Topology of $\mathbb{R}$ Notes

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Topology of $\mathbb{R}$

What is topology of $\mathbb{R}$?

We talk about concepts of open,closed sets, neighborhoods b/c we consider points close to one another. We call it all point set topology.

What does it mean when one point is closer than another point to some fixed point?

Other important concepts are continuity, compactness, connectedness. What are the correct definitions of these topological ideas? Why are they important? They can help to clarify why ,for example, the EVT and IVT theorems are true for continuous functions.

Definition 1. distance in an ordered field

Let F be an ordered field.

Let $a, b \in F$.

The **distance between** a **and** b, denoted $d(a, b)$, is defined by the rule $d(a, b) = |a - b|$.

The distance in an ordered field F is a function from $F \times F$ to F.

Theorem 2. ***properties of the distance function***

Let F be an ordered field. For all $x, y, z \in F$

D1. $d(x, y) \geq 0$.

D2. $d(x, y) = 0$ *iff* $x = y$.

D3. $d(x, y) = d(y, x)$.

D4. $d(x, y) \leq d(x, z) + d(z, y)$.

D1 implies all distances are positive or zero. Therefore, $|x - y| \geq 0$.

D2 implies $d(x, y) \neq 0$ iff $x \neq y$.

Since $d(x, y) \geq 0$, then $d(x, y) > 0$ whenever $x \neq y$.

Therefore, different points are at a positive distance apart.

Thus, $|x - y| > 0$ iff $x \neq y$.

D3 implies distance from x to y is the same as from y to x.

Therefore, distance is symmetric.

Thus, $|x - y| = |y - x|$.

D4 is the triangle inequality.

Therefore, $|x - y| \leq |x - z| + |z - y|$.

An interval is the set of all real numbers between two given real numbers a and b.

Definition 3. interval
Let $I \subset \mathbb{R}$.
Then I is an **interval** iff whenever a and b are elements of I and c is a real number between a and b, then $c \in I$.
In symbols, I is an interval iff $(\forall a, b, c \in \mathbb{R})[(a, b \in I \land a < c < b) \to c \in I]$.

Therefore I is not an interval iff $(\exists a, b, c \in \mathbb{R})[(a, b \in I \land a < c < b) \land c \notin I]$.

Intervals have the property of containing a number between any two of its members.
Let $a, b \in \mathbb{R}$.
Types of intervals:
$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ = closed and bounded
$(a, b) = \{x \in \mathbb{R} : a < x < b\}$ = open and bounded
$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ = closed-open and bounded
$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ = open-closed and bounded
$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$ = closed and unbounded above
$(a, \infty) = \{x \in \mathbb{R} : a < x\}$ = open and unbounded above
$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$ = closed and unbounded below
$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$ = open and unbounded below
$(-\infty, \infty) = \mathbb{R}$

Definition 4. open interval
Let $a, b \in \mathbb{R}$.
The open interval, denoted (a, b), is the set $\{x \in \mathbb{R} : a < x < b\}$.

Therefore, $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

Proposition 5. *Let $a, b \in \mathbb{R}$.*
If $a \geq b$, then $(a, b) = \emptyset$.
If $a < b$, then $(a, b) \neq \emptyset$.

Therefore, the open interval $(a, a) = \{x \in \mathbb{R} : a < x < a\} = \emptyset$ is the empty set.

Definition 6. closed interval
Let $a, b \in \mathbb{R}$.
The closed interval, denoted $[a, b]$, is the set $\{x \in \mathbb{R} : a \leq x \leq b\}$.

Therefore, $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

Proposition 7. *Let $a, b \in \mathbb{R}$.*
Then $(a, b) \subset [a, b]$.

Proposition 8. *Let $a, b \in \mathbb{R}$.*
If $a > b$, then $[a, b] = \emptyset$.
If $a = b$, then $[a, b] = \{a\}$.
If $a < b$, then $[a, b] \neq \emptyset$.

Example 9. The closed interval $[a,a] = \{x \in \mathbb{R} : a \leq x \leq a\} = \{a\}$ is a **singleton set**.

We call an interval that is either empty or a singleton set a **degenerate interval**.

Definition 10. length of an interval
Let $a, b \in \mathbb{R}$ with $a < b$.
The length of the interval from a to b is $b - a$.

Let $a, b \in \mathbb{R}$ with $a < b$.
The length of the open interval (a, b) is $b - a$.
The length of the closed interval $[a, b]$ is $b - a$.

Proposition 11. ***The distance between any two points in the interval*** (a, b) ***is less than*** $b - a$***.***
Let $a, b \in \mathbb{R}$.
Let x, y be any real numbers such that $a < x < b$ and $a < y < b$.
Then $|x - y| < b - a$.

Corollary 12. *Let $a, b \in \mathbb{R}$ with $a < b$.*
Let x, y be any real numbers such that $x \in [a, b]$ and $y \in [a, b]$.
Then $|x - y| \leq b - a$.

Let $a, b \in \mathbb{R}$ with $a < b$.
Let x, y be any real numbers such that $x \in [a, b]$ and $y \in [a, b]$.
Then $|x - y| \leq b - a$.
Therefore, the distance between any two points of the closed interval $[a, b]$ is less than or equal to the length of the interval.

Proposition 13. *Let $I \subset \mathbb{R}$ be an interval.*
If $a \in I$ and $b \in I$ and $a < b$, then $[a, b] \subset I$.

Definition 14. unit interval
The open unit interval is the set $(0, 1)$.
The closed unit interval is the set $[0, 1]$.

Proposition 15. ***intersection of any two intervals is an interval***
If I_1 and I_2 are intervals, then $I_1 \cap I_2$ is an interval.

Proposition 16. ***intersection of a countable collection of intervals is an interval***
If $\{I_n : n \in \mathbb{Z}^+\}$ is a collection of intervals, then $\cap_{n=1}^{\infty} I_n$ is an interval.

Let $S \subset \mathbb{R}$.
Then S is convex iff S is an interval.

Definition 17. δ **neighborhood of a point**
Let δ be a positive real number.
Let $p \in \mathbb{R}$.
The δ **neighborhood of** p, denoted $N(p; \delta)$, is the set $\{x \in \mathbb{R} : d(x, p) < \delta\}$.

Observe that

$$\begin{aligned} N(p;\delta) &= \{x \in \mathbb{R} : d(x,p) < \delta\} \\ &= \{x \in \mathbb{R} : |x-p| < \delta\} \\ &= \{x \in \mathbb{R} : -\delta < x-p < \delta\} \\ &= \{x \in \mathbb{R} : p-\delta < x < p+\delta\} \\ &= (p-\delta, p+\delta). \end{aligned}$$

Therefore, the δ **neighborhood of** $p \in \mathbb{R}$ is the open interval $(p-\delta, p+\delta)$.

Let $\delta > 0$.
Let $N(p;\delta)$ be the δ neighborhood of $p \in \mathbb{R}$.
Then $N(p;\delta) = \{x \in \mathbb{R} : d(x,p) < \delta\}$.
Since $d(p,p) = |p-p| = 0 < \delta$, then $p \in N(p;\delta)$.
Therefore, a point p is in the δ neighborhood of p and $N(p;\delta) \neq \emptyset$.

A δ neighborhood of a point p is an open interval centered at p of radius δ.

Example 18. every nonempty open interval is the ϵ neighborhood of some point
Let $a, b \in \mathbb{R}$.
If $a < b$, then $N(\frac{a+b}{2}; \frac{b-a}{2}) = (a,b)$.

Therefore, if $a < b$, then the open interval (a,b) is the $\frac{b-a}{2}$ neighborhood of the point $\frac{a+b}{2}$, the midpoint of a and b.

Proposition 19. *Let $p \in \mathbb{R}$.*
Let $\delta, \epsilon \in \mathbb{R}$.
If $0 < \delta \leq \epsilon$, then $N(p;\delta) \subset N(p;\epsilon)$.

Definition 20. deleted ϵ neighborhood of a point
Let $\epsilon > 0$.
The **deleted ϵ neighborhood of a point** p, denoted $N'(p;\epsilon)$, is the set of all points in the ϵ neighborhood of p except p.
Therefore, $N'(p;\epsilon) = N(p;\epsilon) - \{p\}$.

Example 21. deleted ϵ neighborhood is a subset of the ϵ neighborhood of a point
Let $\epsilon > 0$.
Let $p \in \mathbb{R}$.
Then $N'(p;\epsilon) \subset N(p;\epsilon)$ and $N'(p;\epsilon) = (p-\epsilon, p) \cup (p, p+\epsilon)$.

Therefore, $N'(p;\epsilon) = (p-\epsilon, p) \cup (p, p+\epsilon) \subset (p-\epsilon, p+\epsilon) = N(p;\epsilon)$.
Since $(p-\epsilon, p) \subset (p-\epsilon, p) \cup (p, p+\epsilon)$, then $(p-\epsilon, p) \subset (p-\epsilon, p+\epsilon)$.
Since $(p, p+\epsilon) \subset (p-\epsilon, p) \cup (p, p+\epsilon)$, then $(p, p+\epsilon) \subset (p-\epsilon, p+\epsilon)$.

Definition 22. neighborhood of a point

A set S is a **neighborhood of a point** p iff some ϵ neighborhood of p is contained in S.

Therefore, a set S is a neighborhood of a point $p \in \mathbb{R}$ iff there exists $\epsilon > 0$ such that $N(p; \epsilon) \subset S$.

Thus, a set S is a neighborhood of a point $p \in \mathbb{R}$ iff there exists $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset S$.

Let S be a neighborhood of $p \in \mathbb{R}$.
Then there exists $\epsilon > 0$ such that $N(p; \epsilon) \subset S$.
Since $p \in N(p; \epsilon)$ and $N(p; \epsilon) \subset S$, then $p \in S$, so $S \neq \emptyset$.
Thus, if S is a neighborhood of a point $p \in \mathbb{R}$, then S is not empty.
Therefore, a neighborhood of a point $p \in \mathbb{R}$ is not empty.

Proposition 23. ***Every ϵ neighborhood of a point is a neighborhood of the point.***
Let $p \in \mathbb{R}$.
Then $N(p; \epsilon)$ is a neighborhood of p for every $\epsilon > 0$.

Types of points in $\mathbb{R}$

Definition 24. interior point

A point p is an **interior point of a set** S iff some ϵ neighborhood of p is contained in S.

Therefore, a point p is an interior point of a set S iff $(\exists \epsilon > 0)(N(p; \epsilon) \subset S)$.

A point p is not an interior point of a set S iff
no ϵ neighborhood of p is contained in S iff
every ϵ neighborhood of p is not contained in S iff
every ϵ neighborhood of p contains some point not in S.
Observe that

$$\begin{aligned} \neg(\exists \epsilon > 0)(N(p; \epsilon) \subset S) &\Leftrightarrow (\forall \epsilon > 0)(N(p; \epsilon) \not\subset S) \\ &\Leftrightarrow (\forall \epsilon > 0)\neg(\forall x)(x \in N(p; \epsilon) \to x \in S) \\ &\Leftrightarrow (\forall \epsilon > 0)(\exists x)(x \in N(p; \epsilon) \wedge x \notin S) \\ &\Leftrightarrow (\forall \epsilon > 0)(\exists x \in N(p; \epsilon))(x \notin S) \end{aligned}$$

Therefore, a point p is not an interior point of a set S iff $(\forall \epsilon > 0)(\exists x \in N(p; \epsilon))(x \notin S)$.

Example 25. $\emptyset$ has no interior points.

There is no interior point of the empty set.

Let S be a set.
Since there is no interior point of $\emptyset$, then if $S = \emptyset$, then there is no interior point of S.
Therefore, if there is some interior point of S, then $S \neq \emptyset$.

Let p be an interior point of a set S.

Then there exists $\epsilon > 0$ such that $N(p;\epsilon) \subset S$.

Since $p \in N(p;\epsilon)$ and $N(p;\epsilon) \subset S$, then $p \in S$.

Thus, if p is an interior point of S, then $p \in S$.

Therefore, every interior point of a set S is an element of S.

Example 26. A singleton set has no interior points.

Let $p \in \mathbb{R}$.

Then p is not an interior point of the set $\{p\}$.

Example 27. Let $a, b \in \mathbb{R}$ with $a < b$.

Every point in the open interval (a, b) is an interior point of the open interval (a, b).

Proposition 28. *Let A and B be sets.*

If p is an interior point of A and $A \subset B$, then p is an interior point of B.

Example 29. Let $a, b \in \mathbb{R}$ with $a < b$.

Every point in the closed interval $[a, b]$ except the end points a and b is an interior point of the closed interval $[a, b]$.

Example 30. No natural number is an interior point of $\mathbb{N}$.

Example 31. No rational number is an interior point of $\mathbb{Q}$.

Example 32. Every real number is an interior point of $\mathbb{R}$.

Definition 33. boundary point

A point p is a **boundary point of a set** S iff every ϵ neighborhood of p contains an element of S and an element not in S.

Definition 34. accumulation point

A point p is an **accumulation point of a set** S iff every ϵ neighborhood of p contains some point of S distinct from p.

Observe that

$$\begin{aligned}
(\forall \epsilon > 0)(\exists x \in S)(x \in N(p;\epsilon) \wedge x \neq p) &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists x \in S)(x \in N'(p;\epsilon)) &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists x \in S \cap N'(p;\epsilon)) &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists x \in N'(p;\epsilon) \cap S) &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists x)(x \in N'(p;\epsilon) \cap S) &\Leftrightarrow \\
(\forall \epsilon > 0)(N'(p;\epsilon) \cap S \neq \emptyset).
\end{aligned}$$

Therefore, a point p is an accumulation point of a set S iff $(\forall \epsilon > 0)(\exists x \in S)(x \in N(p;\epsilon) \wedge x \neq p)$.

Hence, a point p is an accumulation point of a set S iff every deleted ϵ neighborhood of p contains some point of S iff $(\forall \epsilon > 0)(\exists x \in S)(x \in N'(p;\epsilon))$.

Thus, a point p is an accumulation point of a set S iff $(\forall \epsilon > 0)(N'(p;\epsilon) \cap S \neq \emptyset)$.

A point p is not an accumulation point of a set S iff some deleted ϵ neighborhood of p contains no point of S.

Observe that

$$\neg(\forall \epsilon > 0)(N'(p;\epsilon) \cap S \neq \emptyset) \Leftrightarrow (\exists \epsilon > 0)(N'(p;\epsilon) \cap S = \emptyset)$$

Therefore, a point p is not an accumulation point of a set S iff $(\exists \epsilon > 0)(N'(p;\epsilon) \cap S = \emptyset)$.

An accumulation point of a set S does not have to be in S.

Example 35. accumulation point of a set need not lie in the set

Let $S = (0, 1)$.

Then 1 is an accumulation point of S, but $1 \notin S$.

Example 36. point of a set need not be an accumulation point

Let $S = (0, 1) \cup \{2\}$.

Then $2 \in S$, but 2 is not an accumulation point of S.

Example 37. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Then 0 is an accumulation point of S and 1 is not an accumulation point of S.

Example 38. $\emptyset$ has no accumulation points.

There is no accumulation point of the empty set.

Thus, there does not exist a point p such that p is an accumulation point of $\emptyset$.

Hence, for every point p, p is not an accumulation point of $\emptyset$.

Let p be a point.

Let S be a set.

If $S = \emptyset$, then p is not an accumulation point of S.

Therefore, if p is an accumulation point of a set S, then $S \neq \emptyset$.

Example 39. A singleton set has no accumulation points.

Let $x \in \mathbb{R}$.

There is no accumulation point of the set $\{x\}$.

Thus, there does not exist a point p such that p is an accumulation point of $\{x\}$.

Hence, for every point p, p is not an accumulation point of $\{x\}$.

Let p be a point.

Let S be a set.

If $S = \{x\}$, then p is not an accumulation point of S.

Therefore, if p is an accumulation point of a set S, then $S \neq \{x\}$.

Thus, if p is an accumulation point of a set S, then $S \neq \emptyset$ and $S \neq \{x\}$.

Therefore, if p is an accumulation point of a set S, then S contains at least two elements.

Example 40. A finite set has no accumulation points.
There is no accumulation point of a finite set.

Therefore, if S is a finite set, then there is no accumulation point of S.
Thus, if there is an accumulation point of a set S, then S is not finite.
Therefore, if there is an accumulation point of a set S, then S is infinite.

Example 41. Every point in $[a, b]$ is an accumulation point of (a, b).
Let $a, b \in \mathbb{R}$.
If $a < b$, then every point in the closed interval $[a, b]$ is an accumulation point of the open interval (a, b).

Let $a, b \in \mathbb{R}$ with $a < b$.
If $x \in [a, b]$, then x is an accumulation point of (a, b).
Therefore, if $a \leq x \leq b$, then x is an accumulation point of (a, b).

Lemma 42. *Let A and B be sets.*
If p is an accumulation point of A and $A \subset B$, then p is an accumulation point of B.

Example 43. Every point in $[a, b]$ is an accumulation point of $[a, b]$.
Let $a, b \in \mathbb{R}$.
If $a < b$, then every point in the closed interval $[a, b]$ is an accumulation point of the closed interval $[a, b]$.

Let $a, b \in \mathbb{R}$ with $a < b$.
If $x \in [a, b]$, then x is an accumulation point of $[a, b]$.
Therefore, if $a \leq x \leq b$, then x is an accumulation point of $[a, b]$.

Proposition 44. ***Every point in an interval of at least two elements is an accumulation point of the interval.***
Let $I \subset \mathbb{R}$ be an interval with at least two elements.
If $a \in I$, then a is an accumulation point of I.

Example 45. $\mathbb{N}$ has no accumulation points.
No integer is an accumulation point of $\mathbb{N}$.
No point in $\mathbb{R}$ is an accumulation point of $\mathbb{N}$.

Example 46. Every real number is an accumulation point of $\mathbb{Q}$.

Example 47. Every real number is an accumulation point of $\mathbb{R}$.

Proposition 48. ***Every interior point of a set S is an accumulation point of S.***
Let S be a set.
If p is an interior point of S, then p is an accumulation point of S.

Definition 49. isolated point
A point p is an **isolated point of a set** S iff $p \in S$ and there is an ϵ neighborhood of p such that $N(p; \epsilon) \cap S = \{p\}$.

Therefore, a point p is an isolated point of a set S iff $p \in S$ and $(\exists \epsilon > 0)(N(p;\epsilon) \cap S = \{p\})$.

Equivalently, a point p is an isolated point of a set S iff $p \in S$ and there is a deleted ϵ neighborhood of p such that $N'(p;\epsilon) \cap S = \emptyset$.

Therefore, a point p is an isolated point of a set S iff $p \in S$ and $(\exists \epsilon > 0)(N'(p;\epsilon) \cap S = \emptyset)$.

Therefore, a point p is an isolated point of a set S iff $p \in S$ and p is not an accumulation point of S.

Therefore, a point p is not an isolated point of a set S iff either $p \notin S$ or p is an accumulation point of S.

Example 50. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Then each element of S is an isolated point of S.

Example 51. No point is an isolated point of $\emptyset$.

Therefore, the empty set has no isolated points.

Example 52. Let $x \in \mathbb{R}$.

Then x is an isolated point of the singleton set $\{x\}$.

Example 53. Let $a, b \in \mathbb{R}$.

No point in the open interval (a, b) is an isolated point of (a, b).

Example 54. Let $a, b \in \mathbb{R}$.

If $a \neq b$, then no point in the closed interval $[a, b]$ is an isolated point of $[a, b]$.

Example 55. Every natural number is an isolated point of $\mathbb{N}$.

Example 56. No rational number is an isolated point of $\mathbb{Q}$.

Therefore, $\mathbb{Q}$ has no isolated points.

Example 57. No real number is an isolated point of $\mathbb{R}$.

Therefore, $\mathbb{R}$ has no isolated points.

Proposition 58. *Every element of a nonempty set is either an interior point or a boundary point.*

Proposition 59. *Every element of a nonempty set is either an accumulation point or an isolated point.*

Example 60. Since $\mathbb{N}$ is infinite and $\mathbb{N}$ has no accumulation points, then the converse 'if S is infinite, then S has an accumulation point' is false.

Sets in $\mathbb{R}$

Definition 61. interior of a set

The set of all interior points of a set S, denoted S°, is called the **interior of** S.

Let S° be the interior of a nonempty set S.
Then

$$\begin{aligned} S^\circ &= \{x : x \text{ is an interior point of } S\} \\ &= \{x : x \in S \wedge (\exists \delta > 0)(N(x;\delta) \subset S)\} \\ &= \{x \in S : (\exists \delta > 0)(N(x;\delta) \subset S)\} \end{aligned}$$

Thus, if $S \neq \emptyset$, then $S^\circ \subset S$.

Example 62. The interior of $\emptyset$ is empty.

Since there is no interior point of the empty set, then the interior of the empty set is empty.
Therefore, $\emptyset^\circ = \emptyset$.

Since $\emptyset^\circ = \emptyset$ and $\emptyset \subset \emptyset$, then $\emptyset^\circ \subset \emptyset$.
Since $S^\circ \subset S$ if $S \neq \emptyset$, then $S^\circ \subset S$ for every set S.
Therefore, the interior of S is a subset of S for every set S.

Example 63. The interior of a singleton set is empty.

Let $x \in \mathbb{R}$.
Then $\{x\}^\circ \subset \{x\}$, so either $\{x\}^\circ = \emptyset$ or $\{x\}^\circ = \{x\}$.
Since x is not an interior point of $\{x\}$, then $\{x\}^\circ \neq \{x\}$.
Therefore, $\{x\}^\circ = \emptyset$.

Example 64. The interior of $\mathbb{N}$ is empty.

Since no natural number is an interior point of $\mathbb{N}$, then every natural number is not an interior point of $\mathbb{N}$.
Thus, there are no interior points in $\mathbb{N}$, so $\mathbb{N}$ has no interior points.
Therefore, the interior of $\mathbb{N}$ is empty, so $\mathbb{N}^\circ = \emptyset$.

Example 65. The interior of $\mathbb{Q}$ is empty.

Since no rational number is an interior point of $\mathbb{Q}$, then every rational number is not an interior point of $\mathbb{Q}$.
Thus, there are no interior points in $\mathbb{Q}$, so $\mathbb{Q}$ has no interior points.
Therefore, the interior of $\mathbb{Q}$ is empty, so $\mathbb{Q}^\circ = \emptyset$.

Example 66. The interior of $\mathbb{R}$ is $\mathbb{R}$.

Since every real number is an interior point of $\mathbb{R}$, then if $r \in \mathbb{R}$, then $r \in \mathbb{R}^\circ$.
Hence, $\mathbb{R} \subset \mathbb{R}^\circ$.
Since $\mathbb{R}^\circ \subset \mathbb{R}$ and $\mathbb{R} \subset \mathbb{R}^\circ$, then $\mathbb{R}^\circ = \mathbb{R}$.
Therefore, $\mathbb{R}^\circ = \mathbb{R}$.

Proposition 67. *Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.*
If $A \subset B$, then $A^\circ \subset B^\circ$.

Proposition 68. *Let $a, b \in \mathbb{R}$. Then*

1. $(a, b)^\circ = (a, b)$.
2. $[a, b]^\circ = (a, b)$.

Let $a, b \in \mathbb{R}$.

Then the interior of the open interval (a, b) is the open interval (a, b) and the interior of the closed interval $[a, b]$ is the open interval (a, b).

Since $(a, b)^\circ = (a, b)$, then $(a, b) \subset a, b)^\circ$.

Therefore, every point in (a, b) is an interior point of (a, b).

Thus, if $x \in (a, b)$, then x is an interior point of (a, b).

Since $[a, b]^\circ = (a, b)$ and $a \notin (a, b)$ and $b \notin (a, b)$, then the interior of $[a, b]$ is the set of all points in $[a, b]$ except the endpoints a and b.

Proposition 69. *Let $A \subset \mathbb{R}$.*

Then $A^{\circ\circ} = A^\circ$.

Proposition 70. *Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.*

Then $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Proposition 71. *Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.*

Then $(A \cup B)^\circ \supset A^\circ \cup B^\circ$.

Definition 72. open set

A set S is **open** iff every point in S is an interior point of S.

Therefore, a set S is open whenever if $p \in S$, then p is an interior point of S.

Thus, a set S is open iff $(\forall p \in S)(\exists \epsilon > 0)(N(p; \epsilon) \subset S)$.

A set S is not open iff some point in S is not an interior point of S.

Observe that

$$\begin{aligned}
\neg(\forall p \in S)(\exists \epsilon > 0)(N(p; \epsilon) \subset S) &\Leftrightarrow (\exists p \in S)(\forall \epsilon > 0)(N(p; \epsilon) \not\subset S) \\
&\Leftrightarrow (\exists p \in S)(\forall \epsilon > 0)\neg(N(p; \epsilon) \subset S) \\
&\Leftrightarrow (\exists p \in S)(\forall \epsilon > 0)\neg(\forall x)(x \in N(p; \epsilon) \to x \in S) \\
&\Leftrightarrow (\exists p \in S)(\forall \epsilon > 0)(\exists x)(x \in N(p; \epsilon) \wedge x \notin S) \\
&\Leftrightarrow (\exists p \in S)(\forall \epsilon > 0)(\exists x \in N(p; \epsilon))(x \notin S)
\end{aligned}$$

Therefore, a set S is not open iff $(\exists p \in S)(\forall \epsilon > 0)(\exists x \in N(p; \epsilon))(x \notin S)$.

Example 73. $\emptyset$ is open.

The empty set is open.

Proof. Since $\emptyset$ is empty, then there is no element in $\emptyset$, so there is no $x \in \emptyset$ such that x is not an interior point of $\emptyset$.

Therefore, for every $x \in \emptyset$, x is an interior point of $\emptyset$, so $\emptyset$ is open. □

Proof. Since there is no element in $\emptyset$, then the conditional 'if $x \in \emptyset$, then x is an interior point of $\emptyset$' is vacuously true.

Therefore, $\emptyset$ is open. □

Example 74. A singleton set is not open.
Let $x \in \mathbb{R}$.
Then the set $\{x\}$ is not open.

Proof. Since x is not an interior point of $\{x\}$, then $\{x\}$ is not open. □

Example 75. $\mathbb{N}$ is not open.
The set of natural numbers is not open.

Proof. Since no natural number is an interior point of $\mathbb{N}$, then there is no $n \in \mathbb{N}$ such that n is an interior point of $\mathbb{N}$.

Hence, for every $n \in \mathbb{N}$, n is not an interior point of $\mathbb{N}$.

Since $\mathbb{N} \neq \emptyset$, then there exists $n \in \mathbb{N}$ such that n is not an interior point of $\mathbb{N}$.

Therefore, $\mathbb{N}$ is not open. □

Example 76. $\mathbb{Q}$ is not open.
The set of rational numbers is not open.

Proof. Since no rational number is an interior point of $\mathbb{Q}$, then there is no $q \in \mathbb{Q}$ such that q is an interior point of $\mathbb{Q}$.

Hence, for every $q \in \mathbb{Q}$, q is not an interior point of $\mathbb{Q}$.

Since $\mathbb{Q} \neq \emptyset$, then there exists $q \in \mathbb{Q}$ such that q is not an interior point of $\mathbb{Q}$.

Therefore, $\mathbb{Q}$ is not open. □

Example 77. $\mathbb{R}$ is open.
The set of real numbers is open.

Proof. Since every real number is an interior point of $\mathbb{R}$, then $\mathbb{R}$ is open. □

Example 78. Every bounded open interval is open.
Let $a, b \in \mathbb{R}$.
Then the open interval (a, b) is open.

Proof. Since every point in (a, b) is an interior point of (a, b), then (a, b) is open. □

Example 79. An unbounded open interval is open.
Let $a, b \in \mathbb{R}$.
Then the interval (a, ∞) is open and the interval $(-\infty, b)$ is open.

Example 80. A closed bounded interval is not open.
Let $a, b \in \mathbb{R}$ with $a < b$.
Then the closed bounded interval $[a, b]$ is not open.

Proof. Since $a \in [a, b]$, but a is not an interior point of $[a, b]$, then $[a, b]$ is not open. □

Proposition 81. *A set S is open iff $S^\circ = S$.*

Proposition 82. *Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.*

1. *If A is open and B is open, then $A \cup B$ is open.*
2. *If A is open and B is open, then $A \cap B$ is open.*

Theorem 83. *topological properties of open sets in* $\mathbb{R}$

1. *The union of any collection of open sets in $\mathbb{R}$ is open.*
2. *The intersection of any finite collection of open sets in $\mathbb{R}$ is open.*

Corollary 84. $\mathbb{R}$ *is a topological space*

Let τ be the set of all open subsets of $\mathbb{R}$. Then
T1. $\emptyset \in \tau$ and $\mathbb{R} \in \tau$.
T2. The union of any collection of sets in τ is in τ.
T3. The intersection of any finite collection of sets in τ is in τ.

Let τ be the set of all open subsets of $\mathbb{R}$.
Since $T1, T2$, and $T3$ are true, then τ is a topology on $\mathbb{R}$.
Therefore, $(\mathbb{R}, \tau)$ is a topological space.

Let τ be the set of all open subsets of $\mathbb{R}$.
Then $\tau = \{S : S \text{ is an open subset of } \mathbb{R}\}$.
T1.
Since $\emptyset \in \tau$, then $\emptyset$ is an open subset of $\mathbb{R}$, so $\emptyset$ is open.
Since $\mathbb{R} \in \tau$, then $\mathbb{R}$ is an open subset of $\mathbb{R}$, so $\mathbb{R}$ is open.

T2.
Let S be a collection of sets in τ.
Since the union of any collection of sets in τ is in τ, then $\cup S$ is in τ.
Therefore, $\cup S \in \tau$.
We say that τ is closed under arbitrary union.

T3.
Let S be a finite collection of sets in τ.
Since the intersection of any finite collection of sets in τ is in τ, then $\cap S$ is in τ.
Therefore, $\cap S \in \tau$.
We say that τ is closed under finite intersection.

Example 85. union of a countable family of open sets is open

Let $\{S_n : n \in \mathbb{N}\}$ be a family of open subsets of $\mathbb{R}$ indexed by $\mathbb{N}$.
Then $\cup_{n=1}^{\infty} S_n$ is open.

Example 86. intersection of an infinite collection of open sets is not necessarily open

Let $\{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ be a family of open intervals of $\mathbb{R}$ indexed by $\mathbb{N}$.
Then $\cap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n}) = \{0\}$ is not open.
Therefore, at least one collection of open sets in $\mathbb{R}$ is not closed under arbitrary intersection.

Theorem 87. ***characterization of open sets in*** $\mathbb{R}$

A nonempty subset of $\mathbb{R}$ *is open iff it is a union of bounded open intervals.*

Definition 88. derived set

The set of all accumulation points of a set S, denoted S', is called the **derived set of** S.

Example 89. Since $\mathbb{N}$ has no accumulation points, then no point in $\mathbb{R}$ is an accumulation point of $\mathbb{N}$.

Hence, the set of all accumulation points of $\mathbb{N}$ is empty.

Therefore, $\mathbb{N}' = \emptyset$.

Definition 90. closure of a set

The **closure of a set** S, denoted S^-, is the set $S \cup S'$.

Therefore, $S^- = S \cup S'$.

Definition 91. closed set

A set S is **closed** iff every accumulation point of S is an element of S.

Therefore, a set S is closed whenever if p is an accumulation point of S, then $p \in S$.

Therefore, a set S is closed iff S contains all of its accumulation points.

Thus, a set S is not closed iff
some accumulation point of S is not an element of S iff
there exists an accumulation point p of S such that $p \notin S$.

Proposition 92. *A set with no accumulation points is closed.*

Example 93. $\emptyset$ **is closed.**

The empty set is closed.

Proof. Since there is no accumulation point of the empty set, then $\emptyset$ is a set with no accumulation points.

Therefore, $\emptyset$ is closed. □

Example 94. Every finite set is closed.

If S is a finite set, then S is closed.

Proof. Let S be a finite set.

Since S is finite, then there is no accumulation point of S, so S is a set with no accumulation points.

Therefore, S is closed. □

Example 95. $\mathbb{N}$ **is closed.**

The set of natural numbers is closed.

Proof. Since $\mathbb{N}$ has no accumulation points, then $\mathbb{N}$ is a set with no accumulation points.

Therefore, $\mathbb{N}$ is closed. □

Example 96. $\mathbb{Q}$ is not closed.
The set of rational numbers is not closed.

Proof. Since every real number is an accumulation point of $\mathbb{Q}$, then in particular, the real number $\sqrt{2}$ is an accumulation point of $\mathbb{Q}$.
Since $\sqrt{2}$ is irrational, then $\sqrt{2} \notin \mathbb{Q}$.
Therefore, $\mathbb{Q}$ is not closed. □

Example 97. $\mathbb{R}$ is closed.
The set of real numbers is closed.

Proof. Since every real number is an accumulation point of $\mathbb{R}$, then 0 is an accumulation point of $\mathbb{R}$, so $\mathbb{R}$ has at least one accumulation point.
Let x be an arbitrary accumulation point of $\mathbb{R}$.
By definition, every accumulation point of $\mathbb{R}$ is a real number.
Hence, $x \in \mathbb{R}$.
Therefore, $\mathbb{R}$ is closed. □

Example 98. An open interval is not closed.
Let $a, b \in \mathbb{R}$ with $a < b$.
Then the open interval (a, b) is not closed.

Proof. Since $a < b$, then every point in the closed interval $[a, b]$ is an accumulation point of the open interval (a, b).
In particular, $a \in [a, b]$ is an accumulation point of (a, b).
But, $a \notin (a, b)$.
Therefore, the open interval (a, b) is not closed. □

Example 99. Every bounded closed interval is closed.
Let $a, b \in \mathbb{R}$.
Then the closed interval $[a, b]$ is closed.

Since the closed interval $[a, a] = \{a\}$ is closed, then the singleton set $\{a\}$ is closed.

Example 100. An unbounded closed interval is closed.
Let $a, b \in \mathbb{R}$.
Then the interval $[a, \infty)$ is closed and the interval $(-\infty, b]$ is closed.

Proposition 101. *Let $S \subset \mathbb{R}$.*
1. *If S is non empty, closed, and bounded above in $\mathbb{R}$, then $\max S$ exists.*
2. *If S is non empty, closed, and bounded below in $\mathbb{R}$, then $\min S$ exists.*

Let S be a non empty closed bounded set of real numbers.
Since S is bounded in $\mathbb{R}$, then S is bounded above and below in $\mathbb{R}$.
Since S is non empty, closed, and bounded above in $\mathbb{R}$, then $\max S$ exists.
Since S is non empty, closed, and bounded below in $\mathbb{R}$, then $\min S$ exists.
Therefore, if S is a non empty, closed and bounded set of real numbers, then $\max S$ and $\min S$ exist.

Let S be a non empty closed bounded set of real numbers.
Then $\max S$ and $\min S$ exist.
Since $\max S$ exists, then $\sup S = \max S$, so $\sup S \in S$.
Since $\min S$ exists, then $\inf S = \min S$, so $\inf S \in S$.
Therefore, $\sup S \in S$ and $\inf S \in S$ and for every $x \in S$, $\inf S \leq x \leq \sup S$.

Theorem 102. *Let $S \subset \mathbb{R}$.*
Then S is open iff $\mathbb{R} - S$ is closed.

Let $S \subset \mathbb{R}$.
Suppose S is open.
Then $\mathbb{R} - S$ is closed, so the complement of S in $\mathbb{R}$ is closed.
Therefore, the complement of an open set is closed.
Observe that

$$\begin{aligned}
\mathbb{R} - (\mathbb{R} - S) &= \mathbb{R} - (\mathbb{R} \cap \overline{S}) \\
&= \mathbb{R} \cap (\overline{\mathbb{R} \cap \overline{S}}) \\
&= \mathbb{R} \cap (\overline{\mathbb{R}} \cup S) \\
&= (\mathbb{R} \cap \overline{\mathbb{R}}) \cup (\mathbb{R} \cap S) \\
&= \emptyset \cup (\mathbb{R} \cap S) \\
&= \mathbb{R} \cap S \\
&= S.
\end{aligned}$$

Thus, $\mathbb{R} - (\mathbb{R} - S) = S$.
Suppose S is closed.
Then $\mathbb{R} - (\mathbb{R} - S)$ is closed.
Hence, $\mathbb{R} - S$ is open.
Thus, the complement of S is open.
Therefore, the complement of a closed set is open.

In general, a neighborhood of p is a set that has an open subset containing p.

Compactness

Definition 103. covering of a set
A **covering of a set** S is a collection of sets $\mathcal{F}$ such that $S \subset \cup\mathcal{F}$.

Let S be a set.
Let $\mathcal{F}$ be a collection(family) of sets.
Then $\mathcal{F}$ is a covering of S iff $S \subset \cup\mathcal{F}$.
Observe that

$$\begin{aligned}
S \subset \cup\mathcal{F} &\Leftrightarrow (\forall x)(x \in S \to x \in \cup\mathcal{F}) \\
&\Leftrightarrow (\forall x \in S)(x \in \cup\mathcal{F}) \\
&\Leftrightarrow (\forall x \in S)(\exists A \in \mathcal{F})(x \in A).
\end{aligned}$$

Therefore, a collection of sets $\mathcal{F}$ is a covering for a set S iff $(\forall x \in S)(\exists A \in \mathcal{F})(x \in A)$.

Observe that

$$\neg(\forall x \in S)(\exists A \in \mathcal{F})(x \in A) \quad \Leftrightarrow \quad (\exists x \in S)(\forall A \in \mathcal{F})(x \notin A).$$

Therefore, a collection of sets $\mathcal{F}$ is not a covering for a set S iff $(\exists x \in S)(\forall A \in \mathcal{F})(x \notin A)$.

Example 104. Let $S = [1, \infty)$.
Let $\mathcal{F}_1 = \{(0,1), (1,\infty)\}$.
Let $\mathcal{F}_2 = \{(0,1], (1,\infty)\}$.
Then $\mathcal{F}_1$ is not a covering of S, but $\mathcal{F}_2$ is a covering of S.

Definition 105. open covering
An **open covering of a set** S is a covering $\mathcal{F}$ of S such that each set in $\mathcal{F}$ is open.

Definition 106. finite covering
A **finite covering of a set** S is a covering $\mathcal{F}$ of S such that $\mathcal{F}$ is a finite set.

Definition 107. countable covering
A **countable covering of a set** S is a covering $\mathcal{F}$ of S such that $\mathcal{F}$ is a countable set.

Definition 108. subcovering of a covering
A **subcovering of a covering** $\mathcal{F}$ **of a set** S is a covering $\mathcal{G}$ of S such that $\mathcal{G} \subset \mathcal{F}$.

Example 109. a covering of a set is not unique
Let $S = [1, \infty)$.
Let $\mathcal{F}_1 = \{(0, \infty)\}$.
Let $\mathcal{F}_2 = \{(n-1, n+1) : n \in \mathbb{N}\}$.
Then $\mathcal{F}_1$ is a finite covering of S and $\mathcal{F}_2$ is an infinite covering of S.

Example 110. Let $S = [1, \infty)$.
Let $\mathcal{F} = \{(0, n) : n \in \mathbb{N}\}$.
Let $\mathcal{G} = \{(0, n) : n \in \mathbb{N}, n \geq 23\}$.
Then $\mathcal{F}$ is an open covering of S and $\mathcal{G}$ is a subcovering of $\mathcal{F}$.

Definition 111. compact set
A set S is **compact** iff every open covering of S contains a finite subcovering of S.

Therefore a set S is not compact iff some open covering of S contains no finite subcovering of S.

Example 112. A finite set is compact.

Since the empty set is finite, then $\emptyset$ is compact.

Example 113. **$\mathbb{N}$ is not compact**
Define $I_n = (n - \frac{1}{2}, n + \frac{1}{2})$ for each $n \in \mathbb{N}$.
Let $\mathcal{F} = \{I_n : n \in \mathbb{N}\}$.
Then $\mathcal{F}$ is an open covering of $\mathbb{N}$, but $\mathcal{F}$ contains no finite subcovering of $\mathbb{N}$.
Therefore, $\mathbb{N}$ is not compact.

Theorem 114. ***Lindelof covering theorem***
Every open covering of a set S contains a countable subcovering of S.

Theorem 115. ***Heine-Borel covering theorem***
Every open covering of a closed and bounded set S contains a finite subcovering of S.

Let S be a closed and bounded set in $\mathbb{R}$.
Then every open covering of S contains a finite subcovering of S.
Therefore, S is compact.
Thus, if a set S is closed and bounded, then S is compact.
Prove if S is compact, then S is closed and bounded.