Sequences in $\mathbb{R}$ Examples

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Sequences in $\mathbb{R}$

Sequences as Functions

Example 1. The sequence given by $a_n = n^2$ is strictly increasing.

Proof. Let $n \in \mathbb{N}$.

Since $a_n = n^2 < n^2 + 2n + 1 = (n+1)^2 = a_{n+1}$, then (a_n) is strictly increasing. □

Example 2. The sequence given by $a_n = 3 + \frac{(-1)^n}{n}$ is neither increasing nor decreasing.

Proof. Since $2 \in \mathbb{N}$ and $a_2 = \frac{7}{2} > \frac{8}{3} = a_3$, then (a_n) is not increasing.

Since $1 \in \mathbb{N}$ and $a_1 = 2 < \frac{7}{2} = a_2$, then (a_n) is not decreasing.

Therefore, (a_n) is neither increasing nor decreasing. □

Example 3. constant sequence is increasing and decreasing.

Let k be a real number.

The sequence given by $a_n = k$ is monotonic increasing and decreasing, but is neither strictly increasing nor strictly decreasing.

Proof. Since $1 \in \mathbb{N}$ and $a_1 = k \geq k = a_2$, then (a_n) is not strictly increasing.

Since $1 \in \mathbb{N}$ and $a_1 = k \leq k = a_2$, then (a_n) is not strictly decreasing.

Let $n \in \mathbb{N}$.

Since $a_n = k \leq k = a_{n+1}$, then (a_n) is monotonic increasing.

Since $a_n = k \geq k = a_{n+1}$, then (a_n) is monotonic decreasing. □

Example 4. every strictly increasing sequence is (monotonic) increasing.

Let (a_n) be a strictly increasing sequence of real numbers.

Then (a_n) is monotonic increasing.

Proof. Let $n \in \mathbb{N}$.

Since (a_n) is strictly increasing, then $a_n < a_{n+1}$.

Thus, either $a_n < a_{n+1}$ or $a_n = a_{n+1}$, so $a_n \leq a_{n+1}$.

Hence, $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

Therefore, (a_n) is monotonic increasing. □

Example 5. every strictly decreasing sequence is (monotonic) decreasing.

Let (a_n) be a strictly decreasing sequence of real numbers.
Then (a_n) is monotonic decreasing.

Proof. Let $n \in \mathbb{N}$.
Since (a_n) is strictly decreasing, then $a_n > a_{n+1}$.
Thus, either $a_n > a_{n+1}$ or $a_n = a_{n+1}$, so $a_n \geq a_{n+1}$.
Hence, $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.
Therefore, (a_n) is monotonic decreasing. □

Example 6. The sequence defined by $a_n = n^2$ for all $n \in \mathbb{N}$ is bounded below by 0, but is not bounded above.

Proof. Since $a_n = n^2 > 0$ for all $n \in \mathbb{N}$, then 0 is a lower bound of (a_n), so (a_n) is bounded below by 0.
To prove (a_n) is unbounded above, let $M \in \mathbb{R}$ be given.
We must prove there exists $n \in \mathbb{N}$ such that $a_n > M$.
Either $M \leq 0$ or $M > 0$.
We consider these cases separately.
Case 1: Suppose $M \leq 0$.
Since $1 \in \mathbb{N}$ and $M \leq 0 < 1 = a_1$, then $a_1 > M$.
Case 2: Suppose $M > 0$.
Then $\sqrt{M} > 0$.
By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > \sqrt{M}$.
Since $n > \sqrt{M} > 0$, then $a_n = n^2 > M$.
Therefore, (a_n) is unbounded above, so (a_n) is not bounded above. □

Example 7. The sequence defined by $a_n = -n$ for all $n \in \mathbb{N}$ is bounded above by 0, but is not bounded below.

Proof. Let $n \in \mathbb{N}$.
Then $a_n = -n < 0$, so $a_n < 0$.
Hence, $a_n \leq 0$, so 0 is an upper bound of (a_n).
Therefore, (a_n) is bounded above by 0.
Let $m \in \mathbb{R}$ be given.
Then $-m \in \mathbb{R}$.
By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > -m$.
Hence, $a_n = -n < m$.
Thus, there exists $n \in \mathbb{N}$ such that $a_n < m$.
Therefore, (a_n) is unbounded below. □

Example 8. The sequence defined by $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ is bounded above by 1 and is bounded below by 0.

Proof. Let $n \in \mathbb{N}$.
Then $n \geq 1$, so $1 \geq \frac{1}{n} = a_n$.
Hence, $a_n \leq 1$, so 1 is an upper bound of (a_n).

Therefore, (a_n) is bounded above by 1.
Let $n \in \mathbb{N}$.
Then $n \geq 1 > 0$, so $n > 0$.
Hence, $a_n = \frac{1}{n} > 0$, so $a_n > 0$.
Thus, $0 < a_n$, so $0 \leq a_n$.
Therefore, 0 is a lower bound of (a_n), so (a_n) is bounded below by 0.
Since (a_n) is bounded above and below, then (a_n) is bounded. □

Example 9. The sequence defined by $a_n = \cos(n)$ for all $n \in \mathbb{N}$ is bounded above by 1 and is bounded below by -1.

Proof. Let $n \in \mathbb{N}$.
Since $\mathbb{N} \subset \mathbb{R}$, then $n \in \mathbb{R}$.
Since $|\cos(x)| \leq 1$ for all $x \in \mathbb{R}$, then in particular, $|a_n| = |\cos(n)| \leq 1$.
Hence, $|a_n| \leq 1$, so $-1 \leq a_n \leq 1$.
Thus, $-1 \leq a_n \leq 1$ for all $n \in \mathbb{N}$, so 1 is an upper bound and -1 is a lower bound of (a_n).
Therefore, (a_n) is bounded above by 1 and below by -1.
Since (a_n) is bounded above and below, then (a_n) is bounded. □

Example 10. The sequence defined by $b_n = (2n+1)^2$ is a subsequence of the sequence defined by $a_n = n^2$ for all $n \in \mathbb{N}$.

Proof. Since (a_n) is a sequence of real numbers, then there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = a_n = n^2$ for all $n \in \mathbb{N}$.
Let $g : \mathbb{N} \to \mathbb{N}$ be a function defined by $g(n) = 2n+1$ for all $n \in \mathbb{N}$.
To prove (b_n) is a subsequence of (a_n), we must prove g is strictly increasing and $b_n = (f \circ g)(n)$ for all $n \in \mathbb{N}$.
Let $m, n \in \mathbb{N}$ such that $m < n$.
Since $m < n \Rightarrow 2m < 2n \Rightarrow g(m) = 2m+1 < 2n+1 = g(n)$, then $g(m) < g(n)$.
Hence, g is strictly increasing.
Let $n \in \mathbb{N}$.
Then

$$\begin{aligned} (f \circ g)(n) &= f(g(n)) \\ &= f(2n+1) \\ &= (2n+1)^2 \\ &= b_n. \end{aligned}$$

Therefore, $b_n = (f \circ g)(n)$ for all $n \in \mathbb{N}$, as desired. □

Example 11. a sequence is a subsequence of itself
If (a_n) is a sequence, then (a_n) is a subsequence of (a_n).

Proof. Let (a_n) be a sequence of real numbers.

Then there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = a_n$ for all $n \in \mathbb{N}$.

Let $g : \mathbb{N} \to \mathbb{N}$ be a function defined by $g(n) = n$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Since $g(n) = n < n + 1 = g(n + 1)$, then g is strictly increasing.

Observe that

$$\begin{aligned} (f \circ g)(n) &= f(g(n)) \\ &= f(n) \\ &= a_n. \end{aligned}$$

Therefore, (a_n) is a subsequence of (a_n). □

Convergent Sequences in $\mathbb{R}$

Example 12. The sequence $(\frac{1}{n})$ converges to zero in $\mathbb{R}$.

Therefore $\lim_{n\to\infty} \frac{1}{n} = 0$.

Equivalently, $\frac{1}{n} \to 0$.

Proof. Let $\epsilon > 0$ be given.

We must prove there exists a natural number N such that for all natural numbers n, if $n > N$, then $|\frac{1}{n}| < \epsilon$.

Since $\epsilon > 0$, then $\frac{1}{\epsilon} \in \mathbb{R}$.

By the archimedean property of $\mathbb{R}$ there exists a natural number N such that $N > \frac{1}{\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N$ and $N > \frac{1}{\epsilon}$, then $n > \frac{1}{\epsilon}$.

Since $n \in \mathbb{N}$, then $n \geq 1 > 0$, so $n > 0$.

Since $n > \frac{1}{\epsilon}$ and $\epsilon > 0$ and $n > 0$, then $\epsilon > \frac{1}{n} > 0$.

Therefore, $|\frac{1}{n}| = \frac{1}{n} < \epsilon$, as desired. □

Example 13. limit of a constant sequence

For all $k \in \mathbb{R}$, $\lim_{n\to\infty} k = k$. (limit of a constant k is k)

Proof. Let $k \in \mathbb{R}$.

Let $\epsilon > 0$ be given.

To prove prove $\lim_{n\to\infty} k = k$, we must prove there exists a natural number N such that for all natural numbers n, if $n > N$, then $|k - k| < \epsilon$.

Let N be the natural number 1.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $|k - k| = |0| = 0 < \epsilon$, then $|k - k| < \epsilon$.

Hence, the implication $n > N \Rightarrow |k - k| < \epsilon$ is trivially true.

Therefore, $n > N$ implies $|k - k| < \epsilon$ for all $n \in \mathbb{N}$, as desired. □

Example 14. If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n+1} = L$.

Proof. Let (a_n) be a sequence such that $\lim_{n\to\infty} a_n = L$.

Then (a_n) is convergent.

Let (b_n) be a sequence defined by $b_n = a_{n+1}$ for all $n \in \mathbb{N}$.

We must prove $\lim_{n\to\infty} b_n = L$.

Since $b_n = a_{n+1}$ for all $n \in \mathbb{N}$, then (b_n) is the 1-tail of (a_n), so (b_n) is a subsequence of (a_n).

Since (b_n) is a subsequence of (a_n), then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = L$, as desired. ∎

Lemma 15. *Let (a_n) be a sequence defined by $a_n = (-1)^n$ for all $n \in \mathbb{N}$.*

Then $|(-1)^n| = 1$ for all $n \in \mathbb{N}$ and $|(-1)^{n+1} - (-1)^n| = 2$ for all $n \in \mathbb{N}$.

Proof. We prove $|(-1)^n| = 1$ for all $n \in \mathbb{N}$ by induction on n.

Let $p(n) : |(-1)^n| = 1$ be a predicate defined over $\mathbb{N}$.

Basis:

Since $|(-1)^1| = |-1| = 1$, then the statement $p(1)$ is true.

Induction:

Let $n \in \mathbb{N}$ such that $p(n)$ is true.

Then $|(-1)^n| = 1$.

Observe that

$$\begin{aligned} |(-1)^{n+1}| &= |(-1)^n(-1)| \\ &= |(-1)^n||-1| \\ &= 1 \cdot 1 \\ &= 1. \end{aligned}$$

Therefore, $|(-1)^{n+1}| = 1$, so $p(n+1)$ is true.

Hence, $p(n)$ implies $p(n+1)$ for all $n \in \mathbb{N}$.

Since $p(1)$ is true and $p(n)$ implies $p(n+1)$ for all $n \in \mathbb{N}$, then by induction, $p(n)$ is true for all $n \in \mathbb{N}$.

Therefore, $|(-1)^n| = 1$ for all $n \in \mathbb{N}$.

We prove $|(-1)^{n+1} - (-1)^n| = 2$ for all $n \in \mathbb{N}$ by induction on n.

Let $p(n) : |(-1)^{n+1} - (-1)^n| = 2$ be a predicate defined over $\mathbb{N}$.

Basis:

Since $|(-1)^2 - (-1)^1| = |1 + 1| = 2$, then the statement $p(1)$ is true.

Induction:

Let $n \in \mathbb{N}$ such that $p(n)$ is true.

Then $|(-1)^{n+1} - (-1)^n| = 2$.

Observe that

$$\begin{aligned}
|(-1)^{n+2} - (-1)^{n+1}| &= |(-1)^n(-1)^2 - (-1)^n(-1)| \\
&= |(-1)^n((-1)^2 - (-1))| \\
&= |(-1)^n(2)| \\
&= |(-1)^n| \cdot |2| \\
&= 1 \cdot 2 \\
&= 2.
\end{aligned}$$

Therefore, $|(-1)^{n+2} - (-1)^{n+1}| = 2$, so $p(n+1)$ is true.

Hence, $p(n)$ implies $p(n+1)$ for all $n \in \mathbb{N}$.

Since $p(1)$ is true and $p(n)$ implies $p(n+1)$ for all $n \in \mathbb{N}$, then by induction, $p(n)$ is true for all $n \in \mathbb{N}$. Therefore, $|(-1)^{n+1} - (-1)^n| = 2$ for all $n \in \mathbb{N}$. ☐

Example 16. bounded divergent sequence that oscillates

The sequence (a_n) defined by $a_n = (-1)^n$ diverges.

Therefore, $\lim_{n\to\infty}(-1)^n$ does not exist.

Moreover, (a_n) is bounded.

Therefore, (a_n) is a bounded divergent sequence.

Proof. We first prove (a_n) diverges.

Suppose (a_n) does not diverge.

Then (a_n) converges to some real number L.

Hence, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|(-1)^n - L| < \epsilon$ whenever $n > N$.

In particular, let $\epsilon = \frac{1}{2}$.

Then there exists $N \in \mathbb{N}$ such that $|(-1)^n - L| < \frac{1}{2}$ whenever $n > N$.

Observe that $|(-1)^{n+1} - (-1)^n| = 2$ for all $n \in \mathbb{N}$.

Since $N + 1 > N$, then $|(-1)^{N+1} - L| < \frac{1}{2}$.

Since $N + 2 > N$, then $|(-1)^{N+2} - L| < \frac{1}{2}$.

Observe that

$$\begin{aligned}
2 &= |(-1)^{N+2} - (-1)^{N+1}| \\
&= |((-1)^{N+2} - L) + (L - (-1)^{N+1})| \\
&\leq |(-1)^{N+2} - L| + |L - (-1)^{N+1}| \\
&= |(-1)^{N+2} - L| + |(-1)^{N+1} - L| \\
&< \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}$$

Hence, $2 < 1$, a contradiction.

Therefore, (a_n) diverges, as desired.

To prove (a_n) is bounded, we must prove there exists a real number b such that $|(-1)^n| \leq b$ for all $n \in \mathbb{N}$.

Let $b = 1$.

Let $n \in \mathbb{N}$.

Then $|(-1)^n| = 1 \leq b$, as desired. $\square$

Example 17. If (a_n) is a convergent sequence of real numbers, then $\lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty} a_n$.

Proof. Let (a_n) be a convergent sequence of real numbers.

Then there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = L$.

Let (b_n) be a sequence of real numbers defined by $b_n = a_{n-1}$ for all natural numbers $n > 1$.

We must prove $\lim_{n\to\infty} b_n = L$.

Let $n \in \mathbb{N}$ be given.

Since $1 + n \in \mathbb{N}$ and $1 + n > 1$, then $b_{1+n} = a_{(1+n)-1} = a_n$.

Hence, $b_{1+n} = a_n$ for all $n \in \mathbb{N}$, so $(b_{1+n}) = (a_n)$.

Since $(b_{1+n}) = (a_n)$ is convergent, then

$$\begin{aligned}\lim_{n\to\infty} b_n &= \lim_{n\to\infty} b_{1+n} \\ &= \lim_{n\to\infty} a_n \\ &= L.\end{aligned}$$

Therefore, $\lim_{n\to\infty} b_n = L$, as desired. $\square$

Proof. Let (a_n) be a convergent sequence of real numbers.

Let (b_n) be a sequence of real numbers defined by $b_n = a_{n-1}$ for all natural numbers $n > 1$.

We must prove $\lim_{n\to\infty} b_n = L$.

Let $\epsilon > 0$ be given.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < \epsilon$.

Since $N \in \mathbb{N}$, then $N + 1 \in \mathbb{N}$.

Let $n \in \mathbb{N}$ such that $n > N + 1$.

Then $n - 1 > N$.

Hence, $|a_{n-1} - L| < \epsilon$.

Since $N \in \mathbb{N}$, then $N \geq 1$.

Since $n > N + 1 > N \geq 1$, then $n > 1$.

Hence, $b_n = a_{n-1}$.

Observe that

$$\begin{aligned}|b_n - L| &= |a_{n-1} - L| \\ &< \epsilon.\end{aligned}$$

Therefore, $|b_n - L| < \epsilon$, so $\lim_{n\to\infty} b_n = L$, as desired. $\square$

Example 18. a. $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

b. $\lim_{n\to\infty} \frac{1}{n^k} = 0$ for any $k \in \mathbb{N}$.

c. $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$.

Solution. a. Since the sequence $(\frac{1}{n})$ converges to 0 and $(\frac{1}{n})$ is bounded, then the sequence given by $\frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}$ converges to 0.

Therefore, $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

b. We prove $\lim_{n\to\infty} \frac{1}{n^k} = 0$ for any $k \in \mathbb{N}$ by induction on k.

Let $S = \{k \in \mathbb{N} : \lim_{n\to\infty} \frac{1}{n^k} = 0\}$.

Since $\lim_{n\to\infty} \frac{1}{n^1} = \lim_{n\to\infty} \frac{1}{n} = 0$, then $1 \in S$.

Suppose $m \in S$.

Then $m \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{1}{n^m} = 0$.

Since $\lim_{n\to\infty} \frac{1}{n^m} = 0$ and the sequence $(\frac{1}{n})$ is bounded, then the sequence given by $\frac{1}{n^m} \cdot \frac{1}{n} = \frac{1}{n^{m+1}}$ converges to 0.

Thus, $\lim_{n\to\infty} \frac{1}{n^{m+1}} = 0$, so $m + 1 \in S$.

Hence, $m \in S$ implies $m + 1 \in S$.

By PMI, $\lim_{n\to\infty} \frac{1}{n^k} = 0$ for all $k \in \mathbb{N}$, as desired.

c. Since the sequence $(\frac{1}{n})$ converges to 0 and the sequence $(\cos(n))$ is bounded, then the sequence given by $\frac{1}{n} \cdot \cos(n) = \frac{\cos(n)}{n}$ converges to 0.

Therefore, $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$. □

Divergent Sequences

Example 19. unbounded divergent sequence

The sequence (a_n) defined by $a_n = n^2$ diverges to infinity.

Hence, $\lim_{n\to\infty} n^2 = \infty$, so (a_n) is divergent.

Moreover, (a_n) is unbounded.

Therefore, (a_n) is an unbounded divergent sequence.

Solution. Clearly, the sequence (a_n) defined by $a_n = n^2$ does not converge.

To rigorously prove (a_n) diverges, we prove (a_n) diverges to ∞.

That is, we prove for every $M > 0$ there exists $N \in \mathbb{N}$ such that $n^2 > M$ whenever $n > N$.

This will prove (a_n) diverges because the definition of divergence to ∞ implies a sequence diverges.

To prove (a_n) is unbounded, we must prove $(\forall b \in \mathbb{R})(\exists n \in \mathbb{N})(|n^2| > b)$. □

Proof. To prove $\lim_{n\to\infty} n^2 = \infty$, let $M > 0$ be given.

We must prove there exists $N \in \mathbb{N}$ such that if $n > N$, then $n^2 > M$.

Since $M > 0$, then $\sqrt{M} > 0$.

Since $\sqrt{M} \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \sqrt{M}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $n > N > \sqrt{M} > 0$, so $n > \sqrt{M} > 0$.

Thus, $n^2 > M > 0$, so $n^2 > M$, as desired.

Hence, (a_n) diverges to ∞, so (a_n) is divergent.

To prove (a_n) is unbounded, we must prove for every $b > 0$, there exists $n \in \mathbb{N}$ such that $|n^2| > b$.

Let b be a positive real number.

Then $b > 0$, so $\sqrt{b} > 0$.

Since $\sqrt{b} \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > \sqrt{b}$.

Thus, $0 < \sqrt{b} < n$, so $0 < b < n^2 = |n|^2 = |n^2|$.

Therefore, $b < |n^2|$, so $|n^2| > b$, as desired.

Thus, (a_n) is unbounded.

Hence (a_n) is an unbounded divergent sequence. □

Monotone Convergence Theorem

Example 20. sequence of rational numbers that converges to an irrational number

Let (x_n) be a sequence of rational numbers defined recursively by $x_1 = 2$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for all $n \in \mathbb{N}$.

Then $\lim_{n\to\infty} x_n = \sqrt{2}$.

Solution. We compute several terms of the sequence and observe that the terms are all positive.

A graph of this sequence as a function suggests that the function is decreasing and the sequence seems to converge to $\sqrt{2}$, even though $\sqrt{2}$ is not a rational number.

Note that this sequence is obtained by using Newton's method to approximate the zeroes of the real valued function $f(x) = x^2 - 2$. □

Proof. To prove (x_n) is a sequence of all positive terms, we prove $x_n > 0$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : x_n > 0\}$.

Since $1 \in \mathbb{N}$ and $x_1 = 2 > 0$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $x_k > 0$.

Hence, $\frac{1}{x_k} > 0$ and $\frac{x_k}{2} > 0$.

Thus, $x_{k+1} = \frac{x_k}{2} + \frac{1}{x_k} > 0$, so $x_{k+1} > 0$.

Since $k + 1 \in \mathbb{N}$ and $x_{k+1} > 0$, then $k + 1 \in S$.

Thus, $k \in S$ implies $k + 1 \in S$.

By the principle of mathematical induction, $x_n > 0$ for all $n \in \mathbb{N}$, as desired. □

Proof. To prove (x_n) is a sequence of rational numbers, we prove $x_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : x_n \in \mathbb{Q}\}$.

Since $1 \in \mathbb{N}$ and $x_1 = 2 \in \mathbb{Q}$, then $1 \in S$.

Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $x_k \in \mathbb{Q}$.
Since $x_k \in \mathbb{Q}$, then $\frac{x_k}{2} \in \mathbb{Q}$.
Since $k \in \mathbb{N}$, then $x_k > 0$, so $x_k \neq 0$.
Hence, $\frac{1}{x_k} \in \mathbb{Q}$.
Thus, $x_{k+1} = \frac{x_k}{2} + \frac{1}{x_k} \in \mathbb{Q}$, so $x_{k+1} \in \mathbb{Q}$.
Since $k+1 \in \mathbb{N}$ and $x_{k+1} \in \mathbb{Q}$, then $k+1 \in S$.
Thus, $k \in S$ implies $k+1 \in S$.
By the PMI, $x_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$, as desired. $\square$

Proof. We prove $(x_n)^2 - 2 > 0$ for all $n \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
Either $n = 1$ or $n > 1$.
We consider these cases separately.
Case 1: Suppose $n = 1$.
Then $(x_1)^2 - 2 = 2^2 - 2 = 2 > 0$.
Case 2: Suppose $n > 1$.
Then $x_n = \frac{x_{n-1}}{2} + \frac{1}{x_{n-1}}$.
Observe that

$$\begin{aligned}
(x_n)^2 - 2 &= (\frac{x_{n-1}}{2} + \frac{1}{x_{n-1}})^2 - 2 \\
&= (\frac{x_{n-1}}{2})^2 + 2(\frac{x_{n-1}}{2})(\frac{1}{x_{n-1}}) + (\frac{1}{x_{n-1}})^2 - 2 \\
&= (\frac{x_{n-1}}{2})^2 + 1 + (\frac{1}{x_{n-1}})^2 - 2 \\
&= (\frac{x_{n-1}}{2})^2 - 1 + (\frac{1}{x_{n-1}})^2 \\
&= (\frac{x_{n-1}}{2})^2 - 2(\frac{x_{n-1}}{2})(\frac{1}{x_{n-1}}) + (\frac{1}{x_{n-1}})^2 \\
&= (\frac{x_{n-1}}{2} - \frac{1}{x_{n-1}})^2 \\
&\geq 0.
\end{aligned}$$

Thus, $(x_n)^2 - 2 \geq 0$.
Hence, in all cases, $(x_n)^2 - 2 \geq 0$ for all $n \in \mathbb{N}$.
Suppose there exists $k \in \mathbb{N}$ such that $(x_k)^2 - 2 = 0$.
Then $(x_k)^2 = 2$.
Since $x_k \in \mathbb{Q}$, then this implies there is a rational number whose square is 2.
This contradicts the fact that there is no rational number whose square is 2.
Therefore, there is no $k \in \mathbb{N}$ such that $(x_k)^2 - 2 = 0$.
Hence, $(x_k)^2 - 2 \neq 0$ for all $k \in \mathbb{N}$, so $(x_n)^2 - 2 \neq 0$ for all $n \in \mathbb{N}$.
Since $(x_n)^2 - 2 \neq 0$ for all $n \in \mathbb{N}$ and $(x_n)^2 - 2 \geq 0$ for all $n \in \mathbb{N}$, then $(x_n)^2 - 2 > 0$ for all $n \in \mathbb{N}$, as desired. $\square$

Proof. We prove (x_n) is strictly decreasing.

Let $n \in \mathbb{N}$ be given.

Since $x_n > 0$ for all $n \in \mathbb{N}$, then in particular, $x_n > 0$.

Hence, $2x_n > 0$.

Since $(x_n)^2 - 2 > 0$ for all $n \in \mathbb{N}$, then in particular, $(x_n)^2 - 2 > 0$.

Observe that

$$\begin{aligned} x_n - x_{n+1} &= x_n - (\frac{x_n}{2} + \frac{1}{x_n}) \\ &= \frac{x_n}{2} - \frac{1}{x_n} \\ &= \frac{x_n^2 - 2}{2x_n} \\ &> 0. \end{aligned}$$

Thus, $x_n - x_{n+1} > 0$, so $x_n > x_{n+1}$.

Therefore, (x_n) is strictly decreasing, so (x_n) is monotonic. □

Proof. We prove (x_n) is bounded above by 2.

Since $x_1 = 2$ and (x_n) is strictly decreasing, then $2 = x_1 > x_2 > x_3 > x_4 > ....$

Hence, $2 \geq x_n$ for all $n \in \mathbb{N}$.

Therefore, 2 is an upper bound of (x_n).

Since $x_n > 0$ for all $n \in \mathbb{N}$ and $2 \geq x_n$ for all $n \in \mathbb{N}$, then $2 \geq x_n > 0$ for all $n \in \mathbb{N}$.

Hence, $0 < x_n \leq 2$ for all $n \in \mathbb{N}$, so (x_n) is bounded.

Since (x_n) is bounded and monotonic, then by MCT, (x_n) is convergent. □

Proof. We prove $x_n > 1$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : x_n > 1\}$.

Since $1 \in \mathbb{N}$ and $x_1 = 2 > 1$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $x_k > 1$.

Since $x_k > 1$, then $\frac{x_k}{2} > \frac{1}{2}$.

Since 2 is an upper bound of (x_n), then $x_k \leq 2$.

Since all terms of (x_n) are positive, then $x_k > 0$.

Thus, $0 < x_k \leq 2$, so $\frac{1}{x_k} \geq \frac{1}{2}$.

Since $\frac{x_k}{2} > \frac{1}{2}$ and $\frac{1}{x_k} \geq \frac{1}{2}$, then $x_{k+1} = \frac{x_k}{2} + \frac{1}{x_k} > \frac{1}{2} + \frac{1}{2} = 1$, so $x_{k+1} > 1$.

Since $k + 1 \in \mathbb{N}$ and $x_{k+1} > 1$, then $k + 1 \in S$.

Thus, $k \in S$ implies $k + 1 \in S$.

By the PMI, $x_n > 1$ for all $n \in \mathbb{N}$, as desired. □

Proof. We prove (x_n) converges to $\sqrt{2}$.

Since (x_n) is a sequence of rational numbers and $\mathbb{Q} \subset \mathbb{R}$, then (x_n) is a sequence of real numbers.

Since (x_n) is convergent, then there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = L$.

We must prove $L = \sqrt{2}$.

Since $x_n > 1$ for all $n \in \mathbb{N}$, then 1 is a lower bound of (x_n).

Since (x_n) is a decreasing convergent sequence, then L is the greatest lower bound of (x_n).

Hence, $L \geq 1 > 0$, so $L > 0$.

Thus, $L \neq 0$.

Since $x_n > 0$ for all $n \in \mathbb{N}$, then $x_n \neq 0$ for all $n \in \mathbb{N}$.

Observe that

$$\begin{aligned}
L &= \lim_{n\to\infty} x_n \\
&= \lim_{n\to\infty} x_{n+1} \\
&= \lim_{n\to\infty} (\frac{x_n}{2} + \frac{1}{x_n}) \\
&= \lim_{n\to\infty} \frac{x_n}{2} + \lim_{n\to\infty} \frac{1}{x_n} \\
&= \frac{1}{2} \lim_{n\to\infty} x_n + \frac{1}{\lim_{n\to\infty} x_n} \\
&= \frac{L}{2} + \frac{1}{L}.
\end{aligned}$$

Thus, $L = \frac{L}{2} + \frac{1}{L}$, so $\frac{L}{2} = \frac{1}{L}$.

Hence, $L^2 = 2$, so either $L = \sqrt{2}$ or $L = -\sqrt{2}$.

Since $L > 0$, then $L \neq -\sqrt{2}$, so $L = \sqrt{2}$, as desired. □