists x such that x is an accumulation point of S.

Since $\delta > 0$, then $\frac{\delta}{2} > 0$, so $N'(x;\frac{\delta}{2}) \cap S \neq \emptyset$.

Hence, there exists a point a such that $a \in N'(x;\frac{\delta}{2}) \cap S$, so $a \in N'(x;\frac{\delta}{2})$ and $a \in S$.

Since $a \in N'(x;\frac{\delta}{2})$, then $a \in N(x;\frac{\delta}{2})$, so $d(a,x) < \frac{\delta}{2}$.

Since $a \in N(x;\frac{\delta}{2})$ and $a \in S$, then $a \in N(x;\frac{\delta}{2}) \cap S$.

Since x is an accumulation point of S and $\frac{\delta}{2} > 0$, then the set $N(x;\frac{\delta}{2}) \cap S$ is infinite.

Thus, there is at least one other point of the set $N(x;\frac{\delta}{2}) \cap S$.

Therefore, there exists a point b such that $b \in N(x;\frac{\delta}{2}) \cap S$ and $b \neq a$.

Hence, $b \in N(x;\frac{\delta}{2})$ and $b \in S$.

Since $b \in N(x;\frac{\delta}{2})$, then $d(x,b) < \frac{\delta}{2}$.

Since $d(a,x) < \frac{\delta}{2}$ and $d(x,b) < \frac{\delta}{2}$, then $d(a,x) + d(x,b) < \delta$.

By the triangle inequality we have $d(a,x) + d(x,b) \geq d(a,b)$.

Since $d(a,b) \leq d(a,x) + d(x,b)$ and $d(a,x) + d(x,b) < \delta$, then $d(a,b) < \delta$.

Since $a \in S$ and $b \in S$ and $a \neq b$, then $d(a,b) > \delta$.

Thus, we have a contradiction $d(a,b) < \delta$ and $d(a,b) > \delta$.

Therefore, S does not have an accumulation point. ☐

Exercise 27. True or false?

If p is an accumulation point of a set A and a set B, then p is an accumulation point of $A \cap B$.

Proof. This is a false statement.

Here is a counterexample.

Let $A = [0,\infty)$ and $B = (-\infty, 0]$.

Then $A \cap B = \{0\}$.

Observe that 0 is an accumulation point of A and B.

Since $A \cap B$ is a finite set, then $A \cap B$ does not have an accumulation point, so 0 cannot be an accumulation point of $A \cap B$. ☐

Exercise 28. Every interior point is an accumulation point, but not conversely.

Proof. Let S be a set of real numbers.

We must prove every interior point of S is an accumulation point of S.

Either $S = \emptyset$ or $S \neq \emptyset$.

We consider these cases separately.

Case 1: Suppose $S = \emptyset$.

Since there is no interior point of $\emptyset$, then for any x, x is not an interior point of $\emptyset$.

Hence, for any x, the conditional 'x is an interior point of $\emptyset$ implies x is an accumulation point of $\emptyset$' is vacuously true.

Therefore, every interior point of $\emptyset$ is an accumulation point of $\emptyset$, so every interior point of S is an accumulation point of S.

Case 2: Suppose $S \neq \emptyset$.

Then there is at least one element of S.

Let $x \in S$ such that x is an interior point of S.

Then there exists $\delta > 0$ such that $N(x; \delta) \subset S$.

Let $\epsilon > 0$ be given.

To prove x is an accumulation point of S, we must prove $N'(x; \epsilon) \cap S \neq \emptyset$.

Let $M = \min\{\delta, \epsilon\}$.

Then $M \leq \delta$ and $M \leq \epsilon$.

Since $\delta > 0$ and $\epsilon > 0$, then $M > 0$, so $\frac{M}{2} > 0$.

Since $\frac{1}{2} < 1$ and $M > 0$, then $\frac{M}{2} < M$.

Let $p = x + \frac{M}{2}$.

Then $p - x = \frac{M}{2} > 0$, so $d(p, x) = |p - x| = p - x = \frac{M}{2}$.

Since $\frac{M}{2} < M \leq \delta$, then $d(p, x) < \delta$, so $p \in N(x; \delta)$.

Since $N(x; \delta) \subset S$, then $p \in S$.

Since $d(p, x) = \frac{M}{2} < M \leq \epsilon$, then $d(p, x) < \epsilon$, so $p \in N(x; \epsilon)$.

Since $p - x > 0$, then $p > x$, so $p \neq x$.

Hence, $p \in N'(x; \epsilon)$.

Therefore, $p \in N'(x; \epsilon) \cap S$, so $N'(x; \epsilon) \cap S \neq \emptyset$, as desired. □

Proof. To disprove the converse, we must prove 'every accumulation point is an interior point' is false.

Hence, we must prove 'some accumulation point is not an interior point'.

Let $a, b \in \mathbb{R}$ with $a < b$.

Let (a, b) be the open interval.

Since $a < b$, then a is an accumulation point of (a, b).

Since $(a, b) \neq \emptyset$, then every interior point of (a, b) is an element of (a, b).

Hence, if x is an interior point of (a, b), then $x \in (a, b)$, so if $x \notin (a, b)$, then x is not an interior point of (a, b).

In particular, since $a \notin (a, b)$, then a is not an interior point of (a, b).

Therefore, a is an accumulation point of (a, b), but a is not an interior point of (a, b). □

Exercise 29. Let S be a set of real numbers.

If $\sup S$ exists, then either $\sup S \in S$ or $\sup S$ is an accumulation point of S.

Proof. We prove by contrapositive.

Suppose $\sup S$ exists and $\sup S \notin S$.

To prove $\sup S$ is an accumulation point of S, let $\epsilon > 0$ be given.

We must prove there exists $x \in S$ such that $x \in N'(\sup S; \epsilon)$.

Since $\epsilon > 0 = \sup S - \sup S$, then $\epsilon > \sup S - \sup S$, so $\sup S > \sup S - \epsilon$.

Since $\sup S - \epsilon < \sup S$, then $\sup S - \epsilon$ is not an upper bound of S.

Hence, there exists $x \in S$ such that $x > \sup S - \epsilon$.

Since $\sup S$ is an upper bound of S and $x \in S$, then $x \leq \sup S$, so either $x < \sup S$ or $x = \sup S$.

Since $x \in S$ and $\sup S \notin S$, then $x \neq \sup S$.

Thus, $x < \sup S$, so $\sup S - x > 0$.

Since $\sup S - \epsilon < x$, then $\sup S - x < \epsilon$.

Hence, $d(x, \sup S) = |x - \sup S| = |\sup S - x| = \sup S - x < \epsilon$, so $d(x, \sup S) < \epsilon$.

Thus, $x \in N(\sup S; \epsilon)$.

Since $x \neq \sup S$, then $x \in N'(\sup S; \epsilon)$.

Therefore, there exists $x \in S$ such that $x \in N'(\sup S; \epsilon)$, as desired. □

Exercise 30. Let S be a set of real numbers.

If $\inf S$ exists, then either $\inf S \in S$ or $\inf S$ is an accumulation point of S.

Proof. We prove by contrapositive.

Suppose $\inf S$ exists and $\inf S \notin S$.

To prove $\inf S$ is an accumulation point of S, let $\epsilon > 0$ be given.

We must prove there exists $x \in S$ such that $x \in N'(\inf S; \epsilon)$.

Since $\epsilon > 0 = \inf S - \inf S$, then $\epsilon > \inf S - \inf S$, so $\inf S + \epsilon > \inf S$.

Since $\inf S + \epsilon > \inf S$, then $\inf S + \epsilon$ is not a lower bound of S.

Hence, there exists $x \in S$ such that $x < \inf S + \epsilon$.

Since $\inf S$ is a lower bound of S and $x \in S$, then $\inf S \leq x$, so either $\inf S < x$ or $\inf S = x$.

Since $x \in S$ and $\inf S \notin S$, then $x \neq \inf S$.

Thus, $\inf S < x$, so $x - \inf S > 0$.

Since $x < \inf S + \epsilon$, then $x - \inf S < \epsilon$.

Hence, $d(x, \inf S) = |x - \inf S| = x - \inf S < \epsilon$, so $d(x, \inf S) < \epsilon$.

Thus, $x \in N(\inf S; \epsilon)$.

Since $x \neq \inf S$, then $x \in N'(\inf S; \epsilon)$.

Therefore, there exists $x \in S$ such that $x \in N'(\inf S; \epsilon)$, as desired. □

Exercise 31. Let $S \subset \mathbb{R}$ such that at least one point of accumulation of S exists.

Then for every $\epsilon > 0$ there exist points $x, y \in S$ such that $0 < |x - y| < \epsilon$.

Proof. Let $\epsilon > 0$ be given.

Then $\frac{\epsilon}{2} > 0$.

Since at least one accumulation point of S exists, let p be an accumulation point of S.

Then $N'(p; \frac{\epsilon}{2}) \cap S \neq \emptyset$, so there exists x such that $x \in N'(p; \frac{\epsilon}{2}) \cap S$.

Hence, $x \in N'(p; \frac{\epsilon}{2})$ and $x \in S$.

Since $x \in N'(p; \frac{\epsilon}{2})$ and $N'(p; \frac{\epsilon}{2}) \subset N(p; \frac{\epsilon}{2})$, then $x \in N(p; \frac{\epsilon}{2})$, so $x \in N(p; \frac{\epsilon}{2}) \cap S$.

Since p is an accumulation point of S and $\frac{\epsilon}{2} > 0$, then the set $N(p; \frac{\epsilon}{2}) \cap S$ is infinite.

Thus, there exists at least one other point of $N(p; \frac{\epsilon}{2}) \cap S$.

Hence, there exists y such that $y \in N(p; \frac{\epsilon}{2}) \cap S$ and $y \neq x$.

Since $y \in N(p; \frac{\epsilon}{2}) \cap S$, then $y \in N(p; \frac{\epsilon}{2})$ and $y \in S$.

Since $x \in N(p; \frac{\epsilon}{2})$ and $y \in N(p; \frac{\epsilon}{2})$, then $d(p, x) < \frac{\epsilon}{2}$ and $d(p, y) < \frac{\epsilon}{2}$, so $|p - x| < \frac{\epsilon}{2}$ and $|p - y| < \frac{\epsilon}{2}$.

Since $x \neq y$, then $d(x, y) > 0$, so $|x - y| > 0$.

Observe that

$$\begin{aligned} |x - y| &= |(x - p) + (p - y)| \\ &\leq |x - p| + |p - y| \\ &= |p - x| + |p - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, $|x - y| < \epsilon$, so $0 < |x - y| < \epsilon$, as desired. ☐

Exercise 32. Let (a_n) be a sequence of points such that $\lim_{n\to\infty} a_n = L$ for some real number L.

Let L be an interior point of a set S.

Then there is an integer N such that $a_n \in S$ for all $n > N$.

Proof. Since L is an interior point of S, then there exists $\epsilon > 0$ such that $N(L; \epsilon) \subset S$.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < \epsilon$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|a_n - L| < \epsilon$, so $d(a_n, L) < \epsilon$.

Thus, $a_n \in N(L; \epsilon)$.

Since $N(L; \epsilon) \subset S$, then $a_n \in S$.

Hence, $a_n \in S$ for all integers $n > N$.

Therefore, there exists $N \in \mathbb{N}$ such that $a_n \in S$ for all integers $n > N$. ☐

Exercise 33. Let (x_n) be a sequence of real numbers such that $\lim_{n\to\infty} x_n = L$ for some real number L and $x_n \neq L$ for all $n \in \mathbb{N}$.

Then the set $\{x_n : n \in \mathbb{N}\}$ has exactly one accumulation point, L.

Proof. Let S be the range of the sequence (x_n).

Then $S = \{x_n : n \in \mathbb{N}\}$.

We must prove L is the unique accumulation point of S.

Existence:

We prove L is an accumulation point of S.

Let $\epsilon > 0$ be given.

Since $\lim_{n\to\infty} x_n = L$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|x_n - L| < \epsilon$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|x_n - L| < \epsilon$, so $d(x_n, L) < \epsilon$.

Hence, $x_n \in N(L; \epsilon)$.

Since $n \in \mathbb{N}$, then $x_n \neq L$, so $x_n \in N'(L; \epsilon)$.

Since $n \in \mathbb{N}$, then $x_n \in S$, so $x_n \in N'(L; \epsilon) \cap S$.

Therefore, $N'(L; \epsilon) \cap S \neq \emptyset$, so L is an accumulation point of S.

Uniqueness:

Suppose x is an accumulation point of S.

We must prove $x = L$.

Let $\epsilon > 0$ be given.

Then $\frac{\epsilon}{2} > 0$.

Since x is an accumulation point of S, then $N'(x; \frac{\epsilon}{2}) \cap S \neq \emptyset$, so there exists p such that $p \in N'(x; \frac{\epsilon}{2}) \cap S$.

Hence, $p \in N'(x; \frac{\epsilon}{2})$ and $p \in S$.

Since $p \in S$, then there exists $m \in \mathbb{N}$ such that $p = x_m$.

Thus, $x_m \in N'(x; \frac{\epsilon}{2})$, so $x_m \in N(x; \frac{\epsilon}{2})$ and $x_m \neq x$.

Hence, $d(x, x_m) < \frac{\epsilon}{2}$, so $|x - x_m| < \frac{\epsilon}{2}$.

Since $\lim_{n\to\infty} x_n = L$ and $\frac{\epsilon}{2} > 0$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|x_n - L| < \frac{\epsilon}{2}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|x_n - L| < \frac{\epsilon}{2}$.

We're stuck!!!!! $\square$

Exercise 34. The interior of a set is open.

Proof. Let S be a set.

Let S° be the interior of S.

Then $S^\circ = \{x : x \text{ is an interior point of } S\}$.

We must prove S° is open.

Either $S^\circ = \emptyset$ or $S^\circ \neq \emptyset$.

We consider these cases separately.

Case 1: Suppose $S^\circ = \emptyset$.

Since the empty set is open, then S° is open.

Case 2: Suppose $S^\circ \neq \emptyset$.

Then there is at least one element of S°.

Let $x \in S^\circ$.

Then x is an interior point of S, so there exists $\epsilon > 0$ such that $N(x; \epsilon) \subset S$.

To prove S° is open, we must prove x is an interior point of S°.

Since $N(x; \epsilon) \neq \emptyset$, then there is at least one element of $N(x; \epsilon)$.

Let $p \in N(x; \epsilon)$.

Then $p \in (x - \epsilon, x + \epsilon)$.

Since $(x-\epsilon, x+\epsilon)$ is an open interval and every point in an open interval is an interior point, then p is an interior point of $(x-\epsilon, x+\epsilon)$, so p is an interior point of $N(x;\epsilon)$.

Hence, there exists $\delta > 0$ such that $N(p;\delta) \subset N(x;\epsilon)$.

Since $N(x;\epsilon) \subset S$, then $N(p;\delta) \subset S$.

Thus, there exists $\delta > 0$ such that $N(p;\delta) \subset S$, so p is an interior point of S.

Hence, $p \in S^\circ$, so $N(x;\epsilon) \subset S^\circ$.

Therefore, there exists $\epsilon > 0$ such that $N(x;\epsilon) \subset S^\circ$, so x is an interior point of S°, as desired ☐

Exercise 35. Let S be a set.

If $S' = \emptyset$, then S is closed.

Proof. Let S' be the derived set of S.

Then $S' = \{x : x \text{ is an accumulation point of } S\}$.

Suppose $S' = \emptyset$.

Then there is no accumulation point of S.

Therefore, S is a set with no accumulation points, so S is closed. ☐

Exercise 36. Let A and B be closed sets.

Then $A \cup B$ is closed and $A \cap B$ is closed.

Proof. We prove $A \cup B$ is closed.

Let x be an accumulation point of $A \cup B$.

Let $\epsilon > 0$ be given.

Since x is an accumulation point of $A \cup B$, then $N'(x;\epsilon) \cap (A \cup B) \neq \emptyset$, so there exists p such that $p \in N'(x;\epsilon) \cap (A \cup B)$.

Hence, $p \in N'(x;\epsilon)$ and $p \in A \cup B$.

Since $p \in A \cup B$, then either $p \in A$ or $p \in B$.

We consider these cases separately.

Case 1: Suppose $p \in A$.

Since $p \in N'(x;\epsilon)$ and $p \in A$, then $p \in N'(x;\epsilon) \cap A$, so $N'(x;\epsilon) \cap A \neq \emptyset$.

Thus, x is an accumulation point of A.

Since A is closed, then $x \in A$.

Case 2: Suppose $p \in B$.

Since $p \in N'(x;\epsilon)$ and $p \in B$, then $p \in N'(x;\epsilon) \cap B$, so $N'(x;\epsilon) \cap B \neq \emptyset$.

Thus, x is an accumulation point of B.

Since B is closed, then $x \in B$.

Hence, either $x \in A$ or $x \in B$, so $x \in A \cup B$.

Therefore, $A \cup B$ is closed. ☐

Proof. We prove $A \cap B$ is closed.

Let x be an accumulation point of $A \cap B$.

Let $\epsilon > 0$ be given.

Since x is an accumulation point of $A \cap B$, then $N'(x;\epsilon) \cap (A \cap B) \neq \emptyset$, so there exists p such that $p \in N'(x;\epsilon) \cap (A \cap B)$.

Hence, $p \in N'(x;\epsilon)$ and $p \in A \cap B$.
Since $p \in A \cap B$, then $p \in A$ and $p \in B$.
Since $p \in N'(x;\epsilon)$ and $p \in A$, then $p \in N'(x;\epsilon) \cap A$, so $N'(x;\epsilon) \cap A \neq \emptyset$.
Thus, x is an accumulation point of A.
Since A is closed, then $x \in A$.
Since $p \in N'(x;\epsilon)$ and $p \in B$, then $p \in N'(x;\epsilon) \cap B$, so $N'(x;\epsilon) \cap B \neq \emptyset$.
Thus, x is an accumulation point of B.
Since B is closed, then $x \in B$.
Hence, $x \in A$ and $x \in B$, so $x \in A \cap B$.
Therefore, $A \cap B$ is closed. □

Exercise 37. The derived set of a set is closed.

Proof. Let S be a set.
Let S' be the derived set of S.
Then $S' = \{x : x \text{ is an accumulation point of } S\}$.
Let S'' be the derived set of S'.
Then $S'' = \{x : x \text{ is an accumulation point of } S'\}$.
We must prove S' is closed.
Either $S'' = \emptyset$ or $S'' \neq \emptyset$.
We consider these cases separately.
Case 1: Suppose $S'' = \emptyset$.
Then there is no accumulation point of S'.
Therefore, S' is a set with no accumulation points, so S' is closed.
Case 2: Suppose $S'' \neq \emptyset$.
Then there is at least one element of S''.
Let $x \in S''$.
Then x is an accumulation point of S'.
To prove S' is closed, we must prove $x \in S'$, so we must prove x is an accumulation point of S.
Hence, we must prove $N'(x;\epsilon) \cap S \neq \emptyset$ for every $\epsilon > 0$.
Let $\epsilon > 0$ be given.
Since x is an accumulation point of S', then $N'(x;\epsilon) \cap S' \neq \emptyset$.
Thus, there exists p such that $p \in N'(x;\epsilon) \cap S'$, so $p \in N'(x;\epsilon)$ and $p \in S'$.
Since $p \in N'(x;\epsilon)$, then $p \in N(x;\epsilon)$ and $p \neq x$.
Since $p \neq x$, then either $p < x$ or $p > x$.
We consider these cases separately.
Case 2a: Suppose $p > x$.
Since $p \in N(x;\epsilon)$, then $p \in (x-\epsilon, x+\epsilon)$, so $x - \epsilon < p < x + \epsilon$.
Hence, $p < x + \epsilon$.
Since $x < p$ and $p < x + \epsilon$, then $x < p < x + \epsilon$, so $p \in (x, x+\epsilon)$.
Since the open interval $(x, x+\epsilon)$ is open, then p is an interior point of $(x, x+\epsilon)$.
Thus, there exists $\delta > 0$ such that $N(p;\delta) \subset (x, x+\epsilon)$.
Since $p \in S'$, then p is an accumulation point of S, so $N'(p;\delta) \cap S \neq \emptyset$.
Thus, there exists q such that $q \in N'(p;\delta) \cap S$, so $q \in N'(p;\delta)$ and $q \in S$.

Since $q \in N'(p;\delta)$, then $q \in N(p;\delta)$.
Since $N(p;\delta) \subset (x, x+\epsilon) \subset (x-\epsilon, x+\epsilon)$, then $q \in (x-\epsilon, x+\epsilon)$, so $q \in N(x;\epsilon)$.
We prove $q \neq x$ by contradiction.
Suppose $q = x$.
Since $q \in N(p;\delta)$ and $N(p;\delta) \subset (x, x+\epsilon)$, then $q \in (x, x+\epsilon)$, so $x < q < x+\epsilon$.
Thus, $x < q$.
But, this contradicts the assumption $x = q$.
Hence, $q \neq x$.
Since $q \in N(x;\epsilon)$ and $q \neq x$, then $q \in N'(x;\epsilon)$.
Thus, $q \in N'(x;\epsilon)$ and $q \in S$, so $q \in N'(x;\epsilon) \cap S$.
Therefore, $N'(x;\epsilon) \cap S \neq \emptyset$.
Case 2b: Suppose $p < x$.
Since $p \in N(x;\epsilon)$, then $p \in (x - \epsilon, x + \epsilon)$, so $x - \epsilon < p < x + \epsilon$.
Hence, $x - \epsilon < p$.
Since $x - \epsilon < p$ and $p < x$, then $x - \epsilon < p < x$, so $p \in (x - \epsilon, x)$.
Since the open interval $(x - \epsilon, x)$ is open, then p is an interior point of $(x - \epsilon, x)$.
Thus, there exists $\delta > 0$ such that $N(p;\delta) \subset (x - \epsilon, x)$.
Since $p \in S'$, then p is an accumulation point of S, so $N'(p;\delta) \cap S \neq \emptyset$.
Thus, there exists q such that $q \in N'(p;\delta) \cap S$, so $q \in N'(p;\delta)$ and $q \in S$.
Since $q \in N'(p;\delta)$, then $q \in N(p;\delta)$.
Since $N(p;\delta) \subset (x-\epsilon, x) \subset (x-\epsilon, x+\epsilon)$, then $q \in (x-\epsilon, x+\epsilon)$, so $q \in N(x;\epsilon)$.
We prove $q \neq x$ by contradiction.
Suppose $q = x$.
Since $q \in N(p;\delta)$ and $N(p;\delta) \subset (x-\epsilon, x)$, then $q \in (x-\epsilon, x)$, so $x-\epsilon < q < x$.
Thus, $q < x$.
But, this contradicts the assumption $q = x$.
Hence, $q \neq x$.
Since $q \in N(x;\epsilon)$ and $q \neq x$, then $q \in N'(x;\epsilon)$.
Thus, $q \in N'(x;\epsilon)$ and $q \in S$, so $q \in N'(x;\epsilon) \cap S$.
Therefore, $N'(x;\epsilon) \cap S \neq \emptyset$.
Thus, in either case, $N'(x;\epsilon) \cap S \neq \emptyset$, so x is an accumulation point of S, as desired. ☐

Exercise 38. Let S be a set.
If S is closed and $a \in S$, then the set $\{x \in S : x \leq a\}$ is closed.

Proof. Suppose S is closed and $a \in S$.
Let $T = \{x \in S : x \leq a\}$.
Then $T \subset S$.
We must prove T is closed.
Either there is an accumulation point of T or there is not.
We consider these cases separately.
Case 1: Suppose there is no accumulation point of T.
Then T is a set with no accumulation points.
Since every set with no accumulation points is closed, then T is closed.

Case 2: Suppose there is some accumulation point of T.

Then there is at least one accumulation point of T.

Let x be an arbitrary accumulation point of T.

Since x is an accumulation point of T and $T \subset S$, then x is an accumulation point of S.

Since S is closed, then $x \in S$.

We prove $x \leq a$ by contradiction.

Suppose $x > a$.

Then $x - a > 0$.

Since x is an accumulation point of T, then $N'(x; x - a) \cap T \neq \emptyset$, so there exists p such that $p \in N'(x; x - a) \cap T$.

Hence, $p \in N'(x; x - a)$ and $p \in T$.

Since $p \in N'(x; x - a)$, then $p \in N(x; x - a)$, so $p \in (a, 2x - a)$.

Thus, $a < p < 2x - a$, so $a < p$.

Since $p \in T$, then $p \leq a$.

Hence, we have $p \leq a$ and $p > a$, a violation of trichotomy.

Therefore, $x \leq a$.

Since $x \in S$ and $x \leq a$, then $x \in T$, so T is closed. □

Exercise 39. Let $S_1 = (-3, \frac{2}{3})$.

Let $S_2 = (-1, \frac{1}{2})$.

Let $S_3 = (0, \frac{1}{2})$.

Let $S_4 = (\frac{1}{3}, \frac{2}{3})$.

Let $S_5 = (\frac{1}{2}, 1)$.

Let $S_6 = (\frac{9}{10}, 2)$.

Let $S_7 = (\frac{2}{3}, \frac{3}{2})$.

Let $\mathcal{F} = \{S_k : n \in \mathbb{N}, 1 \leq k \leq 7\}$.

Then

a. The collection of sets $\mathcal{F}$ is a finite open covering of $[0, 1]$.

b. The subcollection $\{S_1, S_5, S_6\}$ of $\mathcal{F}$ is a subcover of $[0, 1]$.

c. The subcollection $\{S_1, S_2, S_3, S_7\}$ of $\mathcal{F}$ is not a cover of $[0, 1]$.

Proof. We prove a.

Since $\cup_{\mathcal{F}} = \cup_{k=1}^{7} S_k = S_1 \cup S_2 \cup ... \cup S_7 = (-3, 2) \supset [0, 1]$, then $\mathcal{F}$ is a covering of $[0, 1]$.

Since each open interval S_k of $\mathcal{F}$ is an open set, then $\mathcal{F}$ is an open covering of $[0, 1]$.

Since $\mathcal{F}$ is a finite set, then $\mathcal{F}$ is a finite open covering of $[0, 1]$. □

Proof. We prove b.

Let $\mathcal{G} = \{S_1, S_5, S_6\}$.

Since $\cup_{\mathcal{G}} = S_1 \cup S_5 \cup S_6 = (-3, 2) \supset [0, 1]$, then $\mathcal{G}$ is a covering of $[0, 1]$.

Since $\mathcal{G} = \{S_1, S_5, S_6\} \subset \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\} = \mathcal{F}$, then $\mathcal{G}$ is a subcovering of $[0, 1]$. □

Proof. We prove c.

Let $\mathcal{H} = \{S_1, S_2, S_3, S_7\}$.

Since $\frac{2}{3} \notin (-3, \frac{2}{3})$ and $\frac{2}{3} \notin (\frac{2}{3}, \frac{3}{2})$, then $\frac{2}{3} \notin (-3, \frac{2}{3}) \cup (\frac{2}{3}, \frac{3}{2})$.

Since $\cup_{\mathcal{H}} = S_1 \cup S_2 \cup S_3 \cup S_7 = (-3, \frac{2}{3}) \cup (\frac{2}{3}, \frac{3}{2})$, then $\frac{2}{3} \notin \cup_{\mathcal{H}}$.

Since $\frac{2}{3} \in [0, 1]$, but $\frac{2}{3} \notin \cup_{\mathcal{H}}$, then $[0, 1] \not\subset \mathcal{H}$, so $\mathcal{H}$ is not a covering of $[0, 1]$. □

Exercise 40. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Let $\mathcal{F} = \{N(\frac{1}{n}; \frac{1}{n(n+1)}) : n \in \mathbb{N}\}$.

Then $\mathcal{F}$ is an open covering of S.

Proof. We prove $\mathcal{F}$ is a covering of S.

Since $1 \in \mathbb{N}$ and $\frac{1}{1} = 1 \in S$, then $S \neq \emptyset$.

Let $x \in S$.

Then there exists $n \in \mathbb{N}$ such that $x = \frac{1}{n}$.

Since $n \in \mathbb{N}$, then $n > 0$.

Since $n + 1 > n$ and $n > 0$, then $n + 1 > 0$.

Since $n > 0$ and $n + 1 > 0$, then $n(n + 1) > 0$, so $\frac{1}{n(n+1)} > 0$.

Hence, $\frac{1}{n(n+1)} > \frac{1}{n} - \frac{1}{n}$, so $\frac{1}{n} + \frac{1}{n(n+1)} > \frac{1}{n}$ and $\frac{1}{n} > \frac{1}{n} - \frac{1}{n(n+1)}$.

Thus, $\frac{1}{n} - \frac{1}{n(n+1)} < \frac{1}{n} < \frac{1}{n} + \frac{1}{n(n+1)}$, so $\frac{1}{n} \in (\frac{1}{n} - \frac{1}{n(n+1)}, \frac{1}{n} + \frac{1}{n(n+1)})$.

Therefore, $\frac{1}{n} \in N(\frac{1}{n}; \frac{1}{n(n+1)})$, so $x \in N(\frac{1}{n}; \frac{1}{n(n+1)})$.

Let $A = N(\frac{1}{n}; \frac{1}{n(n+1)})$.

Then $x \in A$.

Since $n \in \mathbb{N}$ and $A = N(\frac{1}{n}; \frac{1}{n(n+1)})$, then $A \in \mathcal{F}$.

Thus, there exists $A \in \mathcal{F}$ such that $x \in A$, so $x \in \cup\mathcal{F}$.

Therefore, $S \subset \cup\mathcal{F}$, so $\mathcal{F}$ is a covering of S.

For each $n \in \mathbb{N}$, the set $N(\frac{1}{n}; \frac{1}{n(n+1)})$ is an open interval, so $N(\frac{1}{n}; \frac{1}{n(n+1)})$ is an open set.

Thus, $N(\frac{1}{n}; \frac{1}{n(n+1)})$ is an open set for each $n \in \mathbb{N}$, so each set in $\mathcal{F}$ is open.

Therefore, $\mathcal{F}$ is an open covering of S. □

Exercise 41. For each $a \in \mathbb{R}$, let $S_a = (a, a + \frac{1}{4})$.

Let I be the index set $[0, \frac{3}{4}]$.

Let $\mathcal{F} = \{S_a : a \in I\}$.

Then

a. The collection of sets $\mathcal{F}$ is an open covering of $(0, 1)$.

b. The subcollection $\{S_0, S_{\frac{1}{8}}, S_{\frac{1}{4}}, S_{\frac{3}{8}}, S_{\frac{1}{2}}, S_{\frac{5}{8}}, S_{\frac{3}{4}}\}$ of $\mathcal{F}$ is a finite subcover of $(0, 1)$.

c. The interval $(0, 1)$ is not compact.

Proof. We prove a.

To prove $\mathcal{F}$ is a covering of the open unit interval $(0, 1)$, we prove $(0, 1) \subset \cup_{\mathcal{F}}$.

Let $x \in (0, 1)$.

Then $0 < x < 1$, so $0 < x$ and $x < 1$.

We must prove there exists $S_a \in \mathcal{F}$ such that $x \in S_a$.

Let $m = \max\{0, x - \frac{1}{4}\}$.

Let $M = \min\{\frac{3}{4}, x\}$.

We first prove $m < M$.

Since either $m = 0$ or $m = x - \frac{1}{4}$ and either $M = \frac{3}{4}$ or $M = x$, then either $m = 0$ and $M = \frac{3}{4}$ or $m = 0$ and $M = x$ or $m = x - \frac{1}{4}$ and $M = \frac{3}{4}$ or $m = x - \frac{1}{4}$ and $M = x$.

Thus, we have 4 cases to consider.

We consider these cases separately.

Case 1: Suppose $m = 0$ and $M = \frac{3}{4}$.

Since $0 < \frac{3}{4}$, then $m < M$.

Case 2: Suppose $m = 0$ and $M = x$.

Since $0 < x$, then $m < M$.

Case 3: Suppose $m = x - \frac{1}{4}$ and $M = \frac{3}{4}$.

Since $x < 1$, then $x - \frac{1}{4} < \frac{3}{4}$, so $m < M$.

Case 4: Suppose $m = x - \frac{1}{4}$ and $M = x$.

Since $x - \frac{1}{4} < x$, then $m < M$.

Thus, in all cases, $m < M$.

Hence, then by density of $\mathbb{R}$, there exists $a \in \mathbb{R}$ such that $m < a < M$, so $m < a$ and $a < M$.

Since $0 \leq m$ and $m < a$, then $0 < a$.

Since $a < M$ and $M \leq \frac{3}{4}$, then $a < \frac{3}{4}$.

Thus, $0 < a < \frac{3}{4}$, so $a \in [0, \frac{3}{4}] = I$.

Hence, $S_a \in \mathcal{F}$.

Since $a < M$ and $M \leq x$, then $a < x$.

Since $x - \frac{1}{4} \leq m$ and $m < a$, then $x - \frac{1}{4} < a$, so $x < a + \frac{1}{4}$.

Thus, $a < x < a + \frac{1}{4}$, so $x \in (a, a + \frac{1}{4}) = S_a$.

Thus, there exists $S_a \in \mathcal{F}$ such that $x \in S_a$, so $\mathcal{F}$ is a covering of $(0, 1)$.

We prove each set in $\mathcal{F}$ is open.

Since $S_0 = (0, \frac{1}{4}) \in \mathcal{F}$, then $\mathcal{F} \neq \emptyset$.

Let S_a be an arbitrary set in $\mathcal{F}$.

Then $S_a = (a, a + \frac{1}{4})$ for some real $a \in [0, \frac{3}{4}]$.

Since the open interval $(a, a + \frac{1}{4})$ is open, then S_a is open.

Thus, every set in $\mathcal{F}$ is open, so $\mathcal{F}$ is an open covering of $(0, 1)$. $\square$

Proof. We prove b.

Let $\mathcal{G} = \{S_0, S_{\frac{1}{8}}, S_{\frac{1}{4}}, S_{\frac{3}{8}}, S_{\frac{1}{2}}, S_{\frac{5}{8}}, S_{\frac{3}{4}}\}$.

We must prove $\mathcal{G}$ is a finite subcover of $(0, 1)$.

To prove $(0, 1) \subset \cup_{\mathcal{G}}$, let $x \in (0, 1)$.

Since $(0, 1) = (0, \frac{1}{4}) \cup \{\frac{1}{4}\} \cup (\frac{1}{4}, \frac{1}{2}) \cup \{\frac{1}{2}\} \cup (\frac{1}{2}, \frac{3}{4}) \cup \{\frac{3}{4}\} \cup (\frac{3}{4}, 1)$, then either $x \in (0, \frac{1}{4})$ or $x = \frac{1}{4}$ or $x \in (\frac{1}{4}, \frac{1}{2})$ or $x = \frac{1}{2}$ or $x \in (\frac{1}{2}, \frac{3}{4})$ or $x = \frac{3}{4}$ or $x \in (\frac{3}{4}, 1)$.

We consider these cases separately.

Case 1: Suppose $x \in (0, \frac{1}{4})$.

Since $(0, \frac{1}{4}) = S_0$, then $x \in S_0$.

Since $S_0 \subset \cup_{\mathcal{G}}$, then $x \in \cup_{\mathcal{G}}$.

Case 2: Suppose $x = \frac{1}{4}$.

Since $\frac{1}{8} < \frac{1}{4} < \frac{3}{8}$, then $\frac{1}{8} < x < \frac{3}{8}$, so $x \in (\frac{1}{8}, \frac{3}{8})$.

Since $(\frac{1}{8}, \frac{3}{8}) = S_{\frac{1}{8}}$, then $x \in S_{\frac{1}{8}}$.
Since $S_{\frac{1}{8}} \subset \cup_{\mathcal{G}}$, then $x \in \cup_{\mathcal{G}}$.

Case 3: Suppose $x \in (\frac{1}{4}, \frac{1}{2})$.
Since $(\frac{1}{4}, \frac{1}{2}) = S_{\frac{1}{4}}$, then $x \in S_{\frac{1}{4}}$.
Since $S_{\frac{1}{4}} \subset \cup_{\mathcal{G}}$, then $x \in \cup_{\mathcal{G}}$.

Case 4: Suppose $x = \frac{1}{2}$.
Since $\frac{3}{8} < \frac{1}{2} < \frac{5}{8}$, then $\frac{3}{8} < x < \frac{5}{8}$, so $x \in (\frac{3}{8}, \frac{5}{8})$.
Since $(\frac{3}{8}, \frac{5}{8}) = S_{\frac{3}{8}}$, then $x \in S_{\frac{3}{8}}$.
Since $S_{\frac{3}{8}} \subset \cup_{\mathcal{G}}$, then $x \in \cup_{\mathcal{G}}$.

Case 5: Suppose $x \in (\frac{1}{2}, \frac{3}{4})$.
Since $(\frac{1}{2}, \frac{3}{4}) = S_{\frac{1}{2}}$, then $x \in S_{\frac{1}{2}}$.
Since $S_{\frac{1}{2}} \subset \cup_{\mathcal{G}}$, then $x \in \cup_{\mathcal{G}}$.

Case 6: Suppose $x = \frac{3}{4}$.
Since $\frac{5}{8} < \frac{3}{4} < \frac{7}{8}$, then $\frac{5}{8} < x < \frac{7}{8}$, so $x \in (\frac{5}{8}, \frac{7}{8})$.
Since $(\frac{5}{8}, \frac{7}{8}) = S_{\frac{5}{8}}$, then $x \in S_{\frac{5}{8}}$.
Since $S_{\frac{5}{8}} \subset \cup_{\mathcal{G}}$, then $x \in \cup_{\mathcal{G}}$.

Case 7: Suppose $x \in (\frac{3}{4}, 1)$.
Since $(\frac{3}{4}, 1) = S_{\frac{3}{4}}$, then $x \in S_{\frac{3}{4}}$.
Since $S_{\frac{3}{4}} \subset \cup_{\mathcal{G}}$, then $x \in \cup_{\mathcal{G}}$.

Thus, in all cases, $x \in \cup_{\mathcal{G}}$.

Hence, $(0, 1) \subset \cup_{\mathcal{G}}$.

Since $(0, 1) \subset \cup_{\mathcal{G}}$ and $\mathcal{G}$ is a finite set and $\mathcal{G} \subset \mathcal{F}$, then $\mathcal{G}$ is a finite subcover of $(0, 1)$. □

Proof. We prove c.

To prove $(0, 1)$ is not compact, □