Sequences in $\mathbb{R}$ Theory

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June 29, 2021

Sequences of Real Numbers

Proposition 1. ***n^{th} term of an arithmetic sequence***
Let $d \in \mathbb{R}$.
The n^{th} term of an arithmetic sequence with common difference d and initial value a_1 is $a_n = a_1 + (n-1)d$.

Proof. We prove $a_n = a_1 + (n-1)d$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : a_n = a_1 + (n-1)d\}$.
Since $a_1 = a_1 + 0 = a_1 + 0d = a_1 + (1-1)d$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $a_k = a_1 + (k-1)d$.
Observe that

$$\begin{aligned} a_{k+1} &= a_k + d \\ &= (a_1 + (k-1)d) + d \\ &= (a_1 + kd - d) + d \\ &= a_1 + kd - d + d \\ &= a_1 + kd \\ &= a_1 + ((k+1)-1)d. \end{aligned}$$

Thus, $a_{k+1} = a_1 + ((k+1)-1)d$, so $k+1 \in S$.
Therefore, by PMI, $S = \mathbb{N}$, so $a_n = a_1 + (n-1)d$ for all $n \in \mathbb{N}$. ☐

Proposition 2. *Let (a_n) be an arithmetic sequence of real numbers with common difference d.*
Then $a_n = \frac{a_{n-1}+a_{n+1}}{2}$ for all integers $n > 1$.

Proof. Let $n \in \mathbb{Z}$ with $n > 1$.
Since (a_n) is an arithmetic sequence, then $a_{n+1} = a_n + d$ for all $n \in \mathbb{Z}^+$.
Since $n > 1 > 0$, then $n > 0$.
Since $n \in \mathbb{Z}$ and $n > 0$, then $n \in \mathbb{Z}^+$, so $a_{n+1} = a_n + d$.
Since $n \in \mathbb{Z}$, then $n - 1 \in \mathbb{Z}$.
Since $n > 1$, then $n - 1 > 0$.
Since $n - 1 \in \mathbb{Z}$ and $n - 1 > 0$, then $n - 1 \in \mathbb{Z}^+$.

Hence, $a_n = a_{n-1} + d$, so $a_{n-1} = a_n - d$.
Therefore,

$$\begin{aligned}\frac{a_{n-1}+a_{n+1}}{2} &= \frac{(a_n-d)+(a_n+d)}{2} \\ &= \frac{2a_n}{2} \\ &= a_n.\end{aligned}$$

□

Proposition 3. ***n^{th} term of a geometric sequence***
Let $r \in \mathbb{R}, r \neq 0$.
The n^{th} term of a geometric sequence with common ratio r and initial value a_1 is $a_n = a_1 r^{n-1}$.

Proof. We prove $a_n = a_1 r^{n-1}$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : a_n = a_1 r^{n-1}\}$.
Since $a_1 = a_1 \cdot 1 = a_1 r^0 = a_1 r^{1-1}$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $a_k = a_1 r^{k-1}$.
Observe that

$$\begin{aligned}a_{k+1} &= a_k \cdot r \\ &= (a_1 r^{k-1})r \\ &= a_1(r^{k-1}r) \\ &= a_1(r^{k-1+1}) \\ &= a_1 r^k \\ &= a_1 r^{(k+1)-1}.\end{aligned}$$

Thus, $a_{k+1} = a_1 r^{(k+1)-1}$, so $k+1 \in S$.
Therefore, by PMI, $S = \mathbb{N}$, so $a_n = a_1 r^{n-1}$ for all $n \in \mathbb{N}$. □

Proposition 4. *Let (a_n) be a geometric sequence of positive real numbers with common ratio positive r.*
Then $a_n = \sqrt{a_{n-1}a_{n+1}}$ for all integers $n > 1$.

Proof. Let $n \in \mathbb{Z}$ with $n > 1$.
Since (a_n) is a geometric sequence, then $a_{n+1} = a_n r$ for all $n \in \mathbb{Z}^+$.
Since (a_n) is a sequence of positive terms, then $a_n > 0$ for all $n \in \mathbb{Z}^+$.
Since $n > 1 > 0$, then $n > 0$.
Since $n \in \mathbb{Z}$ and $n > 0$, then $n \in \mathbb{Z}^+$, so $a_{n+1} = a_n r$ and and $a_n > 0$.
Since $n \in \mathbb{Z}$, then $n - 1 \in \mathbb{Z}$.
Since $n > 1$, then $n - 1 > 0$.
Since $n - 1 \in \mathbb{Z}$ and $n - 1 > 0$, then $n - 1 \in \mathbb{Z}^+$, so $a_n = a_{n-1} r$.
Since $r > 0$, then $r \neq 0$, so $a_{n-1} = \frac{a_n}{r}$.

Therefore,

$$\begin{aligned}\sqrt{a_{n-1}a_{n+1}} &= \sqrt{(\frac{a_n}{r})(a_n r)} \\ &= \sqrt{a_n a_n} \\ &= \sqrt{(a_n)^2} \\ &= |a_n| \\ &= a_n.\end{aligned}$$

□

Sequences as Functions

Proposition 5. ***sum and product of bounded sequences is bounded***
Let (a_n) and (b_n) be bounded sequences of real numbers. Then
1. $(a_n + b_n)$ is bounded.
2. $(a_n b_n)$ is bounded.

Proof. We prove 1.
Suppose (a_n) and (b_n) are bounded.
Since (a_n) is bounded, then there exists $\alpha \in \mathbb{R}$ such that $|a_n| \leq \alpha$ for all $n \in \mathbb{N}$.
Since (b_n) is bounded, then there exists $\beta \in \mathbb{R}$ such that $|b_n| \leq \beta$ for all $n \in \mathbb{N}$.
To prove $(a_n + b_n)$ is bounded, we must prove there exists $\gamma \in \mathbb{R}$ such that $|a_n + b_n| \leq \gamma$ for all $n \in \mathbb{N}$.
Let $\gamma = \alpha + \beta$.
Since $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, then $\alpha + \beta \in \mathbb{R}$, so $\gamma \in \mathbb{R}$.
Let $n \in \mathbb{N}$.
Then $|a_n| \leq \alpha$ and $|b_n| \leq \beta$.
Thus,

$$\begin{aligned}|a_n + b_n| &\leq |a_n| + |b_n| \\ &\leq \alpha + \beta \\ &= \gamma.\end{aligned}$$

Therefore, $|a_n + b_n| \leq \gamma$. □

Proof. We prove 2.
Suppose (a_n) and (b_n) are bounded.
Since (a_n) is bounded, then there exists $\alpha \in \mathbb{R}$ such that $|a_n| \leq \alpha$ for all $n \in \mathbb{N}$.
Since (b_n) is bounded, then there exists $\beta \in \mathbb{R}$ such that $|b_n| \leq \beta$ for all $n \in \mathbb{N}$.
To prove $(a_n b_n)$ is bounded, we must prove there exists $\gamma \in \mathbb{R}$ such that $|a_n b_n| \leq \gamma$ for all $n \in \mathbb{N}$.

Let $\gamma = \alpha\beta$.
Since $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, then $\alpha\beta \in \mathbb{R}$, so $\gamma \in \mathbb{R}$.
Let $n \in \mathbb{N}$.
Then $|a_n| \leq \alpha$ and $|b_n| \leq \beta$.
Since $0 \leq |a_n| \leq \alpha$ and $0 \leq |b_n| \leq \beta$, then $|a_n b_n| = |a_n||b_n| \leq \alpha\beta = \gamma$.
Therefore, $|a_n b_n| \leq \gamma$. □

Proposition 6. ***necessary and sufficient conditions for a monotonic sequence***

Let (a_n) be a sequence of real numbers. Then

1. (a_n) is strictly increasing iff $m < n$ implies $a_m < a_n$ for all $m, n \in \mathbb{N}$.

2. (a_n) is (monotonic) increasing iff $m < n$ implies $a_m \leq a_n$ for all $m, n \in \mathbb{N}$.

3. (a_n) is strictly decreasing iff $m < n$ implies $a_m > a_n$ for all $m, n \in \mathbb{N}$.

4. (a_n) is (monotonic) decreasing iff $m < n$ implies $a_m \geq a_n$ for all $m, n \in \mathbb{N}$.

Proof. We prove 1.

We must prove (a_n) is strictly increasing iff $m < n$ implies $a_m < a_n$ for all $m, n \in \mathbb{N}$.

We prove if $m < n$ implies $a_m < a_n$ for all $m, n \in \mathbb{N}$, then (a_n) is strictly increasing.

Suppose $m < n$ implies $a_m < a_n$ for all $m, n \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
Since $n, n+1 \in \mathbb{N}$ and $n < n+1$, then $a_n < a_{n+1}$.
Therefore, (a_n) is strictly increasing.

Conversely, we prove if (a_n) is strictly increasing, then $m < n$ implies $a_m < a_n$ for all $m, n \in \mathbb{N}$.

Suppose (a_n) is strictly increasing.
Let $m \in \mathbb{N}$ be given.

The statement $m < n$ implies $a_m < a_n$ for all $n \in \mathbb{N}$ means that if n is an arbitrary natural number such that $m < n$, then $a_m < a_n$.

So, if $n \in \mathbb{N}$ such that $m < n$, then we must prove $a_m < a_{m+1}$ and $a_m < a_{m+2}$ and $a_m < a_{m+3}$... etc.

Thus, we must prove $a_m < a_{m+t}$ for every natural number t.
We prove $a_m < a_{m+t}$ for all $t \in \mathbb{N}$ by induction on t.
Let $S = \{t \in \mathbb{N} : a_m < a_{m+t}\}$.
Since (a_n) is strictly increasing, then $a_m < a_{m+1}$.
Hence, $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $a_m < a_{m+k}$.
Since (a_n) is strictly increasing, then $a_{m+k} < a_{m+k+1}$.
Thus, $a_m < a_{m+k} < a_{m+k+1}$, so $a_m < a_{m+k+1}$.
Since $k+1 \in \mathbb{N}$ and $a_m < a_{m+k+1}$, then $k+1 \in S$.
Thus, $k \in S$ implies $k+1 \in S$.

By the principle of mathematical induction, $a_m < a_{m+t}$ for every natural number t.

Hence, $m < n$ implies $a_m < a_n$ for all $n \in \mathbb{N}$.

Since $m \in \mathbb{N}$ is arbitrary, then $m < n$ implies $a_m < a_n$ for all $m, n \in \mathbb{N}$. $\square$

Proof. We prove 2.

We must prove (a_n) is monotonic increasing iff $m < n$ implies $a_m \leq a_n$ for all $m, n \in \mathbb{N}$.

We prove if $m < n$ implies $a_m \leq a_n$ for all $m, n \in \mathbb{N}$, then (a_n) is monotonic increasing.

Suppose $m < n$ implies $a_m \leq a_n$ for all $m, n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Since $n, n+1 \in \mathbb{N}$ and $n < n+1$, then $a_n \leq a_{n+1}$.

Therefore, (a_n) is monotonic increasing.

Conversely, we prove if (a_n) is monotonic increasing, then $m < n$ implies $a_m \leq a_n$ for all $m, n \in \mathbb{N}$.

Suppose (a_n) is monotonic increasing.

Let $m \in \mathbb{N}$ be given.

The statement $m < n$ implies $a_m \leq a_n$ for all $n \in \mathbb{N}$ means that if n is an arbitrary natural number such that $m < n$, then $a_m \leq a_n$.

So, if $n \in \mathbb{N}$ such that $m < n$, then we must prove $a_m \leq a_{m+1}$ and $a_m \leq a_{m+2}$ and $a_m \leq a_{m+3}$... etc.

Thus, we must prove $a_m \leq a_{m+t}$ for every natural number t.

We prove $a_m \leq a_{m+t}$ for all $t \in \mathbb{N}$ by induction on t.

Let $S = \{t \in \mathbb{N} : a_m \leq a_{m+t}\}$.

Since (a_n) is monotonic increasing, then $a_m \leq a_{m+1}$.

Hence, $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $a_m \leq a_{m+k}$.

Since (a_n) is monotonic increasing, then $a_{m+k} \leq a_{m+k+1}$.

Thus, $a_m \leq a_{m+k} \leq a_{m+k+1}$, so $a_m \leq a_{m+k+1}$.

Since $k+1 \in \mathbb{N}$ and $a_m \leq a_{m+k+1}$, then $k+1 \in S$.

Thus, $k \in S$ implies $k+1 \in S$.

By the principle of mathematical induction, $a_m \leq a_{m+t}$ for every natural number t.

Hence, $m < n$ implies $a_m \leq a_n$ for all $n \in \mathbb{N}$.

Since $m \in \mathbb{N}$ is arbitrary, then $m < n$ implies $a_m \leq a_n$ for all $m, n \in \mathbb{N}$. $\square$

Proof. We prove 3.

We must prove (a_n) is strictly decreasing iff $m < n$ implies $a_m > a_n$ for all $m, n \in \mathbb{N}$.

We prove if $m < n$ implies $a_m > a_n$ for all $m, n \in \mathbb{N}$, then (a_n) is strictly decreasing.

Suppose $m < n$ implies $a_m > a_n$ for all $m, n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Since $n, n+1 \in \mathbb{N}$ and $n < n+1$, then $a_n > a_{n+1}$.

Therefore, (a_n) is strictly decreasing.

Conversely, we prove if (a_n) is strictly decreasing, then $m < n$ implies $a_m > a_n$ for all $m, n \in \mathbb{N}$.

Suppose (a_n) is strictly decreasing.

Let $m \in \mathbb{N}$ be given.

The statement $m < n$ implies $a_m > a_n$ for all $n \in \mathbb{N}$ means that if n is an arbitrary natural number such that $m < n$, then $a_m > a_n$.

So, if $n \in \mathbb{N}$ such that $m < n$, then we must prove $a_m > a_{m+1}$ and $a_m > a_{m+2}$ and $a_m > a_{m+3}$... etc.

Thus, we must prove $a_m > a_{m+t}$ for every natural number t.

We prove $a_m > a_{m+t}$ for all $t \in \mathbb{N}$ by induction on t.

Let $S = \{t \in \mathbb{N} : a_m > a_{m+t}\}$.

Since (a_n) is strictly decreasing, then $a_m > a_{m+1}$.

Hence, $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $a_m > a_{m+k}$.

Since (a_n) is strictly decreasing, then $a_{m+k} > a_{m+k+1}$.

Thus, $a_m > a_{m+k} > a_{m+k+1}$, so $a_m > a_{m+k+1}$.

Since $k + 1 \in \mathbb{N}$ and $a_m > a_{m+k+1}$, then $k + 1 \in S$.

Thus, $k \in S$ implies $k + 1 \in S$.

By the principle of mathematical induction, $a_m > a_{m+t}$ for every natural number t.

Hence, $m < n$ implies $a_m > a_n$ for all $n \in \mathbb{N}$.

Since $m \in \mathbb{N}$ is arbitrary, then $m < n$ implies $a_m > a_n$ for all $m, n \in \mathbb{N}$. □

Proof. We prove 4.

We must prove (a_n) is monotonic decreasing iff $m < n$ implies $a_m \geq a_n$ for all $m, n \in \mathbb{N}$.

We prove if $m < n$ implies $a_m \geq a_n$ for all $m, n \in \mathbb{N}$, then (a_n) is monotonic decreasing.

Suppose $m < n$ implies $a_m \geq a_n$ for all $m, n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Since $n, n + 1 \in \mathbb{N}$ and $n < n + 1$, then $a_n \geq a_{n+1}$.

Therefore, (a_n) is monotonic decreasing.

Conversely, we prove if (a_n) is monotonic decreasing, then $m < n$ implies $a_m \geq a_n$ for all $m, n \in \mathbb{N}$.

Suppose (a_n) is monotonic decreasing.

Let $m \in \mathbb{N}$ be given.

The statement $m < n$ implies $a_m \geq a_n$ for all $n \in \mathbb{N}$ means that if n is an arbitrary natural number such that $m < n$, then $a_m \geq a_n$.

So, if $n \in \mathbb{N}$ such that $m < n$, then we must prove $a_m \geq a_{m+1}$ and $a_m \geq a_{m+2}$ and $a_m \geq a_{m+3}$... etc.

Thus, we must prove $a_m \geq a_{m+t}$ for every natural number t.

We prove $a_m \geq a_{m+t}$ for all $t \in \mathbb{N}$ by induction on t.

Let $S = \{t \in \mathbb{N} : a_m \geq a_{m+t}\}$.

Since (a_n) is monotonic decreasing, then $a_m \geq a_{m+1}$.

Hence, $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $a_m \geq a_{m+k}$.
Since (a_n) is monotonic decreasing, then $a_{m+k} \geq a_{m+k+1}$.
Thus, $a_m \geq a_{m+k} \geq a_{m+k+1}$, so $a_m \geq a_{m+k+1}$.
Since $k+1 \in \mathbb{N}$ and $a_m \geq a_{m+k+1}$, then $k+1 \in S$.
Thus, $k \in S$ implies $k+1 \in S$.
By the principle of mathematical induction, $a_m \geq a_{m+t}$ for every natural number t.
Hence, $m < n$ implies $a_m \geq a_n$ for all $n \in \mathbb{N}$.
Since $m \in \mathbb{N}$ is arbitrary, then $m < n$ implies $a_m \geq a_n$ for all $m, n \in \mathbb{N}$. □

Proposition 7. *If $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, then $f(n) \geq n$ for all $n \in \mathbb{N}$.*

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function.
We prove $f(n) \geq n$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : f(n) \geq n\}$.
Since $1 \in \mathbb{N}$, then $f(1) \in \mathbb{N}$, so $f(1) \geq 1$.
Hence, $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $f(k) \geq k$.
Since f is strictly increasing and $k \in \mathbb{N}$ and $k+1 \in \mathbb{N}$, then $f(k) < f(k+1)$.
Thus, $k \leq f(k)$ and $f(k) < f(k+1)$, so $k < f(k+1)$.
Suppose $f(k+1) < k+1$.
Then $k < f(k+1)$ and $f(k+1) < k+1$, so $k < f(k+1) < k+1$.
Since $k, k+1, f(k+1) \in \mathbb{N}$, then this implies there is a natural number between two consecutive natural numbers, an impossibility.
Therefore, $f(k+1) \geq k+1$, so $k+1 \in S$.
Thus, by PMI, $S = \mathbb{N}$, so $f(n) \geq n$ for all $n \in \mathbb{N}$, as desired. □

Proposition 8. ***subsequence preserves monotonicity and boundedness***

1. *Every subsequence of an increasing sequence is increasing.*
2. *Every subsequence of a decreasing sequence is decreasing.*
3. *Every subsequence of a bounded sequence is bounded.*

Proof. We prove 1.
We must prove every subsequence of an increasing sequence is increasing.
Let (a_n) be a monotonic increasing sequence of real numbers.
Since (a_n) is a sequence of real numbers, then there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = a_n$ for all $n \in \mathbb{N}$.
Suppose (b_n) is a subsequence of (a_n).
Then there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $b_n = (f \circ g)(n)$ for all $n \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
Since g is strictly increasing, then $g(n) < g(n+1)$.

Since (a_n) is monotonic increasing and $g(n), g(n+1) \in \mathbb{N}$ and $g(n) < g(n+1)$, then $a_{g(n)} \leq a_{g(n+1)}$.

Observe that

$$\begin{aligned} b_n &= (f \circ g)(n) \\ &= f(g(n)) \\ &= a_{g(n)} \\ &\leq a_{g(n+1)} \\ &= f(g(n+1)) \\ &= (f \circ g)(n+1) \\ &= b_{n+1}. \end{aligned}$$

Therefore, $b_n \leq b_{n+1}$, so (b_n) is increasing, as desired. □

Proof. We prove 2.

We must prove every subsequence of a decreasing sequence is decreasing.

Let (a_n) be a monotonic decreasing sequence of real numbers.

Since (a_n) is a sequence of real numbers, then there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = a_n$ for all $n \in \mathbb{N}$.

Suppose (b_n) is a subsequence of (a_n).

Then there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $b_n = (f \circ g)(n)$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Since g is strictly increasing, then $g(n) < g(n+1)$.

Since (a_n) is monotonic decreasing and $g(n), g(n+1) \in \mathbb{N}$ and $g(n) < g(n+1)$, then $a_{g(n)} \geq a_{g(n+1)}$.

Observe that

$$\begin{aligned} b_n &= (f \circ g)(n) \\ &= f(g(n)) \\ &= a_{g(n)} \\ &\geq a_{g(n+1)} \\ &= f(g(n+1)) \\ &= (f \circ g)(n+1) \\ &= b_{n+1}. \end{aligned}$$

Therefore, $b_n \geq b_{n+1}$, so (b_n) is decreasing, as desired. □

Proof. We prove 3.

We must prove every subsequence of a bounded sequence is bounded.

Let (a_n) be a bounded sequence of real numbers.

Since (a_n) is a sequence of real numbers, then there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = a_n$ for all $n \in \mathbb{N}$.

Suppose (b_n) is a subsequence of (a_n).

Then there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $b_n = (f \circ g)(n)$ for all $n \in \mathbb{N}$.

Since (a_n) is bounded, then there exists $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Then $g(n) \in \mathbb{N}$ and

$$\begin{aligned} |b_n| &= |(f \circ g)(n)| \\ &= |f(g(n))| \\ &= |a_{g(n)}| \\ &\leq M. \end{aligned}$$

Therefore, $|b_n| \leq M$, so (b_n) is bounded, as desired. □

Proposition 9. ***M tail of a sequence is a subsequence of the sequence***

Let (a_n) be a sequence in $\mathbb{R}$.

If (b_n) is an M tail of (a_n), then (b_n) is a subsequence of (a_n).

Proof. Suppose (b_n) is an M tail of (a_n).

Then there exists $M \in \mathbb{N}$ such that $b_n = a_{M+n}$ for all $n \in \mathbb{N}$.

Since (a_n) is a sequence in $\mathbb{R}$, then there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = a_n$ for all $n \in \mathbb{N}$.

Let $g : \mathbb{N} \to \mathbb{N}$ be a function defined by $g(n) = M + n$ for all $n \in \mathbb{N}$.

Let $m, n \in \mathbb{N}$ such that $m < n$.

Then $g(m) = M + m < M + n = g(n)$, so $g(m) < g(n)$.

Hence, g is strictly increasing.

Let $n \in \mathbb{N}$ be given.

Then

$$\begin{aligned} b_n &= a_{M+n} \\ &= f(M+n) \\ &= f(g(n)) \\ &= (f \circ g)(n). \end{aligned}$$

Therefore, (b_n) is a subsequence of (a_n). □

Convergent Sequences in $\mathbb{R}$

Theorem 10. ***uniqueness of a limit of a convergent sequence***

The limit of a convergent sequence of real numbers is unique.

Proof. Let (a_n) be a convergent sequence of real numbers.

Then a limit of (a_n) exists as a real number.

Thus, there is at least one limit of (a_n).

To prove the limit is unique, let $L_1, L_2 \in \mathbb{R}$ such that L_1 is a limit of (a_n) and L_2 is a limit of (a_n).

We must prove $L_1 = L_2$.

Suppose $L_1 \neq L_2$.

Then $L_1 - L_2 \neq 0$, so $|L_1 - L_2| > 0$.

Let $\epsilon = \frac{|L_1 - L_2|}{2}$.

Then $\epsilon > 0$.

Since L_1 is a limit of (a_n) and $\epsilon > 0$, then there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $|a_n - L_1| < \epsilon$.

Since L_2 is a limit of (a_n) and $\epsilon > 0$, then there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|a_n - L_2| < \epsilon$.

Let $N = \max\{N_1, N_2\}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N \geq N_1$, then $n > N_1$.

Hence, $|a_n - L_1| < \epsilon$.

Since $n > N \geq N_2$, then $n > N_2$.

Hence, $|a_n - L_2| < \epsilon$.

Observe that

$$\begin{aligned} |L_1 - L_2| &= |(L_1 - a_n) + (a_n - L_2)| \\ &\leq |L_1 - a_n| + |a_n - L_2| \\ &= |a_n - L_1| + |a_n - L_2| \\ &< \epsilon + \epsilon \\ &= 2\epsilon. \end{aligned}$$

Thus, $|L_1 - L_2| < 2\epsilon$, so $\frac{|L_1 - L_2|}{2} < \epsilon$.

Hence, $\epsilon < \epsilon$, a contradiction.

Therefore, $L_1 = L_2$, as desired. ☐

Proposition 11. ***a difference in a finite number of initial terms does not affect the convergence of a sequence***

Let (a_n) and (b_n) be sequences of real numbers.

If there exists $K \in \mathbb{N}$ such that $b_n = a_n$ for all $n > K$ and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} b_n = L$.

Proof. Suppose there exists $K \in \mathbb{N}$ such that $b_n = a_n$ for all $n > K$ and $\lim_{n\to\infty} a_n = L$.

Let $\epsilon > 0$ be given.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < \epsilon$.

Let $M = \max\{K, N\}$.

Let $n \in \mathbb{N}$ such that $n > M$.

Since $n > M \geq N$, then $n > N$.

Hence, $|a_n - L| < \epsilon$.

Since $n > M \geq K$, then $n > K$.

Hence, $b_n = a_n$.

Thus, $|b_n - L| = |a_n - L| < \epsilon$, so $|b_n - L| < \epsilon$.

Therefore, $\lim_{n\to\infty} b_n = L$. ☐

Proposition 12. *Let $L \in \mathbb{R}$.*

Let (a_n) and $(a_n - L)$ be sequences in $\mathbb{R}$.

Then $\lim_{n\to\infty} a_n = L$ iff $\lim_{n\to\infty}(a_n - L) = 0$.

Proof. Observe that

$$\begin{aligned}\lim_{n\to\infty} a_n = L \quad &\Leftrightarrow\\ (\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |a_n - L| < \epsilon) \quad &\Leftrightarrow\\ (\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |(a_n - L) - 0| < \epsilon) \quad &\Leftrightarrow\\ \lim_{n\to\infty}(a_n - L) = 0. &\end{aligned}$$

∎

Theorem 13. ***every subsequence of a convergent sequence is convergent***

Let (a_n) be a convergent sequence of real numbers.

If (b_n) is a subsequence of (a_n), then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

Proof. Suppose (b_n) is a subsequence of (a_n).

Since (a_n) is convergent, then there exists a real number L such that $\lim_{n\to\infty} a_n = L$.

Let $\epsilon > 0$ be given.

Then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < \epsilon$.

Since (a_n) is a sequence, then there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = a_n$ for all $n \in \mathbb{N}$.

Since (b_n) is a subsequence of (a_n), then there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $b_n = (f \circ g)(n)$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $N < n$.

Since g is strictly increasing, then $g(N) < g(n)$.

Since g is strictly increasing and $N \in \mathbb{N}$, then by a previous proposition, $g(N) \geq N$.

Thus, $N \leq g(N)$ and $g(N) < g(n)$, so $N < g(n)$.

Since $g(n) \in \mathbb{N}$ and $g(n) > N$, then $|a_{g(n)} - L| < \epsilon$.

Observe that

$$\begin{aligned}|b_n - L| &= |(f \circ g)(n) - L|\\ &= |f(g(n)) - L|\\ &= |a_{g(n)} - L|\\ &< \epsilon.\end{aligned}$$

Therefore, $|b_n - L| < \epsilon$, so $\lim_{n\to\infty} b_n = L = \lim_{n\to\infty} a_n$. ∎

Corollary 14. *Let (a_n) be a sequence of real numbers.*

If (b_n) and (c_n) are convergent subsequences of (a_n) such that $\lim_{n\to\infty} b_n \neq \lim_{n\to\infty} c_n$, then (a_n) is divergent.

Proof. Suppose (b_n) and (c_n) are convergent subsequences of (a_n) such that $\lim_{n\to\infty} b_n \neq \lim_{n\to\infty} c_n$.

We prove (a_n) is divergent by contradiction.

Suppose (a_n) is not divergent.

Then (a_n) is convergent.

Hence, there exists a real number L such that $\lim_{n\to\infty} a_n = L$.

Since (b_n) is a subsequence of (a_n) and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} b_n = L$.

Since (c_n) is a subsequence of (a_n) and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} c_n = L$.

Thus, $\lim_{n\to\infty} b_n = L = \lim_{n\to\infty} c_n$, so $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n$.

This contradicts the assumption that $\lim_{n\to\infty} b_n \neq \lim_{n\to\infty} c_n$.

Therefore, (a_n) is divergent. □

Proposition 15. ***M tail of a sequence is convergent iff the sequence is convergent***

Let (a_n) be a sequence of real numbers.

Let $M \in \mathbb{N}$.

If (a_n) is convergent, then $\lim_{n\to\infty} a_{M+n} = \lim_{n\to\infty} a_n$.

If (a_{M+n}) is convergent, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{M+n}$.

Proof. Suppose (a_n) is convergent.

Then there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = L$.

Let (b_n) be a sequence of real numbers defined by $b_n = a_{M+n}$ for all $n \in \mathbb{N}$.

We must prove $\lim_{n\to\infty} b_n = L$.

Since $b_n = a_{M+n}$ for all $n \in \mathbb{N}$, then (b_n) is the M tail of (a_n).

Hence, (b_n) is a subsequence of (a_n).

Since (a_n) is convergent, then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = L$, as esired.

Conversely, suppose (b_n) is convergent.

Then there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} b_n = L$.

We must prove $\lim_{n\to\infty} a_n = L$.

Let $\epsilon > 0$ be given.

Then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|b_n - L| < \epsilon$.

We prove $a_n = b_{n-M}$ for all $n > M$ by induction on n.

Let $S = \{n \in \mathbb{N} : a_n = b_{n-M}, n > M\}$.

Since $M+1 \in \mathbb{N}$ and $M+1 > M$ and $a_{M+1} = b_1 = b_{(M+1)-M}$, then $M+1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $k > M$.

Since $k \in \mathbb{N}$, then $k+1 \in \mathbb{N}$.

Since $k+1 > k$ and $k > M$, then $k+1 > M$.

Since $k+1-M \in \mathbb{N}$ and $b_{k+1-M} = a_{M+(k+1-M)} = a_{k+1}$, then $k+1 \in S$.

Therefore, by PMI, $a_n = b_{n-M}$ for all $n > M$.

Since $M \in \mathbb{N}$ and $N \in \mathbb{N}$, then $M + N \in \mathbb{N}$.
Let $n \in \mathbb{N}$ such that $n > M + N$.
Then $n - M > N$, so $|b_{n-M} - L| < \epsilon$.
Since $N \in \mathbb{N}$, then $N > 0$, so $M + N > M$.
Since $n > M + N$ and $M + N > M$, then $n > M$, so $a_n = b_{n-M}$.
Observe that

$$\begin{aligned} |a_n - L| &= |b_{n-M} - L| \\ &< \epsilon. \end{aligned}$$

Hence, $|a_n - L| < \epsilon$, so $\lim_{n\to\infty} a_n = L$, as desired. □

Algebraic properties of convergent sequences

Theorem 16. ***convergence implies boundedness***
Every convergent sequence of real numbers is bounded.

Proof. Let (a_n) be a convergent sequence of real numbers.
Then there is a real number L such that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n > N$.
Let $\epsilon = 1$.
Then there exists $N \in \mathbb{N}$ such that $|a_n - L| < 1$ whenever $n > N$.
Let $S = \{|a_1|, |a_2|, ..., |a_N|, 1 + |L|\} = \{|a_k| : 1 \leq k \leq N\} \cup \{1 + |L|\}$.
Then $S \subset \mathbb{R}$.
Since $1 + |L| \in S$, then S is not empty.
Since S contains at most $N + 1$ elements, then S is finite.
Hence, S is a nonempty finite set of real numbers.
Therefore, $\max S$ exists.
To prove (a_n) is bounded, we must prove there exists $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.
Let $M = \max S$.
Since $M = \max S \in S$ and $S \subset \mathbb{R}$, then $M \in \mathbb{R}$.
Let $n \in \mathbb{N}$.
Either $n \leq N$ or $n > N$.
We consider these cases separately.
Case 1: Suppose $n \leq N$.
Then $1 \leq n \leq N$, so $|a_n| \in S$.
Therefore, $|a_n| \leq M$.
Case 2: Suppose $n > N$.
Then $|a_n - L| < 1$.
Since $1 + |L| \in S$ and $M = \max S$, then $1 + |L| \leq M$.

Observe that

$$\begin{aligned}|a_n| &= |(a_n - L) + L| \\ &\leq |a_n - L| + |L| \\ &< 1 + |L| \\ &\leq M.\end{aligned}$$

Therefore, $|a_n| < M$, so $|a_n| \leq M$.

Thus, in all cases, $|a_n| \leq M$, so (a_n) is bounded, as desired. □

Proposition 17. *If $\lim_{n\to\infty} a_n = 0$ and (b_n) is bounded, then $\lim_{n\to\infty} a_n b_n = 0$.*

Proof. Suppose $\lim_{n\to\infty} a_n = 0$ and (b_n) is bounded.

Let $\epsilon > 0$ be given.

Since (b_n) is bounded, then there exists $M > 0$ such that $|b_n| < M$ for all $n \in \mathbb{N}$.

Since $\epsilon > 0$ and $M > 0$, then $\frac{\epsilon}{M} > 0$.

Since $\lim_{n\to\infty} a_n = 0$, then there exists $N \in \mathbb{N}$ such that $|a_n| < \frac{\epsilon}{M}$ whenever $n > N$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|a_n| < \frac{\epsilon}{M}$ and $|b_n| < M$.

Since $0 \leq |a_n| < \frac{\epsilon}{M}$ and $0 \leq |b_n| < M$, then

$$\begin{aligned}|a_n b_n| &= |a_n||b_n| \\ &< \frac{\epsilon}{M} \cdot M \\ &= \epsilon.\end{aligned}$$

Therefore, $|a_n b_n| < \epsilon$, so $\lim_{n\to\infty} a_n b_n = 0$. □

Lemma 18. *Let (a_n) be a sequence of real numbers.*

If there exists $L \neq 0$ such that $\lim_{n\to\infty} a_n = L$, then there is a natural number N such that $|a_n| > \frac{|L|}{2}$ for all $n > N$.

Proof. Suppose there exists $L \neq 0$ such that $\lim_{n\to\infty} a_n = L$.

Since $L \neq 0$, then $|L| > 0$, so $\frac{|L|}{2} > 0$.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that $|a_n - L| < \frac{|L|}{2}$ whenever $n > N$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|a_n - L| < \frac{|L|}{2}$.

Since $\frac{|L|}{2} > |a_n - L| \geq |L| - |a_n|$, then $\frac{|L|}{2} > |L| - |a_n|$.

Therefore, $|a_n| > \frac{|L|}{2}$, as desired. □

Lemma 19. *Let (a_n) be a sequence of real numbers.*

If there exists $L \neq 0$ such that $\lim_{n\to\infty} a_n = L$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$.

Proof. Suppose there exists $L \neq 0$ such that $\lim_{n\to\infty} a_n = L$ and $a_n \neq 0$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Since $a_n \neq 0$ for all $n \in \mathbb{N}$, then $a_n \neq 0$, so $\frac{1}{a_n} \in \mathbb{R}$.

Hence, $\frac{1}{a_n} \in \mathbb{R}$ for all $n \in \mathbb{N}$, so the sequence $(\frac{1}{a_n})$ is well defined.

Since $L \neq 0$, then $\frac{1}{L} \in \mathbb{R}$.

To prove $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$, let $\epsilon > 0$ be given.

Since $\epsilon > 0$ and $|L|^2 > 0$, then $\frac{\epsilon|L|^2}{2} > 0$.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$, then $|a_n - L| < \frac{\epsilon|L|^2}{2}$.

Since $\lim_{n\to\infty} a_n = L$ and $L \neq 0$, then by a previous lemma, there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$, then $|a_n| > \frac{|L|}{2}$.

Let $N = \max\{N_1, N_2\}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N \geq N_1$, then $n > N_1$, so $|a_n - L| < \frac{\epsilon|L|^2}{2}$.

Thus, $0 \leq |a_n - L| < \frac{\epsilon|L|^2}{2}$.

Since $n > N \geq N_2$, then $n > N_2$, so $|a_n| > \frac{|L|}{2}$.

Since $L \neq 0$, then $|L| > 0$.

Since $a_n \neq 0$, then $|a_n| > 0$.

Thus, $\frac{2}{|L|} > \frac{1}{|a_n|} > 0$, so $0 < \frac{1}{|a_n|} < \frac{2}{|L|}$.

Observe that

$$\begin{aligned} |\frac{1}{a_n} - \frac{1}{L}| &= |\frac{L - a_n}{a_n L}| \\ &= |\frac{a_n - L}{a_n L}| \\ &= |a_n - L| \cdot \frac{1}{|a_n|} \cdot \frac{1}{|L|} \\ &< \frac{\epsilon|L|^2}{2} \cdot \frac{2}{|L|} \cdot \frac{1}{|L|} \\ &= \epsilon. \end{aligned}$$

Therefore, $|\frac{1}{a_n} - \frac{1}{L}| < \epsilon$, so $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$, as desired. □

Theorem 20. ***algebraic limit rules for convergent sequences***

If (a_n) and (b_n) are convergent sequences of real numbers, then

1. Scalar Multiple Rule

$\lim_{n\to\infty}(\lambda a_n) = \lambda \lim_{n\to\infty} a_n$ *for every* $\lambda \in \mathbb{R}$.

2. Sum Rule (limit of sum equals sum of limits)

$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

3. Difference Rule (limit of difference equals difference of limits)

$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$.

4. Product Rule (limit of product equals product of limits)

$\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$.

5. Quotient Rule (limit of quotient equals quotient of limits)

If $\lim_{n\to\infty} b_n \neq 0$*, then*
$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

Proof. Let (a_n) be a convergent sequence of real numbers.
Then there exists a real number L such that $\lim_{n\to\infty} a_n = L$.
We prove 1.
Let $\lambda \in \mathbb{R}$.
We must prove $\lim_{n\to\infty}(\lambda a_n) = \lambda L$.
Either $\lambda = 0$ or $\lambda \neq 0$.
We consider these cases separately.
Case 1: Suppose $\lambda = 0$.
Observe that

$$\begin{aligned}\lim_{n\to\infty}(0a_n) &= \lim_{n\to\infty} 0 \\ &= 0 \\ &= 0L.\end{aligned}$$

Therefore, $\lim_{n\to\infty}(0a_n) = 0L$, as desired.
Case 2: Suppose $\lambda \neq 0$.
Let $\epsilon > 0$.
Since $|\lambda| \geq 0$ and $\lambda \neq 0$, then $|\lambda| > 0$.
Hence, $\frac{\epsilon}{|\lambda|} > 0$.
Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that $|a_n - L| < \frac{\epsilon}{|\lambda|}$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $|a_n - L| < \frac{\epsilon}{|\lambda|}$.
Observe that

$$\begin{aligned}|\lambda a_n - \lambda L| &= |\lambda(a_n - L)| \\ &= |\lambda||a_n - L| \\ &< |\lambda|\frac{\epsilon}{|\lambda|} \\ &= \epsilon.\end{aligned}$$

Therefore, $\lim_{n\to\infty}(\lambda a_n) = \lambda L$, as desired. ☐

Proof. Let (a_n) and (b_n) be convergent sequences of real numbers.
Then there exist real numbers L and M such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$.
We prove 2.
We must prove $\lim_{n\to\infty}(a_n + b_n) = L + M$.
Let $\epsilon > 0$.
Then $\frac{\epsilon}{2} > 0$.
Since $\lim_{n\to\infty} a_n = L$, then there exists $N_1 \in \mathbb{N}$ such that $|a_n - L| < \frac{\epsilon}{2}$ whenever $n > N_1$.

Since $\lim_{n\to\infty} b_n = M$, then there exists $N_2 \in \mathbb{N}$ such that $|b_n - M| < \frac{\epsilon}{2}$ whenever $n > N_2$.

Let $N = \max\{N_1, N_2\}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N \geq N_1$, then $n > N_1$.

Hence, $|a_n - L| < \frac{\epsilon}{2}$.

Since $n > N \geq N_2$, then $n > N_2$.

Hence, $|b_n - M| < \frac{\epsilon}{2}$.

Observe that

$$\begin{aligned} |(a_n + b_n) - (L + M)| &= |a_n + b_n - L - M| \\ &= |(a_n - L) + (b_n - M)| \\ &\leq |a_n - L| + |b_n - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, $|(a_n + b_n) - (L + M)| < \epsilon$.

Therefore, $\lim_{n\to\infty}(a_n + b_n) = L + M$, as desired. □

Proof. Let (a_n) and (b_n) be convergent sequences of real numbers.

Then there exist real numbers L and M such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$.

We prove 3.

We must prove $\lim_{n\to\infty}(a_n - b_n) = L - M$.

Observe that

$$\begin{aligned} \lim_{n\to\infty}(a_n - b_n) &= \lim_{n\to\infty}[a_n + (-b_n)] \\ &= \lim_{n\to\infty} a_n + \lim_{x\to a} -b_n \\ &= \lim_{n\to\infty} a_n - \lim_{x\to a} b_n \\ &= L - M. \end{aligned}$$

Therefore, $\lim_{n\to\infty}(a_n - b_n) = L - M$, as desired. □

Proof. Let (a_n) and (b_n) be convergent sequences of real numbers.

Then there exist real numbers L and M such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$.

We prove 4.

We must prove $\lim_{n\to\infty}(a_n b_n) = LM$.

Let $\epsilon > 0$.

Since (b_n) is convergent, then (b_n) is bounded.

Hence, there exists $b > 0$ such that $|b_n| < b$ for all $n \in \mathbb{N}$.

Since $\epsilon > 0$ and $b > 0$, then $\frac{\epsilon}{2b} > 0$.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N_1 \in \mathbb{N}$ such that $|a_n - L| < \frac{\epsilon}{2b}$ whenever $n > N_1$.

Since $|L| \geq 0$ and $|L| \geq 0 \Rightarrow |L|+1 \geq 1 \Rightarrow 2(|L|+1) \geq 2 > 0$, then $2(|L|+1) > 0$.

Since $\epsilon > 0$, then $\frac{\epsilon}{2(|L|+1)} > 0$.

Since $\lim_{n\to\infty} b_n = M$, then there exists $N_2 \in \mathbb{N}$ such that $|b_n - M| < \frac{\epsilon}{2(|L|+1)}$ whenever $n > N_2$.

Let $N = \max\{N_1, N_2\}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N \geq N_1$, then $n > N_1$, so $|a_n - L| < \frac{\epsilon}{2b}$.

Since $0 \leq |a_n - L| < \frac{\epsilon}{2b}$ and $0 \leq |b_n| < b$, then $|a_n - L||b_n| < \frac{\epsilon}{2}$.

Since $n > N \geq N_2$, then $n > N_2$, so $|b_n - M| < \frac{\epsilon}{2(|L|+1)}$.

Since $|L| \geq 0$, then $|L||b_n - M| \leq \frac{|L|\epsilon}{2(|L|+1)}$.

Since $0 \leq |L| < |L|+1$, then $0 \leq \frac{|L|}{|L|+1} < 1$, so $\frac{|L|\epsilon}{2(|L|+1)} < \frac{\epsilon}{2}$.

Thus, $|L||b_n - M| \leq \frac{|L|\epsilon}{2(|L|+1)}$ and $\frac{|L|\epsilon}{2(|L|+1)} < \frac{\epsilon}{2}$, so $|L||b_n - M| < \frac{\epsilon}{2}$.

Observe that

$$\begin{aligned}
|a_n b_n - LM| &= |(a_n b_n - L b_n) + (L b_n - LM)| \\
&\leq |a_n b_n - L b_n| + |L b_n - LM| \\
&= |(a_n - L) b_n| + |L(b_n - M)| \\
&= |a_n - L||b_n| + |L||b_n - M| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus, $|a_n b_n - LM| < \epsilon$.

Therefore, $\lim_{n\to\infty}(a_n b_n) = LM$, as desired. □

Proof. Let (a_n) and (b_n) be convergent sequences of real numbers.

Then there exist real numbers L and M such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$.

Suppose $M \neq 0$.

We prove 5.

We must prove $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$.

We first prove $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$.

Either there exists $k \in \mathbb{N}$ such that $b_k = 0$ or there does not exist $k \in \mathbb{N}$ such that $b_k = 0$.

We consider these cases separately.

Case 1: Suppose there does not exist $k \in \mathbb{N}$ such that $b_k = 0$.

Then $b_k \neq 0$ for all $k \in \mathbb{N}$.

Since $M \neq 0$ and $\lim_{n\to\infty} b_n = M$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then by a previous lemma, $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$.

Case 2: Suppose there exists $k \in \mathbb{N}$ such that $b_k = 0$.

Then the expression $\frac{1}{b_k}$ is undefined, so $(\frac{1}{b_n})$ does not define a sequence of real numbers.

We shall show that when the expression is defined, the sequence that results must converge to $\frac{1}{M}$.

Since $M \neq 0$ and $\lim_{n\to\infty} b_n = M$, then by a previous lemma, there exists $N \in \mathbb{N}$ such that $|b_n| > \frac{|M|}{2}$ for all $n > N$.

Let $c_n = b_{N+n}$ for all $n \in \mathbb{N}$.

Then (c_n) is a sequence of real numbers.

Let $g : \mathbb{N} \to \mathbb{N}$ be a function defined by $g(n) = N + n$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Since $g(n) = N + n < N + (n + 1) = g(n + 1)$, then $g(n) < g(n + 1)$, so g is strictly increasing.

Since $c_n = b_{N+n} = b_{g(n)}$, then (c_n) is a subsequence of (b_n).

Since $n \geq 1 > 0$, then $n > 0$.

Since $N + n \in \mathbb{N}$ and $N + n > N$, then $|b_{N+n}| > \frac{|M|}{2}$, so $|c_n| > \frac{|M|}{2}$.

Since $M \neq 0$, then $|M| > 0$, so $\frac{|M|}{2} > 0$.

Thus, $|c_n| > \frac{|M|}{2} > 0$, so $|c_n| > 0$.

Hence, $c_n \neq 0$, so $c_n \neq 0$ for all $n \in \mathbb{N}$.

Thus, $\frac{1}{c_n} \in \mathbb{R}$ for all $n \in \mathbb{N}$, so $(\frac{1}{c_n})$ is a sequence of real numbers.

Since $\lim_{n\to\infty} b_n = M$ and (c_n) is a subsequence of (b_n), then $\lim_{n\to\infty} c_n = M$.

Since $M \neq 0$ and $\lim_{n\to\infty} c_n = M$ and $c_n \neq 0$ for all $n \in \mathbb{N}$, then by a previous lemma, $\lim_{n\to\infty} \frac{1}{c_n} = \frac{1}{M}$.

Therefore, in all cases, $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$.

Observe that

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} (a_n \cdot \frac{1}{b_n}) \\
&= (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} \frac{1}{b_n}) \\
&= L \cdot \frac{1}{M} \\
&= \frac{L}{M}.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$, as desired. ☐

Theorem 21. ***a limit preserves a non strict inequality***

Let (a_n) and (b_n) be convergent sequences of real numbers.

If there exists $K > 0$ such that $a_n \leq b_n$ for all $n > K$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Proof. Suppose there exists $K > 0$ such that $a_n \leq b_n$ for all $n > K$.

Since (a_n) is convergent, then there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = L$.

Since (b_n) is convergent, then there exists $M \in \mathbb{R}$ such that $\lim_{n\to\infty} b_n = M$.

We must prove $L \leq M$.

Suppose for the sake of contradiction $L > M$.

Then $L - M > 0$, so $\frac{L-M}{2} > 0$

Let $\epsilon = \frac{L-M}{2}$.

Then $\epsilon > 0$.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$, then $|a_n - L| < \epsilon$.

Since $\lim_{n\to\infty} b_n = M$, then there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$, then $|b_n - M| < \epsilon$.

Let $N = \max\{N_1, N_2, K\}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N$ and $N \geq K$, then $n > K$, so $a_n \leq b_n$.

Since $n > N$ and $N \geq N_1$, then $n > N_1$, so $|a_n - L| < \epsilon$.

Thus, $-\epsilon < a_n - L < \epsilon$, so $L - \epsilon < a_n < L + \epsilon$.

Since $n > N$ and $N \geq N_2$, then $n > N_2$, so $|b_n - M| < \epsilon$.

Thus, $-\epsilon < b_n - M < \epsilon$, so $M - \epsilon < b_n < M + \epsilon$.

Since $\epsilon = \frac{L-M}{2}$, then $2\epsilon = L - M$, so $\epsilon + \epsilon = L - M$.

Thus, $M + \epsilon = L - \epsilon$.

Therefore, $b_n < M + \epsilon = L - \epsilon < a_n \leq b_n$, so $b_n < b_n$, a contradiction.

Hence, $L \leq M$, as desired. □

Corollary 22. *Let (a_n) and (b_n) be convergent sequences of real numbers.*

If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Proof. Suppose $a_n \leq b_n$ for all $n \in \mathbb{N}$.

By the density of $\mathbb{R}$, there exists $K \in \mathbb{R}$ such that $0 < K < 1$.

Thus, $0 < K$ and $K < 1$.

Since $0 < K$, then $K > 0$.

Let $n \in \mathbb{N}$ be given.

Then $a_n \leq b_n$.

Since $n \in \mathbb{N}$, then $n \geq 1$.

Since $n \geq 1$ and $1 > K$, then $n > K$.

Since n is arbitrary, then $a_n \leq b_n$ for all $n > K$.

Since (a_n) and (b_n) are convergent sequences, then by the inequality rule for convergent sequences, $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$. □

Corollary 23. *Let (a_n) be a convergent sequence in $\mathbb{R}$.*

1. *If M is an upper bound of (a_n), then $\lim_{n\to\infty} a_n \leq M$.*
2. *If m is a lower bound of (a_n), then $m \leq \lim_{n\to\infty} a_n$.*

Proof. We prove 1.

Suppose $M \in \mathbb{R}$ is an upper bound of (a_n).

Then $a_n \leq M$ for all $n \in \mathbb{N}$.

Let (b_n) be the constant sequence defined by $b_n = M$ for all $n \in \mathbb{N}$.

Since (a_n) is convergent and (b_n) is convergent and $a_n \leq b_n$ for all $n \in \mathbb{N}$, then by the inequality rule for convergent sequences, $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Thus,

$$\begin{aligned}\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n &\Leftrightarrow \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} M \\ &\Leftrightarrow \lim_{n\to\infty} a_n \leq M.\end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n \leq M$, as desired. □

Proof. We prove 1.

Since (a_n) is convergent, then there exists a real number L such that $\lim_{n\to\infty} a_n = L$.

Suppose M is an upper bound of (a_n).

We prove $L \leq M$ by contradiction.

Suppose $L > M$.

Then $L - M > 0$.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < L - M$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|a_n - L| < L - M$.

Observe that

$$\begin{aligned} |a_n - L| < L - M &\Leftrightarrow -(L - M) < a_n - L < L - M \\ &\Leftrightarrow M - L < a_n - L < L - M \\ &\Rightarrow M - L < a_n - L \\ &\Leftrightarrow M < a_n \\ &\Leftrightarrow a_n > M. \end{aligned}$$

Thus, $a_n > M$.

Hence, there exists $n \in \mathbb{N}$ such that $a_n > M$.

This contradicts the assumption that M is an upper bound of (a_n).

Therefore, $L \leq M$, as desired. □

Proof. We prove 2.

Suppose $m \in \mathbb{R}$ is a lower bound of (a_n).

Then $m \leq a_n$ for all $n \in \mathbb{N}$.

Let (b_n) be the constant sequence defined by $b_n = m$ for all $n \in \mathbb{N}$.

Since (a_n) is convergent and (b_n) is convergent and $b_n \leq a_n$ for all $n \in \mathbb{N}$, then by the inequality rule for convergent sequences, $\lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n$.

Thus,

$$\begin{aligned} \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n &\Leftrightarrow \lim_{n\to\infty} m \leq \lim_{n\to\infty} a_n \\ &\Leftrightarrow m \leq \lim_{n\to\infty} a_n. \end{aligned}$$

Therefore, $m \leq \lim_{n\to\infty} a_n$, as desired. □

Proof. We prove 2.

Since (a_n) is convergent, then there exists a real number L such that $\lim_{n\to\infty} a_n = L$.

Suppose m is a lower bound of (a_n).

We prove $m \leq L$ by contradiction.

Suppose $m > L$.

Then $m - L > 0$.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < m - L$.

Let $n \in \mathbb{N}$ such that $n > N$.
Then $|a_n - L| < m - L$.
Observe that

$$\begin{aligned} |a_n - L| < m - L &\Leftrightarrow -(m-L) < a_n - L < m - L \\ &\Rightarrow a_n - L < m - L \\ &\Leftrightarrow a_n < m. \end{aligned}$$

Thus, $a_n < m$.
Hence, there exists $n \in \mathbb{N}$ such that $a_n < m$.
This contradicts the assumption that m is a lower bound of (a_n).
Therefore, $m \leq L$, as desired. □

Corollary 24. ***limit of a convergent sequence is between any upper and lower bound of the sequence***
Let (a_n) be a convergent sequence in $\mathbb{R}$.
If there exist real numbers m and M such that $m \leq a_n \leq M$ for all $n \in \mathbb{N}$, then $m \leq \lim_{n\to\infty} a_n \leq M$.

Proof. Suppose there exist real numbers m and M such that $m \leq a_n \leq M$ for all $n \in \mathbb{N}$.
Then $m \leq a_n$ for all $n \in \mathbb{N}$ and $a_n \leq M$ for all $n \in \mathbb{N}$.
Since (a_n) is a convergent sequence, then there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = L$.
We must prove $m \leq L \leq M$.
Since $a_n \leq M$ for all $n \in \mathbb{N}$, then M is an upper bound of (a_n).
Hence, by the previous corollary, $L \leq M$.
Since $m \leq a_n$ for all $n \in \mathbb{N}$, then m is a lower bound of (a_n).
Hence, by the previous corollary, $m \leq L$.
Therefore, $m \leq L$ and $L \leq M$, so $m \leq L \leq M$, as desired. □

Corollary 25. *Let (a_n) be a convergent sequence in $\mathbb{R}$.*
If there exist $K \in \mathbb{N}$ and real numbers m and M such that $m \leq a_n \leq M$ for all $n > K$, then $m \leq \lim_{n\to\infty} a_n \leq M$.

Proof. Suppose there exist $K \in \mathbb{N}$ and real numbers m and M such that $m \leq a_n \leq M$ for all $n > K$.
Let (b_n) be a sequence defined by $b_n = a_{K+n}$ for all $n \in \mathbb{N}$.
Then (b_n) is a K tail of the sequence (a_n), so (b_n) is a subsequence of (a_n).
Since (a_n) is convergent, then (b_n) is convergent, so $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

We prove $m \leq b_n \leq M$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : m \leq b_n \leq M\}$.
Since $K + 1 \in \mathbb{N}$ and $K + 1 > K$, then $m \leq a_{K+1} \leq M$.
Since $b_1 = a_{K+1}$, then $m \leq b_1 \leq M$, so $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$.

Since $K+k+1 \in \mathbb{N}$ and $K+k+1 > K$, then $m \leq a_{K+k+1} \leq M$.
Since $k+1 \in \mathbb{N}$, then $b_{k+1} = a_{K+k+1}$, so $m \leq b_{k+1} \leq M$.
Hence, $k+1 \in S$.
Thus, by PMI, $m \leq b_n \leq M$ for all $n \in \mathbb{N}$.

Since (b_n) is convergent and $m \leq b_n \leq M$ for all $n \in \mathbb{N}$, then by the inequality rule for convergent sequences, $m \leq \lim_{n\to\infty} b_n \leq M$.

Therefore, $m \leq \lim_{n\to\infty} a_n \leq M$, as desired. □

Theorem 26. ***squeeze rule for convergent sequences***

Let $(a_n), (b_n)$, and (c_n) be sequences of real numbers.

If there exists $K \in \mathbb{N}$ such that $a_n \leq c_n \leq b_n$ for all $n > K$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, then $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Proof. Suppose there exists $K \in \mathbb{N}$ such that $a_n \leq c_n \leq b_n$ for all $n > K$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, then there exists a real number L such that $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

We must prove $\lim_{n\to\infty} c_n = L$.

Let $\epsilon > 0$ be given.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$, then $|a_n - L| < \epsilon$.

Since $\lim_{n\to\infty} b_n = L$, then there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$, then $|b_n - L| < \epsilon$.

Let $N = \max\{N_1, N_2, K\}$.
Let $n \in \mathbb{N}$ such that $n > N$.
Since $n > N$ and $N \geq K$, then $n > K$, so $a_n \leq c_n \leq b_n$.
Therefore, $a_n \leq c_n$ and $c_n \leq b_n$.
Since $n > N$ and $N \geq N_1$, then $n > N_1$, so $|a_n - L| < \epsilon$.
Since $n > N$ and $N \geq N_2$, then $n > N_2$, so $|b_n - L| < \epsilon$.
Observe that

$$\begin{aligned} |a_n - L| < \epsilon \quad &\Leftrightarrow \quad -\epsilon < a_n - L < \epsilon \\ &\Rightarrow \quad -\epsilon < a_n - L \\ &\Leftrightarrow \quad L - \epsilon < a_n. \end{aligned}$$

Since $L - \epsilon < a_n$ and $a_n \leq c_n$, then $L - \epsilon < c_n$, so $-\epsilon < c_n - L$.

Observe that

$$\begin{aligned} |b_n - L| < \epsilon \quad &\Leftrightarrow \quad -\epsilon < b_n - L < \epsilon \\ &\Rightarrow \quad b_n - L < \epsilon \\ &\Leftrightarrow \quad b_n < L + \epsilon. \end{aligned}$$

Since $c_n \leq b_n$ and $b_n < L + \epsilon$, then $c_n < L + \epsilon$, so $c_n - L < \epsilon$.

Since $-\epsilon < c_n - L$ and $c_n - L < \epsilon$, then $-\epsilon < c_n - L < \epsilon$, so $|c_n - L| < \epsilon$.

Therefore, $\lim_{n\to\infty} c_n = L$, as desired. □

Corollary 27. *Let $(a_n), (b_n)$, and (c_n) be sequences of real numbers.*

If $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, then $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Proof. Suppose $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Let $K = 1$.

Then $K \in \mathbb{N}$.

Let $n \in \mathbb{N}$ such that $n > K$.

Since $n \in \mathbb{N}$, then $a_n \leq c_n \leq b_n$.

Since n is arbitrary, then $a_n \leq c_n \leq b_n$ for all $n > K$.

Thus, there exists $K \in \mathbb{N}$ such that $a_n \leq c_n \leq b_n$ for all $n > K$.

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, then by the squeeze rule for convergent sequences, $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, as desired. □

Proposition 28. ***limit of an absolute value equals absolute value of a limit***

Let (a_n) be a convergent sequence.

Then the sequence $(|a_n|)$ is convergent and $\lim_{n\to\infty} |a_n| = |\lim_{n\to\infty} a_n|$.

Proof. Let (a_n) be a convergent sequence.

Then there exists a real number L such that $\lim_{n\to\infty} a_n = L$.

We must prove $\lim_{n\to\infty} |a_n| = |L|$.

Let $\epsilon > 0$ be given.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n > N$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|a_n - L| < \epsilon$.

Hence, $||a_n| - |L|| \leq |a_n - L| < \epsilon$, so $||a_n| - |L|| < \epsilon$.

Therefore, $\lim_{n\to\infty} |a_n| = |L|$. □

Lemma 29. *Let $a, b, c, d \in \mathbb{R}$.*

If $0 \leq a < b$ and $0 < c < d$, then $ac < bd$.

Proof. Suppose $0 \leq a < b$ and $0 < c < d$.

Then $0 \leq a$ and $a < b$ and $0 < c$ and $c < d$.

Since $a \geq 0$, then either $a > 0$ or $a = 0$.

We consider these cases separately.

Case 1: Suppose $a > 0$.

Since $0 < a$ and $a < b$, then $0 < a < b$.

Since $0 < a < b$ and $0 < c < d$, then $0 < ac < bd$.

Therefore, $ac < bd$.

Case 2: Suppose $a = 0$.

Then $ac = 0c = 0$.

Since $b > a$ and $a = 0$, then $b > 0$.

Since $d > c$ and $c > 0$, then $d > 0$.

Since $b > 0$ and $d > 0$, then $bd > 0$.

Therefore, $ac = 0 < bd$, so $ac < bd$. □

Lemma 30. ***sequence converging to a positive real number eventually has positive terms***

Let (a_n) be a sequence of real numbers.

If $\lim_{n\to\infty} a_n$ exists and is positive, then there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n > N$, then $a_n > 0$.

Proof. Suppose $\lim_{n\to\infty} a_n$ exists and is positive.

Then there exists a real number L such that $\lim_{n\to\infty} a_n = L$ and $L > 0$.

Since $\lim_{n\to\infty} a_n = L$ and $L > 0$, then there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n > N$, then $|a_n - L| < L$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|a_n - L| < L$, so $-L < a_n - L < L$.

Hence, $-L < a_n - L$, so $0 < a_n$.

Therefore, $a_n > 0$. $\square$

Proposition 31. ***limit of a square root equals square root of a limit***

Let (a_n) be a sequence of real numbers.

If $\lim_{n\to\infty} a_n$ exists and is positive, then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{\lim_{n\to\infty} a_n}$.

Proof. Suppose $\lim_{n\to\infty} a_n$ exists and is positive.

Then there is a real number L such that $\lim_{n\to\infty} a_n = L$ and $L > 0$.

To prove $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{\lim_{n\to\infty} a_n}$, we must prove $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.

Let $\epsilon > 0$ be given.

Since $L > 0$, then $\sqrt{L} > 0$, so $\epsilon\sqrt{L} > 0$.

Since $\lim_{n\to\infty} a_n = L$ and $\epsilon\sqrt{L} > 0$, then there exists $N_1 \in \mathbb{N}$ such that $|a_n - L| < \epsilon\sqrt{L}$ whenever $n > N_1$.

Since $\lim_{n\to\infty} a_n = L$ and $L > 0$, then by the previous lemma, there exists $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n > N_2$, then $a_n > 0$.

Let $N = \max\{N_1, N_2\}$.

Then either $N = N_1$ or $N = N_2$ and $N \geq N_1$ and $N \geq N_2$.

Since either $N = N_1$ or $N = N_2$ and $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$, then $N \in \mathbb{N}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N$ and $N \geq N_1$, then $n > N_1$, so $|a_n - L| < \epsilon\sqrt{L}$.

Hence, $0 \leq |a_n - L| < \epsilon\sqrt{L}$.

Since $n > N$ and $N \geq N_2$, then $n > N_2$, so $a_n > 0$.

Thus, $\sqrt{a_n} > 0$.

Since $\sqrt{a_n} > 0$ and $\sqrt{L} > 0$, then $\sqrt{a_n} + \sqrt{L} > 0$ and $\sqrt{a_n} + \sqrt{L} > \sqrt{L} > 0$.

Since $\sqrt{a_n} + \sqrt{L} > 0$, then $\sqrt{a_n} + \sqrt{L} \neq 0$.

Since $0 < \sqrt{L} < \sqrt{a_n} + \sqrt{L}$, then $0 < \frac{1}{\sqrt{a_n}+\sqrt{L}} < \frac{1}{\sqrt{L}}$.

Observe that

$$\begin{aligned}
|\sqrt{a_n} - \sqrt{L}| &= |\sqrt{a_n} - \sqrt{L} \cdot \frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}}| \\
&= |\frac{a_n - L}{\sqrt{a_n} + \sqrt{L}}| \\
&= |a_n - L \cdot \frac{1}{\sqrt{a_n} + \sqrt{L}}| \\
&= |a_n - L| \cdot |\frac{1}{\sqrt{a_n} + \sqrt{L}}| \\
&= |a_n - L| \cdot \frac{1}{|\sqrt{a_n} + \sqrt{L}|} \\
&= |a_n - L| \cdot \frac{1}{\sqrt{a_n} + \sqrt{L}} \\
&< \epsilon\sqrt{L} \cdot \frac{1}{\sqrt{L}} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|\sqrt{a_n} - \sqrt{L}| < \epsilon$, as desired. □

Divergent Sequences

Proposition 32. ***divergence to ∞ implies divergence***

A sequence that diverges to ∞ is divergent.

Proof. Let (a_n) be a sequence of real numbers.

We must prove if (a_n) diverges to ∞, then (a_n) diverges.

Suppose $a_n \to \infty$.

To prove (a_n) diverges, let $L \in \mathbb{R}$ be given.

We must prove $(\exists \epsilon > 0)(\forall n \in \mathbb{N})(\exists N' \in \mathbb{N})(N' > n \wedge |s_{N'} - L| \geq \epsilon)$.

Either $L > 0$ or $L = 0$ or $L < 0$.

We consider these cases separately.

Case 1: Suppose $L = 0$.

Let $\epsilon = 1$.

Then $\epsilon > 0$.

Since $a_n \to \infty$, then there exists $N \in \mathbb{N}$ such that $a_n > 1$ whenever $n > N$.

Let $n \in \mathbb{N}$.

We must prove there exists $N' \in \mathbb{N}$ such that $N' > n$ and $|s_{N'}| \geq 1$.

Let $N' = N + n$.

Then $N' \in \mathbb{N}$.

Since $N + n > n$, then $N' > n$.

Since $N + n > N$, then $N' > N$.

Hence, $s_{N'} > 1 > 0$.

Thus, $|s_{N'}| = s_{N'} > 1$, so $|s_{N'}| > 1$.

Therefore, $|s_{N'}| \geq 1$, as desired.
Case 2: Suppose $L > 0$.
Then $2L > 0$.
Since $a_n \to \infty$, then there exists $N \in \mathbb{N}$ such that $a_n > 2L$ whenever $n > N$.
Let $\epsilon = L$.
Then $\epsilon > 0$.
Let $n \in \mathbb{N}$.
We must prove there exists $N' \in \mathbb{N}$ such that $N' > n$ and $|s_{N'} - L| \geq L$.
Let $N' = N + n$.
Then $N' \in \mathbb{N}$.
Since $N + n > n$, then $N' > n$.
Since $N + n > N$, then $N' > N$.
Thus, $s_{N'} > 2L$.
Hence, $s_{N'} - L > L > 0$.
Thus, $|s_{N'} - L| = s_{N'} - L > L$, so $|s_{N'} - L| > L$.
Therefore, $|s_{N'} - L| \geq L$, as desired.
Case 3: Suppose $L < 0$.
Then $-L > 0$.
Since $a_n \to \infty$, then there exists $N \in \mathbb{N}$ such that $a_n > -L$ whenever $n > N$.
Let $\epsilon = -2L$.
Then $\epsilon > 0$.
Let $n \in \mathbb{N}$.
We must prove there exists $N' \in \mathbb{N}$ such that $N' > n$ and $|s_{N'} - L| \geq -2L$.
Let $N' = N + n$.
Then $N' \in \mathbb{N}$.
Since $N + n > n$, then $N' > n$.
Since $N + n > N$, then $N' > N$.
Thus, $s_{N'} > -L$.
Hence, $s_{N'} - L > -2L > 0$.
Thus, $|s_{N'} - L| = s_{N'} - L > -2L$, so $|s_{N'} - L| > -2L$.
Therefore, $|s_{N'} - L| \geq -2L$, as desired. □

Proposition 33. ***sequences that diverge to infinity are unbounded***
Let (a_n) be a sequence of real numbers.
1. If $\lim_{n\to\infty} a_n = \infty$, then (a_n) is unbounded above.
2. If $\lim_{n\to\infty} a_n = -\infty$, then (a_n) is unbounded below.

Proof. We prove 1.
Suppose $\lim_{n\to\infty} a_n = \infty$.
To prove (a_n) is unbounded above, we must prove $(\forall M)(\exists n \in \mathbb{N})(a_n > M)$.
Let $M \in \mathbb{R}$.
Either $M > 0$ or $M \leq 0$.
We consider these cases separately.
Case 1: Suppose $M > 0$.
Since $\lim_{n\to\infty} a_n = \infty$, then there exists $N \in \mathbb{N}$ such that $a_n > M$ whenever $n > N$.

Let $n \in \mathbb{N}$ such that $n > N$.
Then $a_n > M$.
Case 2: Suppose $M \leq 0$.
Since $1 > 0$ and $\lim_{n\to\infty} a_n = \infty$, then there exists $N \in \mathbb{N}$ such that $a_n > 1$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $a_n > 1$.
Since $a_n > 1 > 0 \geq M$, then $a_n > M$.
Therefore, (a_n) is unbounded above, as desired. □

Proof. We prove 2.
Suppose $\lim_{n\to\infty} a_n = -\infty$.
Then $\lim_{n\to\infty} -a_n = \infty$.
To prove (a_n) is unbounded below, we must prove $(\forall M)(\exists n \in \mathbb{N})(a_n < M)$.
Let $M \in \mathbb{R}$.
Either $M \geq 0$ or $M < 0$.
We consider these cases separately.
Case 1: Suppose $M < 0$.
Then $-M > 0$.
Since $\lim_{n\to\infty} -a_n = \infty$, then there exists $N \in \mathbb{N}$ such that $-a_n > -M$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $-a_n > -M$.
Hence, $a_n < M$.
Case 2: Suppose $M \geq 0$.
Since $1 > 0$ and $\lim_{n\to\infty} -a_n = \infty$, then there exists $N \in \mathbb{N}$ such that $-a_n > 1$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $-a_n > 1$.
Hence, $a_n < -1$.
Since $a_n < -1 < 0 \leq M$, then $a_n < M$.
Therefore, (a_n) is unbounded below, as desired. □

Monotone Convergence Theorem

Theorem 34. ***Monotone convergence theorem***
Let (a_n) be a sequence of real numbers.
1. *If (a_n) is increasing and bounded above, then $\lim_{n\to\infty} a_n = \sup(a_n)$.*
2. *If (a_n) is increasing and unbounded above, then $\lim_{n\to\infty} a_n = \infty$.*
3. *If (a_n) is decreasing and bounded below, then $\lim_{n\to\infty} a_n = \inf(a_n)$.*
4. *If (a_n) is decreasing and unbounded below, then $\lim_{n\to\infty} a_n = -\infty$.*

Proof. We prove 1.
Suppose (a_n) is increasing and bounded above.
Let $S = \{a_n : n \in \mathbb{N}\}$.
Since (a_n) is a sequence of real numbers, then $S \subset \mathbb{R}$.

Since $a_1 \in S$, then $S \neq \emptyset$.
Since (a_n) is bounded above, then S is bounded above.
Thus, S is a nonempty subset of $\mathbb{R}$ bounded above in $\mathbb{R}$, so by the completeness axiom of $\mathbb{R}$, $\sup S$ exists.
Let $\epsilon > 0$ be given.
Since $\sup S$ is the least upper bound of (a_n), then there exists $N \in \mathbb{N}$ such that $a_N > \sup S - \epsilon$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $N < n$.
Since (a_n) is increasing, then $a_N \leq a_n$.
Since $\sup S$ is an upper bound of (a_n), then $a_n \leq \sup S$.
Observe that

$$\begin{aligned}
\sup S - \epsilon < a_N \leq a_n \leq \sup S < \sup S + \epsilon \quad &\Rightarrow \quad \sup S - \epsilon < a_n < \sup S + \epsilon \\
&\Leftrightarrow \quad -\epsilon < a_n - \sup S < \epsilon \\
&\Leftrightarrow \quad |a_n - \sup S| < \epsilon.
\end{aligned}$$

Thus, $\lim_{n\to\infty} a_n = \sup S = \sup(a_n)$. □

Proof. We prove 2.
Suppose (a_n) is increasing and unbounded above.
Let $M > 0$ be given.
Since (a_n) is unbounded above, then there exists $N \in \mathbb{N}$ such that $a_N > M$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $N < n$.
Since (a_n) is increasing, then $a_N \leq a_n$.
Since $a_n \geq a_N > M$, then $a_n > M$.
Therefore, $\lim_{n\to\infty} a_n = \infty$. □

Proof. We prove 3.
Suppose (a_n) is decreasing and bounded below.
Let $S = \{a_n : n \in \mathbb{N}\}$.
Since (a_n) is a sequence of real numbers, then $S \subset \mathbb{R}$.
Since $a_1 \in S$, then $S \neq \emptyset$.
Since (a_n) is bounded below, then S is bounded below.
Thus, S is a nonempty subset of $\mathbb{R}$ bounded below in $\mathbb{R}$, so by the completeness of $\mathbb{R}$, $\inf S$ exists.
Let $\epsilon > 0$ be given.
Since $\inf S$ is the greatest lower bound of (a_n), then there exists $N \in \mathbb{N}$ such that $a_N < \inf S + \epsilon$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $N < n$.
Since (a_n) is decreasing, then $a_N \geq a_n$.
Since $\inf S$ is a lower bound of (a_n), then $\inf S \leq a_n$.

Observe that

$$\begin{aligned} \inf S - \epsilon < \inf S \le a_n \le a_N < \inf S + \epsilon \quad &\Rightarrow \quad \inf S - \epsilon < a_n < \inf S + \epsilon \\ &\Leftrightarrow \quad -\epsilon < a_n - \inf S < \epsilon \\ &\Leftrightarrow \quad |a_n - \inf S| < \epsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty} a_n = \inf S = \inf(a_n)$. $\square$

Proof. We prove 4.
Suppose (a_n) is decreasing and unbounded below.
Let $M > 0$ be given.
Since $M \in \mathbb{R}$, then $-M \in \mathbb{R}$.
Since (a_n) is unbounded below, then there exists $N \in \mathbb{N}$ such that $a_N < -M$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $N < n$.
Since (a_n) is decreasing, then $a_N \ge a_n$.
Since $a_n \le a_N < -M$, then $a_n < -M$, so $-a_n > M$.
Therefore, $\lim_{n\to\infty} -a_n = \infty$, so $\lim_{n\to\infty} a_n = -\infty$. $\square$

Lemma 35. *Let $r \in \mathbb{R}$.*
1. If $r > 0$, then $r^n > 0$ for all $n \in \mathbb{N}$.
2. If $r > 1$, then $r^n \ge (r-1)n + 1$ for all $n \in \mathbb{N}$.

Proof. We prove 1.
Suppose $r > 0$.
We prove $r^n > 0$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : r^n > 0\}$.
Since $r^1 = r > 0$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $r^k > 0$.
Since $r > 0$ and $r^k > 0$, then $r^{k+1} = r^k r > 0$.
Thus, $r^{k+1} > 0$, so $k + 1 \in S$.
Therefore, by PMI, $S = \mathbb{N}$, so $r^n > 0$ for all $n \in \mathbb{N}$. $\square$

Proof. We prove 2.
Suppose $r > 1$.
Then $r - 1 > 0$.
Let $c = r - 1$.
Then $c > 0$.
We prove $r^n \ge cn + 1$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : r^n \ge cn + 1\}$.
Since $r^1 = r = (r-1)\cdot 1 + 1 = c \cdot 1 + 1$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $r^k \ge ck + 1$.
Since $k \in \mathbb{N}$, then $k \ge 1 > 0$, so $k > 0$.
Since $c > 0$ and $k > 0$, then $ck > 0$.
Since $r^k \ge ck + 1 > 1$, then $r^k > 1$, so $r^k c > c$.

Observe that

$$\begin{aligned}
r^{k+1} &= r^k \cdot r \\
&= r^k(c+1) \\
&= r^k c + r^k \\
&\geq r^k c + ck + 1 \\
&> c + ck + 1 \\
&= c(k+1) + 1.
\end{aligned}$$

Thus, $r^{k+1} > c(k+1) + 1$, so $k+1 \in S$.
Therefore, by PMI, $S = \mathbb{N}$, so $r^n \geq cn + 1$ for all $n \in \mathbb{N}$. ☐

Proposition 36. ***convergence behavior of a geometric sequence***
Let $r \in \mathbb{R}$.
Let (r^n) be a geometric sequence.
1. If $r > 1$, then $\lim_{n\to\infty} r^n = \infty$.
2. If $r = 1$, then $\lim_{n\to\infty} r^n = 1$.
3. If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$.
4. If $r = -1$, then (r^n) is divergent (oscillates).
5. If $r < -1$, then (r^n) is divergent.

Proof. We prove 1.
Suppose $r > 1$.

We prove (r^n) is strictly increasing.
Let $n \in \mathbb{N}$ be given.
Since $r > 1 > 0$, then $r > 0$.
Since $n \in \mathbb{N}$ and $r > 0$, then by a previous lemma, $r^n > 0$.
Since $r^n > 0$ and $r > 1$, then $r^{n+1} = r^n r > r^n \cdot 1 = r^n$, so $r^{n+1} > r^n$.
Thus, $r^n < r^{n+1}$, so (r^n) is strictly increasing.

Let $M > 0$ be given.
Let $c = r - 1$.
Since $r > 1$, then $r - 1 > 0$, so $c > 0$.
Since $c \neq 0$, then $\frac{M-1}{c} \in \mathbb{R}$.
By the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{M-1}{c}$.
Thus, $cN + 1 > M$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $N < n$.
Since (r^n) is strictly increasing, then $r^N < r^n$.
Since $r > 1$ and $N \in \mathbb{N}$, then by a previous lemma, $r^N \geq cN + 1$.
Hence, $M < cN + 1 \leq r^N < r^n$, so $M < r^n$.
Therefore, $r^n > M$, so $\lim_{n\to\infty} r^n = \infty$, as desired. ☐

Proof. We prove 2.

Suppose $r = 1$.

Then $1 = \lim_{n\to\infty} 1 = \lim_{n\to\infty} 1^n = \lim_{n\to\infty} r^n$, so $\lim_{n\to\infty} r^n = 1$. ☐

Proof. We prove 3.

Suppose $|r| < 1$.

Since $|r| \geq 0$, then either $|r| > 0$ or $|r| = 0$.

We consider these cases separately.

Case 1: Suppose $|r| = 0$.

Then $r = 0$.

Since $n \in \mathbb{N}$, then $0 = \lim_{n\to\infty} 0 = \lim_{n\to\infty} 0^n = \lim_{n\to\infty} r^n$.

Therefore, $\lim_{n\to\infty} r^n = 0$.

Case 2: Suppose $|r| > 0$.

Since $0 < |r|$ and $|r| < 1$, then $0 < |r| < 1$, so $\frac{1}{|r|} > 1 > 0$.

Thus, $\frac{1}{|r|} - 1 > 0$.

Let $c = \frac{1}{|r|} - 1$.

Then $c > 0$, so $c \neq 0$.

Let $\epsilon > 0$ be given.

Then $\epsilon \neq 0$, so $\frac{1}{\epsilon} \in \mathbb{R}$ and $\frac{1}{\epsilon} > 0$.

Since $c \neq 0$, then $\frac{\frac{1}{\epsilon}-1}{c} \in \mathbb{R}$, so by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{\frac{1}{\epsilon}-1}{c}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N > \frac{\frac{1}{\epsilon}-1}{c}$, then $n > \frac{\frac{1}{\epsilon}-1}{c}$, so $cn > \frac{1}{\epsilon} - 1$.

Thus, $cn + 1 > \frac{1}{\epsilon} > 0$, so $\epsilon > \frac{1}{cn+1}$.

Since $|r| > 0$ and $n \in \mathbb{N}$, then by a previous lemma, $|r|^n > 0$.

Since $\frac{1}{|r|} > 1$ and $n \in \mathbb{N}$, then by a previous lemma, $(\frac{1}{|r|})^n \geq cn + 1$.

Since $cn + 1 > \frac{1}{\epsilon} > 0$, then $cn + 1 > 0$.

Observe that

$$\begin{aligned}(\frac{1}{|r|})^n \geq cn + 1 \quad &\Leftrightarrow \quad \frac{1}{|r|^n} \geq cn + 1 \\ &\Leftrightarrow \quad \frac{1}{cn+1} \geq |r|^n.\end{aligned}$$

Thus, $\frac{1}{cn+1} \geq |r|^n$.

Since $|r| = 0$ iff $r = 0$, then $|r| \neq 0$ iff $r \neq 0$.

Since $|r| \neq 0$, then $r \neq 0$.

Thus, $|r^n| = |r|^n \leq \frac{1}{cn+1} < \epsilon$, so $|r^n| < \epsilon$.

Therefore, $\lim_{n\to\infty} r^n = 0$. ☐

Proof. We prove 4.

Suppose $r = -1$.

Then $r^n = (-1)^n$.

The sequence given by $r^n = (-1)^n$ for all $n \in \mathbb{N}$ was previously proven in the examples to diverge. ☐

Proof. We prove 5.

Let $r < -1$.

Suppose (r^n) is bounded.

Then there exists $M > 0$ such that $|r^n| < M$ for all $n \in \mathbb{N}$.

Since $r < -1$ and $-1 < 0$, then $-r > 1$ and $r < 0$, so $|r| = -r > 1$.

Hence, $|r| > 1$, so $|r| - 1 > 0$.

Thus, $|r| - 1 \neq 0$, so $\frac{M-1}{|r|-1} \in \mathbb{R}$.

By the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{M-1}{|r|-1}$.

Hence, $(|r| - 1)N > M - 1$, so $(|r| - 1)N + 1 > M$.

Since $|r| > 1$ and $N \in \mathbb{N}$, then by a previous lemma, $|r|^N \geq (|r| - 1)N + 1$.

Since $r < 0$, then $r \neq 0$.

Thus,

$$\begin{aligned} |r^N| &= |r|^N \\ &\geq (|r| - 1)N + 1 \\ &> M. \end{aligned}$$

Hence, there exists $N \in \mathbb{N}$ such that $|r^N| > M$.

This contradicts the assumption that (r^n) is bounded.

Therefore, (r^n) is unbounded.

Since every unbounded sequence is divergent, then (r^n) is divergent, as desired. $\square$

Bolzano-Weierstrass theorem

Theorem 37. ***Nested intervals theorem***

Let (I_n) be a sequence of nonempty closed, bounded intervals in $\mathbb{R}$ such that $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$. Then there exists $\alpha \in \mathbb{R}$ such that $\alpha \in I_n$ for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$.

Since I_n is a closed and bounded interval, then there exist $a_n, b_n \in \mathbb{R}$ such that $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$.

Since I_n is not empty, then there exists $x \in I_n$.

Hence, $x \in \mathbb{R}$ and $a_n \leq x \leq b_n$.

Thus, $a_n \leq b_n$.

Since n is arbitrary, then $a_n \leq b_n$ for all $n \in \mathbb{N}$.

Since $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, then (a_n) is a sequence of real numbers.

Since $b_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, then (b_n) is a sequence of real numbers.

Let $n \in \mathbb{N}$.

Then $I_{n+1} \subset I_n$.

Since $I_{n+1} = [a_{n+1}, b_{n+1}]$, then $a_{n+1} \in I_{n+1}$ and $b_{n+1} \in I_{n+1}$.

Since $a_{n+1} \in I_{n+1}$ and $I_{n+1} \subset I_n$, then $a_{n+1} \in I_n$, so $a_n \leq a_{n+1} \leq b_n$.

Since $b_{n+1} \in I_{n+1}$ and $I_{n+1} \subset I_n$, then $b_{n+1} \in I_n$, so $a_n \leq b_{n+1} \leq b_n$.

Since $a_n \leq a_{n+1} \leq b_n$, then $a_n \leq a_{n+1}$.

Since $a_n \leq b_{n+1} \leq b_n$, then $b_{n+1} \leq b_n$.

Since $a_n \leq a_{n+1}$ and n is arbitrary, then $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, so (a_n) is increasing.

Since $b_n \geq b_{n+1}$ and n is arbitrary, then $b_n \geq b_{n+1}$ for all $n \in \mathbb{N}$, so (b_n) is decreasing. ☐

Proof. We prove $a_m \leq b_n$ for all $m \in \mathbb{N}$ and all $n \in \mathbb{N}$.

Let $m, n \in \mathbb{N}$ be given.

Either $m < n$ or $m = n$ or $m > n$.

We consider these cases separately.

Case 1: Suppose $m = n$.

Since $a_n \leq b_n$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$, then $a_m \leq b_m = b_n$.

Case 2: Suppose $m < n$.

Since (a_n) is increasing, then $a_m \leq a_n$.

Since $a_n \leq b_n$ for all $n \in \mathbb{N}$ and $n \in \mathbb{N}$, then $a_n \leq b_n$.

Since $a_m \leq a_n$ and $a_n \leq b_n$, then $a_m \leq b_n$.

Case 3: Suppose $m > n$.

Since (b_n) is decreasing and $n < m$, then $b_n \geq b_m$.

Since $a_n \leq b_n$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$, then $a_m \leq b_m$.

Since $a_m \leq b_m$ and $b_m \leq b_n$, then $a_m \leq b_n$.

Therefore, in all cases, $a_m \leq b_n$, as desired. ☐

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$.

Since $a_1 \in A$, then $A \neq \emptyset$.

Since $a_m \leq b_n$ for all $m \in \mathbb{N}$ and all $n \in \mathbb{N}$ and $1 \in \mathbb{N}$, then $a_m \leq b_1$ for all $m \in \mathbb{N}$.

Hence, b_1 is an upper bound of A, so A is bounded above in $\mathbb{R}$

Since $A \neq \emptyset$ and is bounded above in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\sup A$ exists.

Let $B = \{b_n : n \in \mathbb{N}\}$.

Since $b_1 \in B$, then $B \neq \emptyset$.

Since $a_m \leq b_n$ for all $m \in \mathbb{N}$ and all $n \in \mathbb{N}$ and $1 \in \mathbb{N}$, then $a_1 \leq b_n$ for all $n \in \mathbb{N}$.

Hence, a_1 is a lower bound of B, so B is bounded below in $\mathbb{R}$

Since $B \neq \emptyset$ and is bounded below in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\inf B$ exists. ☐

Proof. We prove $\sup A \leq \inf B$.

Let $b \in B$.

Then $b = b_n$ for some $n \in \mathbb{N}$.

Since $a_m \leq b_n$ for all $m, n \in \mathbb{N}$ and $n \in \mathbb{N}$, then $a_m \leq b_n$ for all $m \in \mathbb{N}$.

Hence, b_n is an upper bound of A.

Since $\sup A$ is the least upper bound of A, then $\sup A \leq b_n$.

Since b is arbitrary, then $\sup A \leq b_n$ for all $n \in \mathbb{N}$.

Thus, $\sup A$ is a lower bound of B.

Since $\inf B$ is the greatest lower bound of B, then $\sup A \leq \inf B$. ☐

Proof. Since $\sup A$ is an upper bound of A, then $a_n \leq \sup A$ for all $n \in \mathbb{N}$.

Since $\sup A \leq \inf B$, then $\sup A$ is a lower bound of B, so $\sup A \leq b_n$ for all $n \in \mathbb{N}$.

Since $a_n \leq \sup A$ for all $n \in \mathbb{N}$ and $\sup A \leq b_n$ for all $n \in \mathbb{N}$, then $a_n \leq \sup A \leq b_n$ for all $n \in \mathbb{N}$.

Let $\alpha = \sup A$.

Then $\alpha \in \mathbb{R}$ and $a_n \leq \alpha \leq b_n$ for all $n \in \mathbb{N}$, so $\alpha \in [a_n, b_n]$ for all $n \in \mathbb{N}$.

Hence, $\alpha \in I_n$ for all $n \in \mathbb{N}$.

Therefore, there exists $\alpha \in \mathbb{R}$ such that $\alpha \in I_n$ for all $n \in \mathbb{N}$, as desired. □

Theorem 38. ***Bolzano-Weierstrass theorem***

Every bounded sequence in $\mathbb{R}$ *has a convergent subsequence.*

Proof. Let (x_n) be a bounded sequence of real numbers.

Then there exists $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$, so there exists $M > 0$ such that $-M \leq x_n \leq M$ for all $n \in \mathbb{N}$.

Let $I_1 = [-M, M]$.

... Since (I_n) is a sequence of nonempty closed bounded intervals such that $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$, then by the Nested Intervals theorem, there exists $\alpha \in \mathbb{R}$ such that $\alpha \in I_n$ for all $n \in \mathbb{N}$.

We then show there is a sequence (y_n) such that $y_n \in I_n$ for each $n \in \mathbb{N}$.

We then show that (y_n) is a subsequence of (x_n).

We should show that (y_n) is increasing sequence and is bounded above.

We then show that $\lim_{n\to\infty} y_n = \alpha$. □

Proof. Let (x_n) be a bounded sequence of real numbers.

Then there exists a real number $M > 0$ such that $-M < x_n < M$ for all $n \in \mathbb{N}$.

Let $I_1 = [-M, M]$.

Since (x_n) has infinitely many terms, then A_1 contains infinitely many terms of (x_n).

Hence, I_1 is not empty.

Since $M > 0$, then $-M < 0$, so M and $-M$ are distinct real numbers.

Thus, there exists a unique midpoint of the interval I_1.

Hence, there exist exactly two subintervals of I_1 of equal length.

Let $B_1 = [-M, 0]$ and $C_1 = [0, M]$ be these two subintervals of I_1.

Then $I_1 = B_1 \cup C_1$.

Suppose B_1 and C_1 contain finitely many terms of (x_n).

Then the number of terms in A_1 is $|A_1| = |B_1| + |C_1|$, a finite number.

But, this contradicts the fact that A_1 contains infinitely many terms of (x_n).

Hence, either B_1 contains infinitely many terms of (x_n) or C_1 contains infinitely many terms of (x_n).

Thus, at least one of these closed, bounded subintervals of I_1 contains infinitely many terms of (x_n).

Let I_2 be one of these closed, bounded subintervals of I_1 that contains infinitely many terms of (x_n).

Since I_2 contains infinitely many terms of (x_n), then I_2 is not empty.

Since I_2 is a subinterval of I_1, then $I_2 \subset I_1$.

Since $0 < M$, then there is a unique midpoint of the interval I_2.

Thus, there exist exactly two subintervals of I_2 of equal length.

Let B_2 and C_2 be these two closed, bounded subintervals of I_2 of equal length.

Again, at least one of these two subintervals of I_2 contains infinitely many terms of (x_n).

Let I_3 be one of these subintervals of I_2 that contains infinitely many terms of (x_n).

Since I_3 contains infinitely many terms of (x_n), then I_3 is not empty.

Since I_3 is a subinterval of I_2, then $I_3 \subset I_2$.

We repeat this process.

Since we can continue to always choose a closed, bounded subinterval of a given interval I_k that always contains infinitely many terms of (x_n), then this process never ends.

Therefore, we have a sequence of nested nonempty, closed, bounded intervals such that $I_1 \supset I_2 \supset I_3 \supset ....$

Hence, by the Nested intervals theorem, there exists a real number α such that $\alpha \in I_n$ for all $n \in \mathbb{N}$.

Since $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$, then if $m < n$, then $I_m \supset I_n$. Prove this! $\square$

Proof. We must prove there exists a convergent subsequence (y_n) of (x_n).

Define function $g : \mathbb{N} \to \mathbb{N}$ by $g(n) = k_n$ such that $g(n) < g(n+1)$ for all $n \in \mathbb{N}$ with $g(1) = 1$.

Since $g(n) < g(n+1)$ for all $n \in \mathbb{N}$, then g is strictly increasing.

Let (y_n) be a subsequence of (x_n) such that $y_n = x_{g(n)} = x_{k_n}$ for all $n \in \mathbb{N}$.

Then $y_1 = x_1 = x_{k_1}$.

We need to rigorously show that there exists $k_2 \in \mathbb{N}$ such that $x_{k_2} \in I_2$ and $k_2 > 1$.

Similarly, we need to show that there exists $k_3 \in \mathbb{N}$ such that $x_{k_3} \in I_3$ and $k_3 > k_2$.

etc.

In general, we have to show for each $n > 1$ there exists $k_n \in \mathbb{N}$ such that $x_{k_n} \in I_n$ and $k_n > k_{n-1}$. We should try to prove by induction.

Let $S = \{n \in \mathbb{N} : (\exists k_n \in \mathbb{N})(x_{k_n} \in I_n)(k_n > k_{n-1}\}$ for $n > 1$ and $k_1 = 1$.

Suppose $m \in S$.

Then $m \in \mathbb{N}$ and there exists $k_m \in \mathbb{N}$ such that $a_{k_m} \in A_m$ and $k_m > k_{m-1}$ and $m > 1$.

To prove $m + 1 \in S$, prove there exists $k_{m+1} \in \mathbb{N}$ such that $a_{k_{m+1}} \in A_{m+1}$ and $k_{m+1} > k_m$ and $m + 1 > 1$.

Since A_{m+1} contains infinitely many terms of (a_n), then in particular, A_{m+1} contains at least $m + 1$ elements.

Thus, there exist natural numbers $r_1, r_2, ..., r_{m+1}$ such that $a_{r_1}, a_{r_2}, a_{r_3}, ..., a_{r_{m+1}} \in A_{m+1}$.

It is because each A_n contains infinitely many terms.

So, for example for A_2.

We should show that there exists $k_2 \in \mathbb{N}$ such that $a_{k_2} \in A_2$ and $k_2 > k_1 = 1$.

Suppose there does not exist $k_2 \in \mathbb{N}$ such that $a_{k_2} \in A_2$ and $k_2 > k_1 = 1$.

This is equivalent to supposing that there does not exist $m \in \mathbb{N}$ such that $a_m \in A_2$ and $m > 1$.

Since A_2 contains infinitely many terms of (a_n), then in particular, A_2 contains at least 2 elements.

Call these a_r and a_s, so $a_r \in A_2$ and $a_s \in A_2$ and $r, s \in \mathbb{N}$.

We'd like to show that either r or s must be greater than 1.

So, assume $r \leq 1$.

We must prove $s > 1$.

Since $r \in \mathbb{N}$, then $r \geq 1$.

Since $r \geq 1$ and $r \leq 1$, then $r = 1$.

Since r and s are distinct natural numbers, then $s \neq r$.

Thus, $s \neq 1$.

Since $s \in \mathbb{N}$, then $s \geq 1$.

Hence, $s > 1$, as desired.

What if we want to show there exists $k \in \mathbb{N}$ such that $a_k \in A_2$ and $k > 2$? $\square$

Proof. We now prove the subsequence (y_n) converges to α.

The length of the interval I_n is $2^{2-n}M$. Prove this!

Consider the sequence defined by $2^{2-n}M$ for all $n \in \mathbb{N}$.

This sequence is $4M$ times the geometric sequence $(\frac{1}{2})^n$ which converges to zero.

Thus, $2^{2-n}M$ converges to zero. (I.e. the lengths of the intervals eventually get smaller and closer to zero).

We must prove that the sequence $(2^{2-n}M)$ is decreasing and converges to 0, using any method we wish, such as by proving the sequence is 4 times the geometric sequence $(\frac{1}{2})^n$ which converges to zero.

So, this means the sequence $(2^{2-n}M)$ converges to $4 * 0 = 0$.

Let $\epsilon > 0$ be given.

Since the sequence $(2^{2-n}M)$ is decreasing and converges to 0, then 0 is the greatest lower bound of $(2^{2-n}M)$.

Hence, $\epsilon > 0$ is not a lower bound of $(2^{2-n}M)$, so there exists $N \in \mathbb{N}$ such that $2^{2-N}M < \epsilon$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $N < n$.

Thus, $I_N \supset I_n$.

Since $a_{k_n} \in I_n$ and $I_n \subset I_N$, then $a_{k_n} \in I_N$.

Since $a_{k_n} = y_n$, then $y_n \in I_N$. $\square$

Proof. Let $n \in \mathbb{N}$.

We must show that the length of the n^{th} subinterval I_n is $2^{2-n}M$.

We must show that $y_n \in I_n$.

Since $\alpha \in I_n$ for all $n \in \mathbb{N}$ and $n \in \mathbb{N}$, then $\alpha \in I_n$.

Since $y_n \in I_n$ and $\alpha \in I_n$ and I_n is a nonempty closed bounded interval with length $2^{2-n}M$, then $|y_n - \alpha| \leq 2^{2-n}M$.

Therefore, $|y_n - \alpha| \leq 2^{2-n}M$ for all $n \in \mathbb{N}$. □

Proof. We prove $\lim_{n\to\infty} y_n = \alpha$.

Let $\epsilon > 0$ be given.

By the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > 2 + \frac{M}{\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N$ and $N > 2 + \frac{M}{\epsilon}$, then $n > 2 + \frac{M}{\epsilon}$, so $n - 2 > \frac{M}{\epsilon}$.

Since $n - 2 > 0$, then $\epsilon > \frac{M}{n-2}$.

Since $2^n > n > 0$ for all $n \in \mathbb{N}$, then $\frac{1}{2^n} < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Since $n - 2 \in \mathbb{N}$, then this implies $\frac{1}{2^{n-2}} < \frac{1}{n-2}$.

Since $M > 0$, then $\frac{M}{2^{n-2}} < \frac{M}{n-2}$.

Since $\frac{M}{2^{n-2}} < \frac{M}{n-2}$ and $\frac{M}{n-2} < \epsilon$, then $\frac{M}{2^{n-2}} < \epsilon$.

Since $n - 2 > 0$, then $2^{2-n}M < \epsilon$.

Since $|y_n - \alpha| \leq 2^{2-n}M$ for all $n \in \mathbb{N}$ and $n \in \mathbb{N}$, then $|y_n - \alpha| \leq 2^{2-n}M$.

Thus, $|y_n - \alpha| \leq 2^{2-n}M < \epsilon$, so $|y_n - \alpha| < \epsilon$.

Therefore, $\lim_{n\to\infty} y_n = \alpha$. □

Cauchy sequences

Lemma 39. *Every convergent sequence in $\mathbb{R}$ is a Cauchy sequence.*

Proof. Let (a_n) be a convergent sequence of real numbers.

To prove (a_n) is a Cauchy sequence, let $\epsilon > 0$ be given.

Then $\frac{\epsilon}{2} > 0$.

Since (a_n) is convergent, then there exists a real number L and there exists $N \in \mathbb{N}$ such that $|a_n - L| < \frac{\epsilon}{2}$ whenever $n > N$.

Let $m, n > N$.

Since $m > N$, then $|a_m - L| < \frac{\epsilon}{2}$.

Since $n > N$, then $|a_n - L| < \frac{\epsilon}{2}$.

Observe that

$$\begin{aligned} |a_m - a_n| &= |(a_m - L) + (L - a_n)| \\ &\leq |a_m - L| + |L - a_n| \\ &= |a_m - L| + |a_n - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $|a_m - a_n| < \epsilon$, as desired. □

Lemma 40. *Every Cauchy sequence in $\mathbb{R}$ is bounded.*

Proof. Let (a_n) be a Cauchy sequence of real numbers.

Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, if $m, n > N$, then $|a_m - a_n| < \epsilon$.

Let $\epsilon = 1$.

Then there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, if $m, n > N$, then $|a_m - a_n| < 1$.

Let $S = \{|a_1|, |a_2|, ..., |a_N|, 1 + |a_{N+1}|\} = \{|a_k| : 1 \leq k \leq N\} \cup \{1 + |a_{N+1}|\}$.

Then $S \subset \mathbb{R}$.

Since $1 + |a_{N+1}| \in S$, then S is not empty.

Since S contains at most $N + 1$ elements, then S is finite.

Hence, S is a nonempty finite set of real numbers.

Therefore, $\max S$ exists.

To prove (a_n) is bounded, we must prove there exists $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Let $M = \max S$.

Since $M = \max S \in S$ and $S \subset \mathbb{R}$, then $M \in \mathbb{R}$.

Let $n \in \mathbb{N}$.

Either $n \leq N$ or $n > N$.

We consider these cases separately.

Case 1: Suppose $n \leq N$.

Then $1 \leq n \leq N$, so $|a_n| \in S$.

Therefore, $|a_n| \leq M$.

Case 2: Suppose $n > N$.

Since $n > N$ and $N + 1 > N$, then $|a_n - a_{N+1}| < 1$.

Since $1 + |a_{N+1}| \in S$ and $M = \max S$, then $1 + |a_{N+1}| \leq M$.

Observe that

$$\begin{aligned} |a_n| &= |(a_n - a_{N+1}) + a_{N+1}| \\ &\leq |a_n - a_{N+1}| + |a_{N+1}| \\ &< 1 + |a_{N+1}| \\ &\leq M. \end{aligned}$$

Therefore, $|a_n| < M$, so $|a_n| \leq M$.

Thus, in all cases, $|a_n| \leq M$, so (a_n) is bounded, as desired. □

Theorem 41. ***Cauchy convergence criterion for sequences***

A sequence in $\mathbb{R}$ is convergent iff it is a Cauchy sequence.

Proof. Let (a_n) be a sequence of real numbers.

Suppose (a_n) is convergent.

Then, by a previous lemma, (a_n) is a Cauchy sequence.

Conversely, suppose (a_n) is a Cauchy sequence.

Then, by a previous lemma, (a_n) is bounded.

Thus, by the Bolzano-Weierstrass theorem, (a_n) has a convergent subsequence.

Let (b_n) be a convergent subsequence of (a_n).

Since (b_n) is convergent, then there exists a real number L such that $\lim_{n\to\infty} b_n = L$.

We prove $\lim_{n\to\infty} a_n = L$.
Let $\epsilon > 0$ be given.
Then $\frac{\epsilon}{2} > 0$.
Since $\lim_{n\to\infty} b_n = L$, then there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$, then $|b_n - L| < \frac{\epsilon}{2}$.
Since (a_n) is Cauchy, then there exists $N_2 \in \mathbb{N}$ such that if $m, n > N_2$, then $|a_m - a_n| < \frac{\epsilon}{2}$.
Let $N = \max\{N_1, N_2\}$.
Let $n \in \mathbb{N}$ such that $n > N$.
Since $n > N \geq N_1$, then $n > N_1$, so $|b_n - L| < \frac{\epsilon}{2}$.
Since $n > N \geq N_2$, then $n > N_2$.
Since (b_n) is a subsequence of (a_n), then there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{g(n)}$ for all $n \in \mathbb{N}$.
Since $N_2 < n$ and g is strictly increasing, then $g(N_2) < g(n)$ and $g(N_2) \geq N_2$.
Thus, $g(n) > g(N_2) \geq N_2$, so $g(n) > N_2$.
Since $n > N_2$ and $g(n) > N_2$, then $|a_n - a_{g(n)}| < \frac{\epsilon}{2}$, so $|a_n - b_n| < \frac{\epsilon}{2}$.
Hence,

$$\begin{aligned}
|a_n - L| &= |(a_n - b_n) + (b_n - L)| \\
&\leq |a_n - b_n| + |b_n - L| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus, $|a_n - L| < \epsilon$, so $\lim_{n\to\infty} a_n = L$.
Therefore, (a_n) is convergent. $\square$