Sequences in $\mathbb{R}$ Exercises

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Sequences in $\mathbb{R}$

Exercise 1. Let (a_n) be a sequence defined recursively by $a_{n+1} = 2a_n - a_{n-1} + 2$ for $n > 1$ with $a_1 = 3, a_2 = 6$.

Then $a_n = n^2 + 2$ for all $n \in \mathbb{N}$.

Solution. We prove the proposition $a_n = n^2 + 2$ for all natural numbers n by induction on n.

Let $S = \{n \in \mathbb{N} : a_n = n^2 + 2\}$.

We prove by strong induction:

1. $1 \in S$.
2. $1, 2, ..., k \in S$ implies $k + 1 \in S$. □

Proof. We prove by strong induction on n.

Let $S = \{n \in \mathbb{N} : a_n = n^2 + 2\}$.

Since $a_1 = 3 = 1^2 + 2$, then $1 \in S$.

Suppose $1, 2, ..., k \in S$.

Either $k > 1$ or $k = 1$.

We consider each case separately.

Case 1: Suppose $k = 1$.

Since $a_2 = 6 = 2^2 + 2$, then $2 = k + 1 \in S$.

Case 2: Suppose $k > 1$.

Since $k \in S$, then $a_k = k^2 + 2$.

Since $k > 1$, then $k - 1 \in S$. so $a_{k-1} = (k-1)^2 + 2$.

Since $k > 1$, then

$$
\begin{aligned}
a_{k+1} &= 2a_k - a_{k-1} + 2 \\
&= 2(k^2 + 2) - [(k-1)^2 + 2] + 2 \\
&= 2k^2 + 4 - (k^2 - 2k + 3) + 2 \\
&= k^2 + 2k + 3 \\
&= k^2 + 2k + 1 + 2 \\
&= (k+1)^2 + 2.
\end{aligned}
$$

Hence, $a_{k+1} = (k+1)^2 + 2$, so $k + 1 \in S$.

Thus, by the principle of strong mathematical induction, $a_n = n^2 + 2$ for all $n \in \mathbb{N}$, as desired. $\square$

Exercise 2. Let (a_n) be a sequence defined recursively by $a_{n+1} = a_n + (n+1)^2$ with $a_1 = 1$.

Then $a_n = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : \frac{n(n+1)(2n+1)}{6}\}$.

Since $a_1 = 1 = \frac{1*2*3}{6} = \frac{1(1+1)(2*1+1)}{6}$, then $1 \in S$.

Let $k \in S$ be given.

Then $a_k = \frac{k(k+1)(2k+1)}{6}$.

Observe that

$$\begin{aligned} a_{k+1} &= a_k + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1)[\frac{k(2k+1)}{6} + (k+1)] \\ &= (k+1)\frac{2k^2 + k + 6k + 6}{6} \\ &= (k+1)\frac{2k^2 + 7k + 6}{6} \\ &= (k+1)\frac{(k+2)(2k+3)}{6} \\ &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}. \end{aligned}$$

Hence, $a_{k+1} = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$, so $k+1 \in S$.

Thus, $k \in S$ implies $k+1 \in S$.

By the principle of mathematical induction, $a_n = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$, as desired. $\square$

Exercise 3. Let (a_n) be a sequence defined recursively by $a_{n+1} = a_n + 2(n+1)$ with $a_1 = 2$.

Then $a_n = \sum_{k=1}^{n}(2k)$ for all $n \in \mathbb{N}$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : a_n = \sum_{k=1}^{n}(2k)\}$.

Since $a_1 = 2 = 2 * 1 = \sum_{k=1}^{1}(2k)$, then $1 \in S$.

Suppose $m \in S$.

Then $m \in \mathbb{N}$ and $a_m = \sum_{k=1}^{m}(2k)$.

Observe that

$$\begin{aligned} a_{m+1} &= a_m + 2(m+1) \\ &= \sum_{k=1}^{m}(2k) + 2(m+1) \\ &= (2*1 + 2*2 + 2*3 + ... + 2m) + 2(m+1) \\ &= 2*1 + 2*2 + 2*3 + ... + 2m + 2(m+1) \\ &= \sum_{k=1}^{m+1}(2k). \end{aligned}$$

Since $m+1 \in \mathbb{N}$ and $a_{m+1} = \sum_{k=1}^{m+1}(2k)$, then $m+1 \in S$.

Thus, $m \in S$ implies $m+1 \in S$.

By the principle of mathematical induction, $a_n = \sum_{k=1}^{n}(2k)$ for all $n \in \mathbb{N}$, as desired. □

Exercise 4. Let (a_n) be a sequence defined recursively by $a_{n+1} = a_n + 2n + 3$ with $a_1 = 5$.

Then $a_n = (n+1)^2 + 1$ for all $n \in \mathbb{N}$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : a_n = (n+1)^2 + 1\}$.

Since $a_1 = 5 = (1+1)^2 + 1$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $a_k = (k+1)^2 + 1$.

Observe that

$$\begin{aligned} a_{k+1} &= a_k + 2k + 3 \\ &= [(k+1)^2 + 1] + 2k + 3 \\ &= (k^2 + 2k + 2) + 2k + 3 \\ &= k^2 + 4k + 5 \\ &= (k^2 + 4k + 4) + 1 \\ &= (k+2)^2 + 1 \\ &= [(k+1) + 1]^2 + 1. \end{aligned}$$

Since $k+1 \in \mathbb{N}$ and $a_{k+1} = [(k+1)+1]^2 + 1$, then $k+1 \in S$.

Thus, $k \in S$ implies $k+1 \in S$.

By the principle of mathematical induction, $a_n = (n+1)^2 + 1$ for all $n \in \mathbb{N}$, as desired. □

Exercise 5. Let (a_n) be a sequence defined recursively by $a_{n+1} = a_n + 2^n$ with $a_1 = 1$.

Then $a_n = \sum_{k=0}^{n-1}(2^k) = 2^n - 1$ for all $n \in \mathbb{N}$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : a_n = 2^n - 1\}$.

Since $1 \in \mathbb{N}$ and $a_1 = 1 = 2^1 - 1$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $a_k = 2^k - 1$.

Observe that

$$\begin{aligned} a_{k+1} &= a_k + 2^k \\ &= (2^k - 1) + 2^k \\ &= 2 * 2^k - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Since $k + 1 \in \mathbb{N}$ and $a_{k+1} = 2^{k+1} - 1$, then $k + 1 \in S$.

Thus, $k \in S$ implies $k + 1 \in S$.

By the principle of mathematical induction, $a_n = 2^n - 1$ for all $n \in \mathbb{N}$, as desired. $\square$

Exercise 6. Let (a_n) be a sequence defined recursively by $a_{n+1} = a_n + 3^n$ with $a_1 = 1$.

Then $a_n = \sum_{k=0}^{n-1}(3^k) = \frac{3^n-1}{2}$ for all $n \in \mathbb{N}$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : a_n = \frac{3^n-1}{2}\}$.

Since $1 \in \mathbb{N}$ and $a_1 = 1 = \frac{3^1-1}{2}$, then $1 \in S$.

Suppose $m \in S$.

Then $m \in \mathbb{N}$ and $a_m = \frac{3^m-1}{2}$.

Observe that

$$\begin{aligned} a_{m+1} &= a_m + 3^m \\ &= \frac{3^m - 1}{2} + 3^m \\ &= \frac{3^m - 1 + 2 * 3^m}{2} \\ &= \frac{3 * 3^m - 1}{2} \\ &= \frac{3^{m+1} - 1}{2}. \end{aligned}$$

Since $m + 1 \in \mathbb{N}$ and $a_{m+1} = \frac{3^{m+1}-1}{2}$, then $m + 1 \in S$.

Thus, $m \in S$ implies $m + 1 \in S$.

By the principle of mathematical induction, $a_n = \frac{3^n-1}{2}$ for all $n \in \mathbb{N}$, as desired. $\square$

Exercise 7. Let r be a fixed real number with $r \neq 1$.

Let (a_n) be a sequence defined recursively by $a_{n+1} = a_n + r^n$ with $a_1 = 1$.

Then $a_n = \sum_{k=0}^{n-1}(r^k) = \frac{r^n-1}{r-1}$ for all $n \in \mathbb{N}$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : a_n = \frac{r^n - 1}{r-1}\}$.

Since $1 \in \mathbb{N}$ and $a_1 = 1 = \frac{r^1 - 1}{r-1}$ and $r \neq 1$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $a_k = \frac{r^k - 1}{r-1}$.

Observe that

$$\begin{aligned} a_{k+1} &= a_k + r^k \\ &= \frac{r^k - 1}{r-1} + r^k \\ &= \frac{r^k - 1 + r^k(r-1)}{r-1} \\ &= \frac{r^k - 1 + r^{k+1} - r^k}{r-1} \\ &= \frac{r^{k+1} - 1}{r-1}. \end{aligned}$$

Since $k+1 \in \mathbb{N}$ and $a_{k+1} = \frac{r^{k+1} - 1}{r-1}$, then $k+1 \in S$.

Thus, $k \in S$ implies $k+1 \in S$.

By the principle of mathematical induction, $a_n = \frac{r^n - 1}{r-1}$ for all $n \in \mathbb{N}$, as desired. ☐

Exercise 8. Let c, r, a_0 be fixed real numbers with $r \neq 1$.

Let (a_n) be a sequence defined recursively by $a_{n+1} = c + ra_n$ with $a_1 = c + ra_0$.

Then $a_n = \frac{r^n - 1}{r-1}c + r^n a_0$ for all $n \in \mathbb{N}$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : a_n = \frac{r^n - 1}{r-1}c + r^n a_0\}$.

Since $1 \in \mathbb{N}$ and $a_1 = c + ra_0 = \frac{r^1 - 1}{r-1}c + r^1 a_0$ and $r \neq 1$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $a_k = \frac{r^k - 1}{r-1}c + r^k a_0$.

Observe that

$$\begin{aligned} a_{k+1} &= c + ra_k \\ &= c + r[\frac{r^k - 1}{r-1}c + r^k a_0] \\ &= c + \frac{r^{k+1} - r}{r-1}c + r^{k+1}a_0 \\ &= (1 + \frac{r^{k+1} - r}{r-1})c + r^{k+1}a_0 \\ &= \frac{r - 1 + r^{k+1} - r}{r-1}c + r^{k+1}a_0 \\ &= \frac{r^{k+1} - 1}{r-1}c + r^{k+1}a_0. \end{aligned}$$

Since $k+1 \in \mathbb{N}$ and $a_{k+1} = \frac{r^{k+1}-1}{r-1}c + r^{k+1}a_0$, then $k+1 \in S$.

Thus, $k \in S$ implies $k+1 \in S$.

By the principle of mathematical induction, $a_n = \frac{r^n-1}{r-1}c + r^n a_0$ for all $n \in \mathbb{N}$, as desired. □

Exercise 9. Let (a_n) be a sequence defined recursively by $a_{n+1} = 2a_n - a_{n-1} + 6n$ for $n > 1$ with $a_1 = 1$ and $a_2 = 8$.

Then $a_n = n^3$ for all $n \in \mathbb{N}$.

Proof. We prove by strong induction on n.

Let $S = \{n \in \mathbb{N} : a_n = n^3\}$.

Since $1 \in \mathbb{N}$ and $a_1 = 1 = 1^3$, then $1 \in S$.

Suppose $1, 2, ..., k \in S$.

Since $k \in S$, then $k \in \mathbb{N}$ and $a_k = k^3$.

Since $k \in \mathbb{N}$, then $k \geq 1$, so either $k > 1$ or $k = 1$.

We consider these cases separately.

Case 1: Suppose $k > 1$.

Then $k - 1 > 0$.

Since $k \in \mathbb{N}$, then $k - 1 \in \mathbb{Z}$, so $k - 1 \geq 1$.

Since $k - 1 < k$, then $k - 1 \in S$, so $a_{k-1} = (k-1)^3$.

Observe that

$$\begin{aligned} a_{k+1} &= 2a_k - a_{k-1} + 6k \\ &= 2k^3 - (k-1)^3 + 6k \\ &= 2k^3 - (k^3 - 3k^2 + 3k - 1) + 6k \\ &= k^3 + 3k^2 + 3k + 1 \\ &= (k+1)^3. \end{aligned}$$

Since $k+1 \in \mathbb{N}$ and $a_{k+1} = (k+1)^3$, then $k+1 \in S$.

Case 2: Suppose $k = 1$.

Since $a_2 = 8 = 2^3$ and $k + 1 = 2$, then $k+1 \in S$.

Therefore, in all cases, $k+1 \in S$.

Hence, by the principle of mathematical induction(strong), $a_n = n^3$ for all $n \in \mathbb{N}$, as desired. □

Exercise 10. Let (a_n) be a sequence defined recursively by $a_{n+1} = 3a_n - 3a_{n-1} + a_{n-2}$ for $n > 2$ with $a_1 = 1$, $a_2 = 4$, and $a_3 = 9$.

Then $a_n = n^2$ for all $n \in \mathbb{N}$.

Proof. We prove by strong induction on n.

Let $S = \{n \in \mathbb{N} : a_n = n^2\}$.

Since $1 \in \mathbb{N}$ and $a_1 = 1 = 1^2$, then $1 \in S$.

Suppose $1, 2, ..., k \in S$.

Since $k \in S$, then $k \in \mathbb{N}$ and $a_k = k^2$.

Since $k \in \mathbb{N}$, then either $k = 1$ or $k = 2$ or $k > 2$.

We consider these cases separately.

Case 1: Suppose $k = 1$.
Since $a_2 = 4 = 2^2$ then $2 = k + 1 \in S$.
Case 2: Suppose $k = 2$.
Since $a_3 = 9 = 3^2$ then $3 = k + 1 \in S$.
Case 3: Suppose $k > 2$.
Then

$$\begin{aligned} a_{k+1} &= 3a_k - 3a_{k-1} + a_{k-2} \\ &= 3k^2 - 3(k-1)^2 + (k-2)^2 \\ &= 3k^2 - 3(k^2 - 2k + 1) + (k^2 - 4k + 4) \\ &= k^2 + 2k + 1 \\ &= (k+1)^2. \end{aligned}$$

Since $k + 1 \in \mathbb{N}$ and $a_{k+1} = (k+1)^2$, then $k + 1 \in S$.
Therefore, in all cases, $k + 1 \in S$.
Hence, by the principle of mathematical induction(strong), $a_n = n^2$ for all $n \in \mathbb{N}$, as desired. □

Exercise 11. Let (a_n) be a sequence defined recursively by $a_{n+1} = a_n + \frac{1}{(n+1)(n+2)}$ with $a_1 = \frac{1}{2}$.
Then $a_n = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

Proof. We prove $a_n = \frac{n}{n+1}$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : a_n = \frac{n}{n+1}\}$.
Since $1 \in \mathbb{N}$ and $a_1 = \frac{1}{2} = \frac{1}{1+1}$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $a_k = \frac{k}{k+1}$.
Observe that

$$\begin{aligned} a_{k+1} &= a_k + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2} \\ &= \frac{k+1}{(k+1)+1}. \end{aligned}$$

Since $k + 1 \in \mathbb{N}$ and $a_{k+1} = \frac{k+1}{(k+1)+1}$, then $k + 1 \in S$.

Thus, $k \in S$ implies $k+1 \in S$.

By the principle of mathematical induction, $a_n = \frac{n}{n+1}$ for all $n \in \mathbb{N}$, as desired. □

Exercise 12. Let (a_n) be a sequence defined recursively by $a_{n+2} = \frac{a_{n+1}+a_n}{2}$ for all $n \in \mathbb{Z}^+$ and $a_1 = 1$ and $a_2 = 2$.

Then $1 \leq a_n \leq 2$ for all $n \in \mathbb{Z}^+$.

Proof. We prove $1 \leq a_n \leq 2$ for all $n \in \mathbb{Z}^+$ by strong induction on n.

Let $S = \{n \in \mathbb{Z}^+ : 1 \leq a_n \leq 2\}$.

Since $1 \in \mathbb{Z}^+$ and $1 = a_1 < 2$, then $1 \in S$.

Let $k \in \mathbb{Z}^+$ such that $\{1, 2, ..., k\} \subset S$.

Since $k \in \mathbb{Z}^+$, then $k+1 \in \mathbb{Z}^+$.

Since $\{1, 2, ..., k\} \subset S$, then $k-1 \in S$ and $k \in S$.

Since $k-1 \in S$, then $k-1 \in \mathbb{Z}^+$ and $1 \leq a_{k-1} \leq 2$.

Since $k \in S$, then $k \in \mathbb{Z}^+$ and $1 \leq a_k \leq 2$.

Since $1 \leq a_k \leq 2$ and $1 \leq a_{k-1} \leq 2$, then $2 \leq a_k + a_{k-1} \leq 4$, so $1 \leq \frac{a_k+a_{k-1}}{2} \leq 2$.

Since $k-1 \in \mathbb{Z}^+$, then $a_{k+1} = \frac{a_k+a_{k-1}}{2}$, so $1 \leq a_{k+1} \leq 2$.

Since $k+1 \in \mathbb{Z}^+$ and $1 \leq a_{k+1} \leq 2$, then $k+1 \in S$.

Therefore, by PMI, $1 \leq a_n \leq 2$ for all $n \in \mathbb{Z}^+$. □

Sequences as Functions

Exercise 13. Let (a_n) be a sequence of real numbers defined by $a_n = \frac{3n+7}{n}$ for all $n \in \mathbb{N}$.

Then 3 is a lower bound of (a_n) and 10 is an upper bound of (a_n).

Proof. To prove 10 is an upper bound of (a_n), let $n \in \mathbb{N}$ be given.

Then $n \geq 1 > 0$, so $n \geq 1$ and $n > 0$.

Since $n \geq 1$, then $7n \geq 7$, so $10n \geq 3n + 7$.

Since $n > 0$, then $10 \geq \frac{3n+7}{n} = a_n$, so $a_n \leq 10$, as desired. □

Proof. To prove 3 is a lower bound of (a_n), let $n \in \mathbb{N}$ be given.

Then $n \geq 1 > 0$, so $n > 0$.

Since $0 < 7$, then $3n < 3n + 7$.

Since $n > 0$, then $3 < \frac{3n+7}{n} = a_n$, so $3 < a_n$.

Thus, $3 \leq a_n$, as desired. □

Exercise 14. Let (a_n) be a sequence of real numbers.

Then $(|a_n|)$ is bounded iff (a_n) is bounded.

Proof. Suppose (a_n) is bounded.

Then there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Then $|a_n| \leq M$.

Since $a_n \in \mathbb{R}$, then $|a_n| \geq 0$.

Thus, $||a_n|| = |a_n| \leq M$.

Hence, $||a_n|| \leq M$, so $(|a_n|)$ is bounded.

Conversely, suppose $(|a_n|)$ is bounded.

Then there exists $M > 0$ such that $||a_n|| \leq M$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Then $||a_n|| \leq M$.

Since $a_n \in \mathbb{R}$, then $|a_n| \geq 0$.

Thus, $|a_n| = ||a_n|| \leq M$.

Hence, $|a_n| \leq M$, so (a_n) is bounded. ∎

Exercise 15. Let (a_n) be a sequence defined recursively by $a_{n+1} = \frac{4a_n+3}{a_n+2}$ with $a_1 = 4$.

Then $a_n = \frac{3 \cdot 5^n+1}{5^n-1}$ and (a_n) is strictly decreasing, so that (a_n) is monotonic decreasing.

Solution. We can actually show that $a_n = \frac{3a_1 * 5^{n-1} + a_1 + 3 * 5^{n-1} - 3}{a_1 * 5^{n-1} - a_1 + 5^{n-1} + 3}$.

We must prove

1. $a_n = \frac{3 \cdot 5^n+1}{5^n-1}$ for all $n \in \mathbb{N}$.

2. To prove (a_n) is strictly decreasing, we must prove $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

We compute terms of the sequence (such as using a computer) and observe that all the terms are greater than 3.

So we conjecture that $a_n > 3$ for all $n \in \mathbb{N}$. ∎

Proof. We prove $a_n = \frac{3 \cdot 5^n+1}{5^n-1}$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : a_n = \frac{3 \cdot 5^n+1}{5^n-1}\}$.

Since $1 \in \mathbb{N}$ and $a_1 = 4 = \frac{16}{4} = \frac{3 \cdot 5^1+1}{5^1-1}$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $a_k = \frac{3 \cdot 5^k+1}{5^k-1}$.

Since $k \in \mathbb{N}$, then $k \geq 1$.

Since $k \geq 1 \Rightarrow k \neq 0 \Rightarrow 5^k \neq 1 \Rightarrow 5^k - 1 \neq 0$, then $5^k - 1 \neq 0$.

Observe that

$$\begin{aligned}
a_{k+1} &= \frac{4a_k+3}{a_k+2} \\
&= \frac{4 * \frac{3 \cdot 5^k+1}{5^k-1} + 3}{\frac{3 \cdot 5^k+1}{5^k-1} + 2} \\
&= \frac{4(3 * 5^k + 1) + 3(5^k - 1)}{(3 * 5^k + 1) + 2(5^k - 1)} \\
&= \frac{15 * 5^k + 1}{5 * 5^k - 1} \\
&= \frac{3 * 5 * 5^k + 1}{5^{k+1} - 1} \\
&= \frac{3 * 5^{k+1} + 1}{5^{k+1} - 1}.
\end{aligned}$$

Since $k+1 \in \mathbb{N}$ and $a_{k+1} = \frac{3 * 5^{k+1}+1}{5^{k+1}-1}$, then $k+1 \in S$.

Thus, $k \in S$ implies $k+1 \in S$.

By the principle of mathematical induction, $a_n = \frac{3 \cdot 5^n + 1}{5^n - 1}$ for all $n \in \mathbb{N}$, as desired. $\square$

Proof. We prove $a_n > 3$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : a_n > 3\}$.

Since $1 \in \mathbb{N}$ and $a_1 = 4 > 3$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $a_k > 3$.

Observe that

$$\begin{aligned} a_k > 3 \quad &\Rightarrow \quad a_k + 2 > 5 \\ &\Rightarrow \quad 1 > \frac{5}{a_k + 2} \\ &\Rightarrow \quad 4 - 3 > \frac{5}{a_k + 2} \\ &\Rightarrow \quad 4 - \frac{5}{a_k + 2} > 3 \\ &\Rightarrow \quad \frac{4a_k + 3}{a_k + 2} > 3 \\ &\Rightarrow \quad a_{k+1} > 3. \end{aligned}$$

Since $k+1 \in \mathbb{N}$ and $a_{k+1} > 3$, then $k+1 \in S$.

Thus, $k \in S$ implies $k+1 \in S$.

By the principle of mathematical induction, $a_n > 3$ for all $n \in \mathbb{N}$, as desired. $\square$

Proof. We prove $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Since $a_n > 3$ for all $n \in \mathbb{N}$, then $a_n > 3$.

Hence, $a_n + 2 > 5 > 0$, so $a_n + 2 > 0$.

Since $a_n > 3$, then $a_n - 2 > 1$.

Since $a_n > 3 > 0$ and $a_n - 2 > 1 > 0$, then $a_n(a_n - 2) > 3 * 1 = 3$, so $a_n^2 - 2a_n > 3$.

Hence, $a_n^2 - 2a_n - 3 > 0$.

Observe that

$$\begin{aligned} a_n - a_{n+1} \quad &= \quad a_n - \frac{4a_n + 3}{a_n + 2} \\ &= \quad \frac{a_n(a_n + 2) - (4a_n + 3)}{a_n + 2} \\ &= \quad \frac{a_n^2 - 2a_n - 3}{a_n + 2} \\ &> \quad 0. \end{aligned}$$

Hence, $a_n - a_{n+1} > 0$, so $a_n > a_{n+1}$, as desired. $\square$

Exercise 16. Let (a_n) be a sequence defined by $a_n = n^2 - n$ for all $n \in \mathbb{N}$.

Then (a_n) is strictly increasing and is bounded below by 0 and is not bounded above.

Proof. We prove (a_n) is strictly increasing.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1 > 0$, so $n > 0$.

Thus, $-n < 0 < n$, so $-n < n$.

Hence, $a_n = n^2 - n < n^2 + n = (n^2 + 2n + 1) - (n + 1) = (n + 1)^2 - (n + 1) = a_{n+1}$.

Therefore, $a_n < a_{n+1}$, so (a_n) is strictly increasing.

Thus, (a_n) is monotonic increasing, so (a_n) is monotonic. $\square$

Proof. We prove (a_n) is bounded below by 0.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1$.

Since $n \geq 1 > 0$, then $n > 0$.

Since $n \geq 1$, then $n - 1 \geq 0$.

Thus, $a_n = n^2 - n = n(n - 1) \geq 0$.

Therefore, $a_n \geq 0$, so $a_n \geq 0$ for all $n \in \mathbb{N}$.

Hence, 0 is a lower bound of (a_n), so (a_n) is bounded below by 0. $\square$

Proof. We prove (a_n) is unbounded above.

Let $M > 0$ be given.

Since

$$\begin{aligned} M > 0 &\Rightarrow 4M + 1 > 0 \\ &\Rightarrow \sqrt{4M+1} > 0 \\ &\Rightarrow \frac{\sqrt{4M+1}+1}{2} > 0, \end{aligned}$$

then $\frac{\sqrt{4M+1}+1}{2} > 0$.

Since $\frac{\sqrt{4M+1}+1}{2} \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > \frac{\sqrt{4M+1}+1}{2}$.

Thus, $n > \frac{\sqrt{4M+1}+1}{2} > 0$.

Since $M > 0$ and

$$\begin{aligned} M > 0 &\Rightarrow 4M > 0 \\ &\Rightarrow 4M + 1 > 1 \\ &\Rightarrow \sqrt{4M+1} > 1 \\ &\Rightarrow \sqrt{4M+1} - 1 > 0 \\ &\Rightarrow \frac{\sqrt{4M+1}-1}{2} > 0, \end{aligned}$$

then $\frac{\sqrt{4M+1}-1}{2} > 0$.

Since 0 is a lower bound of (a_n), then $0 \leq a_n$ for all $n \in \mathbb{N}$.
In particular, $0 \leq a_n = |a_n|$.
Since $n > \frac{\sqrt{4M+1}+1}{2}$, then $n - 1 > \frac{\sqrt{4M+1}-1}{2}$.
Since $n > \frac{\sqrt{4M+1}+1}{2} > 0$ and $n - 1 > \frac{\sqrt{4M+1}-1}{2} > 0$, then

$$\begin{aligned} |a_n| &= a_n \\ &= n^2 - n \\ &= n(n-1) \\ &> \frac{\sqrt{4M+1}+1}{2} * \frac{\sqrt{4M+1}-1}{2} \\ &= \frac{(4M+1)-1}{4} \\ &= M. \end{aligned}$$

Hence, $|a_n| > M$.
Thus, there exists $n \in \mathbb{N}$ such that $|a_n| > M$ for all $M > 0$.
Therefore, (a_n) is unbounded above. □

Exercise 17. Let (a_n) be a sequence defined by $a_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$.
Then (a_n) is strictly decreasing and is bounded below by 0 and is bounded above by $\frac{1}{2}$.

Proof. We prove (a_n) is strictly decreasing.
Let $n \in \mathbb{N}$ be given.
Then $n \geq 1 > 0$, so $n > 0$.
Since $0 < n < n + 1 < n + 2$, then $a_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = a_n$.
Hence, $a_{n+1} < a_n$.
Therefore, $a_n > a_{n+1}$, so (a_n) is strictly decreasing.
Thus, (a_n) is monotonic decreasing, so (a_n) is monotonic. □

Proof. We prove (a_n) is bounded below by 0.
Let $n \in \mathbb{N}$ be given.
Then $n \geq 1 > 0$, so $n > 0$.
Hence, $n + 1 > 0$, so $a_n = \frac{1}{n+1} > 0$.
Thus, $a_n > 0$.
Therefore, 0 is a lower bound of (a_n), so (a_n) is bounded below by 0. □

Proof. We prove (a_n) is bounded above by $\frac{1}{2}$.
Let $n \in \mathbb{N}$ be given.
Then $n \geq 1$.
Since $n \geq 1 \Rightarrow n + 1 \geq 2 > 0 \Rightarrow \frac{1}{n+1} \leq \frac{1}{2}$, then $a_n = \frac{1}{n+1} \leq \frac{1}{2}$.
Hence, $a_n \leq \frac{1}{2}$.
Therefore, $\frac{1}{2}$ is an upper bound of (a_n), so (a_n) is bounded above by $\frac{1}{2}$. □

Exercise 18. Let (a_n) be a sequence defined by $a_n = \frac{(-1)^n}{n^2}$ for all $n \in \mathbb{N}$.
Then (a_n) is neither increasing nor decreasing and is bounded below by -1 and is bounded above by $\frac{1}{4}$.

Proof. Since $2 \in \mathbb{N}$ and $a_2 = \frac{1}{4} > \frac{-1}{9} = a_3$, then (a_n) is not increasing.

Since $1 \in \mathbb{N}$ and $a_1 = -1 < \frac{1}{4} = a_2$, then (a_n) is not decreasing.

Since (a_n) is neither increasing nor decreasing, then (a_n) is not monotonic.

To prove (a_n) is bounded below by -1 and above by $\frac{1}{4}$, we prove $-1 \leq a_n \leq \frac{1}{4}$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Either n is even or odd.

We consider these cases separately.

Case 1: Suppose n is even.

Then $n \geq 2 > 0$, so $n^2 \geq 4 > 0$.

Hence, $\frac{1}{n^2} \leq \frac{1}{4}$.

Since $n^2 \geq 4 > 0$, then $n^2 > 0$, so $\frac{1}{n^2} > 0$.

Since $(-1)^n = 1$ for all even n, then $a_n = \frac{(-1)^n}{n^2} = \frac{1}{n^2} \leq \frac{1}{4}$ and $a_n = \frac{(-1)^n}{n^2} = \frac{1}{n^2} > 0 > -1$.

Thus, $a_n \leq \frac{1}{4}$ and $a_n > -1$.

Since $-1 < a_n$ and $a_n \leq \frac{1}{4}$, then $-1 < a_n \leq \frac{1}{4}$.

Case 2: Suppose n is odd.

Then $n \geq 1 > 0$, so $n^2 \geq 1 > 0$.

Hence, $\frac{1}{n^2} \leq 1$, so $\frac{-1}{n^2} \geq -1$.

Since $n^2 \geq 1 > 0$, then $n^2 > 0$, so $\frac{1}{n^2} > 0$.

Hence, $\frac{-1}{n^2} < 0$.

Since $(-1)^n = -1$ for all odd n, then $a_n = \frac{(-1)^n}{n^2} = \frac{-1}{n^2} < 0 < \frac{1}{4}$ and $a_n = \frac{(-1)^n}{n^2} = \frac{-1}{n^2} \geq -1$.

Thus, $a_n < \frac{1}{4}$ and $a_n \geq -1$.

Since $-1 \leq a_n$ and $a_n < \frac{1}{4}$, then $-1 \leq a_n < \frac{1}{4}$.

Therefore, $-1 \leq a_n \leq \frac{1}{4}$, as desired. □

Exercise 19. Let (a_n) be a sequence defined by $a_{n+1} = a_n + \frac{1}{n}$ for $n > 1$ with $a_1 = 1$.

Then (a_n) is strictly increasing.

Proof. Let $n \in \mathbb{N}$ be given.

Then $n \geq 1$, so either $n > 1$ or $n = 1$.

We consider these cases separately.

Case 1: Suppose $n = 1$.

Then $a_1 = 1 < 2 = 1 + 1 = a_1 + \frac{1}{1} = a_2$, so $a_1 < a_2$.

Case 2: Suppose $n > 1$.

Then $a_{n+1} = a_n + \frac{1}{n}$.

Since $n > 1 > 0$, then $n > 0$, so $\frac{1}{n} > 0$.

Thus, $a_{n+1} - a_n = \frac{1}{n} > 0$, so $a_{n+1} - a_n > 0$.

Hence, $a_{n+1} > a_n$, so $a_n < a_{n+1}$.

Therefore, $a_n < a_{n+1}$ for all $n \in N$, so (a_n) is strictly increasing.

Thus, (a_n) is monotonic increasing, so (a_n) is monotonic. □

Exercise 20. Let (a_n) be a sequence of real numbers defined recursively by $a_1 = 3$ and $a_{n+1} = \frac{3a_n+2}{a_n+2}$ for all $n \in \mathbb{N}$.

Then $a_n > 2$ for all $n \in \mathbb{N}$ and (a_n) is strictly decreasing.

Solution. We compute several terms of the sequence and observe that the terms are all greater than 2.

In fact, the terms seems to be approaching 2 (converging to 2).

We also graph this sequence as a function which suggests that the function is strictly decreasing.

Thus, we must prove these observations of this sequence are true. □

Proof. We prove $a_n > 2$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : a_n > 2\}$.

Since $1 \in \mathbb{N}$ and $a_1 = 3 > 2$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $a_k > 2$.

Observe that $a_k > 2 \Rightarrow a_k > 2 > -2 \Rightarrow a_k > -2 \Rightarrow a_k + 2 > 0$.

Hence, $a_k + 2 > 0$.

Thus,

$$\begin{aligned} a_k > 2 \quad &\Rightarrow \quad a_k + 2 > 4 \\ &\Rightarrow \quad 3a_k + 2 > 2a_k + 4 = 2(a_k + 2) \\ &\Rightarrow \quad 3a_k + 2 > 2(a_k + 2) \\ &\Rightarrow \quad \frac{3a_k + 2}{a_k + 2} > 2 \\ &\Rightarrow \quad a_{k+1} > 2. \end{aligned}$$

Therefore, $a_{k+1} > 2$.

Since $k + 1 \in \mathbb{N}$ and $a_{k+1} > 2$, then $k + 1 \in S$.

Thus, $k \in S$ implies $k + 1 \in S$.

By the principle of mathematical induction, $a_n > 2$ for all $n \in \mathbb{N}$, as desired. □

Proof. We prove (a_n) is strictly decreasing.

Let $n \in \mathbb{N}$ be given.

Since $a_n > 2$ for all $n \in \mathbb{N}$, then in particular, $a_n > 2$.

Since $a_n > 2 \Rightarrow a_n - 2 > 0$, then $a_n - 2 > 0$.

Since $a_n > 2 \Rightarrow a_n + 1 > 3 > 0 \Rightarrow a_n + 1 > 0$, then $a_n + 1 > 0$.

Since $a_n > 2 \Rightarrow a_n + 2 > 4 > 0 \Rightarrow a_n + 2 > 0$, then $a_n + 2 > 0$.

Observe that

$$\begin{aligned} a_n - a_{n+1} &= a_n - \frac{3a_n + 2}{a_n + 2} \\ &= \frac{a_n(a_n + 2) - (3a_n + 2)}{a_n + 2} \\ &= \frac{a_n^2 - a_n - 2}{a_n + 2} \\ &= \frac{(a_n - 2)(a_n + 1)}{a_n + 2} \\ &> 0. \end{aligned}$$

Thus, $a_n - a_{n+1} > 0$, so $a_n > a_{n+1}$.
Therefore, (a_n) is strictly decreasing.
Thus, (a_n) is monotonic decreasing, so (a_n) is monotonic. □

Convergent Sequences in $\mathbb{R}$

Exercise 21. Show that the sequence (x_n) defined by $x_n = \frac{n+1}{2n}$ is convergent.

Proof. To prove (x_n) is convergent, we prove $\lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$.
Let $\epsilon > 0$ be given.
Then $\frac{1}{2\epsilon} \in \mathbb{R}$.
Thus, by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{2\epsilon}$.
Let $n \in \mathbb{N}$ such that $n > N$.
Since $n \in \mathbb{N}$, then $n \geq 1 > 0$, so $n > 0$.
Since $n > N > \frac{1}{2\epsilon}$, then $n > \frac{1}{2\epsilon}$.
Since $n > \frac{1}{2\epsilon}$ and $\epsilon > 0$ and $n > 0$, then $\epsilon > \frac{1}{2n} > 0$.
Observe that

$$\begin{aligned} \left|\frac{n+1}{2n} - \frac{1}{2}\right| &= \left|\frac{n+1-n}{2n}\right| \\ &= \left|\frac{1}{2n}\right| \\ &= \frac{1}{2n} \\ &< \epsilon, \text{ as desired.} \end{aligned}$$

□

Exercise 22. Show that the sequence (x_n) defined by $x_n = \frac{3n+2}{n+1}$ is convergent.

Proof. To prove (x_n) is convergent, we prove $\lim_{n\to\infty} \frac{3n+2}{n+1} = 3$.
Let $\epsilon > 0$ be given.
Then $\frac{1}{\epsilon} \in \mathbb{R}$.
Thus, by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$.
Let $n \in \mathbb{N}$ such that $n > N$.
Since $n > N > \frac{1}{\epsilon}$, then $n > \frac{1}{\epsilon}$.
Since $n \in \mathbb{N}$, then $n \geq 1 > 0$, so $n > 0$.
Since $n > \frac{1}{\epsilon}$ and $\epsilon > 0$ and $n > 0$, then $\epsilon > \frac{1}{n}$.
Since $n + 1 > n > 0$, then $\frac{1}{n} > \frac{1}{n+1} > 0$.

Observe that

$$\begin{aligned}
|\frac{3n+2}{n+1}-3| &= |\frac{(3n+2)-3(n+1)}{n+1}| \\
&= |\frac{3n+2-3n-3}{n+1}| \\
&= |\frac{-1}{n+1}| \\
&= \frac{1}{n+1} \\
&< \frac{1}{n} \\
&< \epsilon, \text{ as desired.}
\end{aligned}$$

□

Exercise 23. The sequence (a_n) defined by $a_n = 5 + \frac{1}{n}$ is convergent.

Proof. To prove (a_n) is convergent, we prove $\lim_{n\to\infty} 5 + \frac{1}{n} = 5$.

Let $\epsilon > 0$ be given.

Then, by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N\epsilon > 1$, so $N > \frac{1}{\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $n > N > \frac{1}{\epsilon}$, so $n > \frac{1}{\epsilon}$.

Hence, $\epsilon > \frac{1}{n}$.

Since $n \in \mathbb{N}$, then $n > 0$, so $\frac{1}{n} > 0$.

Observe that

$$\begin{aligned}
|(5+\frac{1}{n})-5| &= |\frac{1}{n}| \\
&= \frac{1}{n} \\
&< \epsilon, \text{ as desired.}
\end{aligned}$$

□

Exercise 24. Show that $\lim_{n\to\infty} \frac{2-2n}{n} = -2$.

Proof. Let $\epsilon > 0$ be given.

Then $\frac{2}{\epsilon} \in \mathbb{R}$.

By the archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > \frac{2}{\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N > \frac{2}{\epsilon}$, then $n > \frac{2}{\epsilon}$, so $\epsilon > \frac{2}{n}$.

Observe that

$$\begin{aligned} |\frac{2-2n}{n}+2| &= |\frac{2}{n}-2+2| \\ &= |\frac{2}{n}| \\ &= \frac{2}{n} \\ &< \epsilon, \text{ as desired.} \end{aligned}$$

□

Exercise 25. Show that the sequence (a_n) defined by $a_n = \frac{1}{n^2+1}$ is convergent.

Proof. To prove (a_n) is convergent, we prove $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$.

Let $\epsilon > 0$ be given.

Then $\frac{1}{\epsilon} > 0$.

Hence, by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$.

Since $n \in \mathbb{N}$, then $N > 0$.

Since $N > \frac{1}{\epsilon}$ and $\epsilon > 0$ and $N > 0$, then $\epsilon > \frac{1}{N}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $0 < N$ and $N < n$, then $0 < N < n$, so $\frac{1}{n} < \frac{1}{N}$.

Since $n \in \mathbb{N}$, then $n \geq 1 > 0$, so $n^2 \geq n$.

Since $1 > 0$, then $n^2 + 1 > n^2$.

Since $n^2 + 1 > n^2 \geq n > 0$, then $n^2 + 1 > n > 0$, so $\frac{1}{n} > \frac{1}{n^2+1} > 0$.

Thus, $\frac{1}{n} > \frac{1}{n^2+1}$ and $\frac{1}{n^2+1} > 0$.

Therefore, $|\frac{1}{n^2+1}| = \frac{1}{n^2+1} < \frac{1}{n} < \frac{1}{N} < \epsilon$, so $|\frac{1}{n^2+1}| < \epsilon$, as desired. □

Exercise 26. Show that the sequence (x_n) defined by $x_n = \sqrt{n+1} - \sqrt{n}$ is convergent.

Proof. To prove (x_n) is convergent, we prove $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = 0$.

Let $\epsilon > 0$ be given.

Then $\frac{1}{\epsilon^2} \in \mathbb{R}$.

Thus, by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^2}$.

Since $N \in \mathbb{N}$, then $N > 0$.

Since $N > \frac{1}{\epsilon^2}$ and $\epsilon > 0$ and $N > 0$, then $\epsilon^2 > \frac{1}{N} > 0$, so $\epsilon > \frac{1}{\sqrt{N}} > 0$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N > 0$, then $\sqrt{n} > \sqrt{N} > 0$, so $\frac{1}{\sqrt{N}} > \frac{1}{\sqrt{n}} > 0$.

Since $0 < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \epsilon$, then $\frac{1}{\sqrt{n}} < \epsilon$.

Since $n \in \mathbb{N}$, then $n \geq 1 > 0$, so $n > 0$.

Since $n + 1 > n > 0$, then $\sqrt{n+1} > \sqrt{n} > 0$, so $\sqrt{n+1} > 0$ and $\sqrt{n} > 0$.

Since $\sqrt{n+1} > 0$, then $\sqrt{n+1} + \sqrt{n} > \sqrt{n}$.

Since $\sqrt{n+1} + \sqrt{n} > \sqrt{n} > 0$, then $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}+\sqrt{n}} > 0$ and $\sqrt{n+1} + \sqrt{n} > 0$.

Since $\sqrt{n+1} + \sqrt{n} > 0$, then $\sqrt{n+1} + \sqrt{n} \neq 0$.
Observe that

$$\begin{aligned}
|\sqrt{n+1} - \sqrt{n}| &= |(\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}| \\
&= |\frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}}| \\
&= |\frac{1}{\sqrt{n+1} + \sqrt{n}}| \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&< \frac{1}{\sqrt{n}} \\
&< \epsilon, \text{ as desired.}
\end{aligned}$$

□

Exercise 27. Show that $\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}$.

Proof. Let $\epsilon > 0$ be given.
Then $2\epsilon > 0$, so by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N(2\epsilon) > 1$, so $N > \frac{1}{2\epsilon}$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $n > N > \frac{1}{2\epsilon}$, so $n > \frac{1}{2\epsilon}$.
Since $n \in \mathbb{N}$, then $n > 0$, so $\epsilon > \frac{1}{2n}$.
Since $n \in \mathbb{N}$, then $n \geq 1$, so $n^2 \geq n \geq 1 > 0$.
Hence, $n^2 > 0$, so $n^2 + 1 > 1 > 0$.
Thus, $n^2 + 1 > 0$, so $2n^2 + 1 > n^2 \geq n > 0$.
Therefore, $2n^2 + 1 > n > 0$, so $\frac{1}{n} > \frac{1}{2n^2+1} > 0$.
Observe that

$$\begin{aligned}
|\frac{n^2}{2n^2+1} - \frac{1}{2}| &= |\frac{2n^2 - (2n^2+1)}{2(2n^2+1)}| \\
&= |\frac{-1}{2(2n^2+1)}| \\
&= \frac{1}{2(2n^2+1)} \\
&< \frac{1}{2n} \\
&< \epsilon, \text{ as desired.}
\end{aligned}$$

□

Exercise 28. Show that $\lim_{n\to\infty} 2^{-n} = 0$.

Proof. Let $\epsilon > 0$ be given.
Then $\frac{1}{\epsilon} > 0$, so $\frac{1}{\epsilon} - 1 \in \mathbb{R}$.
Thus, by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon} - 1$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $n > N > \frac{1}{\epsilon} - 1$, so $n > \frac{1}{\epsilon} - 1$.
Hence, $n + 1 > \frac{1}{\epsilon} > 0$, so $\epsilon > \frac{1}{n+1}$.

We prove $2^n \geq n + 1$ for all $n \in \mathbb{N}$ by induction.
Basis:
Since $2^1 = 2 = 1 + 1$, then the statement holds for $n = 1$.
Induction:
Let $k \in \mathbb{N}$ such that $2^k \geq k + 1$.
Since $k \in \mathbb{N}$, then $k > 0$.
Thus, $2^{k+1} = 2^k \cdot 2 = 2 \cdot 2^k \geq 2(k + 1) = 2k + 2 > k + 2 = (k + 1) + 1$, so $2^{k+1} > (k + 1) + 1$.
Hence, $2^{k+1} \geq (k + 1) + 1$.
Therefore, by PMI, $2^n \geq n + 1$ for all $n \in \mathbb{N}$.

We prove $n + 1 > 0$ for all $n \in \mathbb{N}$.
Let $n \in \mathbb{N}$ be given.
Then $n \geq 1 > -1$, so $n > -1$.
Hence, $n + 1 > 0$, so $n + 1 > 0$ for all $n \in \mathbb{N}$.

Since $2^n \geq n+1$ for all $n \in \mathbb{N}$ and $n+1 > 0$ for all $n \in \mathbb{N}$, then $2^n \geq n+1 > 0$ for all $n \in \mathbb{N}$.

Since $n \in \mathbb{N}$, then $2^n \geq n + 1 > 0$, so $\frac{1}{n+1} \geq \frac{1}{2^n}$.
Observe that

$$\begin{aligned} |2^{-n} - 0| &= |2^{-n}| \\ &= |\frac{1}{2^n}| \\ &= \frac{1}{2^n} \\ &\leq \frac{1}{n+1} \\ &< \epsilon\text{, as desired.} \end{aligned}$$

□

Exercise 29. Show that $\lim_{n\to\infty} \frac{3n}{2n+1} = \frac{3}{2}$.

Proof. Let $\epsilon > 0$ be given.
Then $\frac{3}{2\epsilon} \in \mathbb{R}$.

By the archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > \frac{3}{2\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N > \frac{3}{2\epsilon}$, then $n > \frac{3}{2\epsilon}$, so $\epsilon > \frac{3}{2n}$.

Since $n \in \mathbb{N}$, then $n > 0$, so $2n > n$.

Hence, $2n + 1 > n + 1 > n > 0$, so $2n + 1 > n > 0$.

Thus, $\frac{1}{n} > \frac{1}{2n+1}$, so $\frac{3}{2n} > \frac{3}{2(2n+1)}$.

Observe that

$$\begin{aligned}
|\frac{3n}{2n+1} - \frac{3}{2}| &= |\frac{(3n)2 - (2n+1)3}{(2n+1)2}| \\
&= |\frac{6n - 6n - 3}{2(2n+1)}| \\
&= |\frac{-3}{2(2n+1)}| \\
&= \frac{3}{2(2n+1)} \\
&< \frac{3}{2n} \\
&< \epsilon \text{ , as desired.}
\end{aligned}$$

□

Exercise 30. Show that $\lim_{n\to\infty} \frac{2n+3}{3n+4}$ exists.

Solution. Let (s_n) be a sequence defined by $s_n = \frac{2n+3}{3n+4}$ for all $n \in \mathbb{N}$.

To prove $\lim_{n\to\infty} \frac{2n+3}{3n+4}$ exists, we prove $\lim_{n\to\infty} \frac{2n+3}{3n+4} = \frac{2}{3}$.

Thus, we must prove $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |\frac{2n+3}{3n+4} - \frac{2}{3}| < \epsilon$. □

Proof. Let $\epsilon > 0$ be given.

Then $\frac{1}{9\epsilon} \in \mathbb{R}$.

By the archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > \frac{1}{9\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N > \frac{1}{9\epsilon}$, then $n > \frac{1}{9\epsilon}$, so $\epsilon > \frac{1}{9n}$.

Observe that

$$\begin{aligned}
|\frac{2n+3}{3n+4} - \frac{2}{3}| &= |\frac{(2n+3)3-(3n+4)2}{(3n+4)3}| \\
&= |\frac{1}{3(3n+4)}| \\
&= \frac{1}{3(3n+4)} \\
&= \frac{1}{9n+12} \\
&< \frac{1}{9n} \\
&< \epsilon \text{ , as desired.}
\end{aligned}$$

□

Exercise 31. Show that the sequence defined by $a_n = \frac{1}{n}$ does not converge to 0.01.

Solution. We know that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Since $\lim_{n\to\infty} \frac{1}{n} = 0.01$ iff $(\forall\epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |\frac{1}{n} - 0.01| < \epsilon)$, then $\lim_{n\to\infty} \frac{1}{n} \neq 0.01$ iff $(\exists\epsilon > 0)(\forall N \in \mathbb{N})(\exists n \in \mathbb{N})(n > N \wedge |\frac{1}{n} - 0.01| \geq \epsilon)$.

After trying various values of ϵ, we see that any $\epsilon < 0.01$ will work because this will guarantee that for each N there is some term a_n with $n > N$ such that a_n lies outside the ϵ band. So, we just pick some $\epsilon < 0.01$, such as $\epsilon = \frac{1}{2} \cdot 0.01 = 0.005$.

We want to find conditions that will guarantee that $|\frac{1}{n} - 0.01| \geq 0.005$.

Suppose $|\frac{1}{n} - 0.01| \geq 0.005$.

Then either $\frac{1}{n} - 0.01 \geq 0.005$ or $\frac{1}{n} - 0.01 \leq -0.005$, so either $\frac{1}{n} \geq 0.015$ or $\frac{1}{n} \leq 0.005$.

Thus, either $\frac{1}{0.015} \geq n$ or $\frac{1}{0.005} \leq n$, so either $66.67 \geq n$ or $200 \leq n$.

Hence, either $n \leq 66.67$ or $n \geq 200$.

We drop $n \leq 66.67$ condition since we want to consider some subsequence of (a_n).

Thus, we want $n \geq 200$.

We can associate 1 to a_{200} and 2 to a_{201} and 3 to a_{202}, etc.

Thus, we define a function $f : \mathbb{N} \to \mathbb{R}$ by $f(n) = a_{n+199}$ for all $n \in \mathbb{N}$.

Or, we could do the following since we want $n \geq 200$ for any $n \in \mathbb{N}$.

Let $N \in \mathbb{N}$.

If $N \in \mathbb{N}$, then if $n > N$, either $n \geq 200$ or $n < 200$.

If $n < 200$, we want to pick 200 since $200 \geq 200$.

If $n \geq 200$, we just pick n since $n \geq 200$.

Thus, we just pick the maximum of n and 200. □

Proof. Choose $\epsilon = 0.005$.

Let $N \in \mathbb{N}$.

Let $M = \max\{N, 200\}$.

Since $M \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > M$.

Since $n > M \geq N$, then $n > N$.

Since $n > M \geq 200$, then $n > 200$.

Hence, $\frac{1}{n} < \frac{1}{200}$, so $-\frac{1}{n} > -\frac{1}{200} = -0.005$.

Thus, $-\frac{1}{n} > -0.005$.

Since

$$\begin{aligned} |\frac{1}{n} - 0.01| &\geq |0.01| - |\frac{1}{n}| \\ &= 0.01 - \frac{1}{n} \\ &> 0.005 \\ &= \epsilon, \end{aligned}$$

then $|\frac{1}{n} - 0.01 > \epsilon$, so $|\frac{1}{n} - 0.01 \geq \epsilon$. □

Exercise 32. Show that $\lim_{n\to\infty} \frac{1}{n} \neq \frac{1}{4}$.

Solution. We know that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Since $\lim_{n\to\infty} \frac{1}{n} = \frac{1}{4}$ iff $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |\frac{1}{n} - \frac{1}{4}| < \epsilon)$, then $\lim_{n\to\infty} \frac{1}{n} \neq \frac{1}{4}$ iff $(\exists \epsilon > 0)(\forall N \in \mathbb{N})(\exists n \in \mathbb{N})(n > N \wedge |\frac{1}{n} - \frac{1}{4}| \geq \epsilon)$.

After trying various values of ϵ, we see that any $\epsilon < \frac{1}{4}$ will work because this will guarantee that for each N there is some term a_n with $n > N$ such that a_n lies outside the ϵ band.

So, we just pick some $\epsilon < \frac{1}{4}$, such as $\epsilon = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.

We want to find conditions that will guarantee that $|\frac{1}{n} - \frac{1}{4}| \geq \frac{1}{8}$.

Suppose $|\frac{1}{n} - \frac{1}{4}| \geq \frac{1}{8}$.

Then either $\frac{1}{n} - \frac{1}{4} \geq \frac{1}{8}$ or $\frac{1}{n} - \frac{1}{4} \leq -\frac{1}{8}$, so either $\frac{1}{n} \geq \frac{3}{8}$ or $\frac{1}{n} \leq \frac{1}{8}$.

Thus, either $\frac{8}{3} \geq n$ or $8 \leq n$, so either $2.66 \geq n$ or $n \geq 8$.

We drop $2.66 \geq n$ condition since we want to consider some subsequence of (a_n).

Thus, we want $n \geq 8$.

We can associate 1 to a_8 and 2 to a_9 and 3 to a_{10}, etc.

Thus, we define a function $f : \mathbb{N} \to \mathbb{R}$ by $f(n) = a_{n+7}$ for all $n \in \mathbb{N}$. □

Proof. Choose $\epsilon = \frac{1}{8}$.

Let $N \in \mathbb{N}$ be given.

Then $N \geq 1$.

Since $N \geq 1 \Rightarrow N + 7 \geq 8 \Rightarrow \frac{1}{N+7} \leq \frac{1}{8} \Rightarrow \frac{-1}{N+7} \geq \frac{-1}{8}$, then $\frac{-1}{N+7} \geq \frac{-1}{8}$.

Let $n = N + 7$.

Then $n \in \mathbb{N}$.

Since $N + 7 > N$, then $n > N$.

Observe that

$$\begin{aligned}
|\frac{1}{N+7} - \frac{1}{4}| &\geq |\frac{1}{4}| - |\frac{1}{N+7}| \\
&= \frac{1}{4} - \frac{1}{N+7} \\
&\geq \frac{1}{8} \\
&= \epsilon.
\end{aligned}$$

Thus, $|\frac{1}{N+7} - \frac{1}{4}| \geq \epsilon$, as desired. ☐

Exercise 33. Show that $\lim_{n\to\infty} \frac{2n+1}{3n-1} = \frac{2}{3}$.

Proof. Let $\epsilon > 0$ be given.

Then $\frac{1}{\epsilon} \in \mathbb{R}$, so $\frac{5}{3\epsilon} \in \mathbb{R}$.

Thus, by the Archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > \frac{5}{3\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N > \frac{5}{3\epsilon}$, then $n > \frac{5}{3\epsilon}$, so $\frac{3\epsilon}{5} > \frac{1}{n}$.

Since $n \in \mathbb{N}$, then $n \geq 1 > \frac{1}{2}$, so $n > \frac{1}{2}$.

Thus, $2n > 1$, so $2n - 1 > 0$.

Hence, $3n - 1 > n > 0$, so $\frac{3\epsilon}{5} > \frac{1}{n} > \frac{1}{3n-1} > 0$.

Thus, $0 < \frac{1}{3n-1} < \frac{1}{n} < \frac{3\epsilon}{5}$, so $\frac{1}{3n-1} < \frac{3\epsilon}{5}$.

Observe that

$$\begin{aligned}
|\frac{2n+1}{3n-1} - \frac{2}{3}| &= |\frac{3(2n+1) - 2(3n-1)}{3(3n-1)}| \\
&= |\frac{5}{3(3n-1)}| \\
&= \frac{5}{3} \cdot \frac{1}{3n-1} \\
&< \frac{5}{3} \cdot \frac{3\epsilon}{5} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|\frac{2n+1}{3n-1} - \frac{2}{3}| < \epsilon$, as desired. ☐

Proof. Let (a_n) be a sequence of real numbers defined by $a_n = \frac{2n+1}{3n-1}$ for all $n \in \mathbb{N}$.

We prove (a_n) is strictly decreasing.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1$.

Since $n \geq 1 > \frac{1}{3} \Rightarrow n > \frac{1}{3} \Rightarrow 3n > 1 \Rightarrow 3n - 1 > 0$, then $3n - 1 > 0$.

Since $n \geq 1 > 0 \Rightarrow n > 0 \Rightarrow 3n > 0 \Rightarrow 3n + 2 > 2 > 0 \Rightarrow 3n + 2 > 0$, then $3n + 2 > 0$.

Observe that

$$\begin{aligned}
a_n - a_{n+1} &= \frac{2n+1}{3n-1} - \frac{2(n+1)+1}{3(n+1)-1} \\
&= \frac{2n+1}{3n-1} - \frac{2n+3}{3n+2} \\
&= \frac{(2n+1)(3n+2)-(3n-1)(2n+3)}{(3n-1)(3n+2)} \\
&= \frac{(6n^2+7n+2)-(6n^2+7n-3)}{(3n-1)(3n+2)} \\
&= \frac{5}{(3n-1)(3n+2)} \\
&> 0.
\end{aligned}$$

Therefore, $a_n - a_{n+1} > 0$, so $a_n > a_{n+1}$, as desired. ☐

Exercise 34. Show that $\lim_{n\to\infty} \frac{2n^2+3n-5}{5n^2+2n+1} = \frac{2}{5}$.

Solution. Let (a_n) be a sequence of real numbers defined by $a_n = \frac{2n^2+3n-5}{5n^2+2n+1}$ for all $n \in \mathbb{N}$.

To prove $\lim_{n\to\infty} a_n = \frac{2}{5}$, we must prove $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |\frac{2n^2+3n-5}{5n^2+2n+1} - \frac{2}{5}| < \epsilon)$.

We work backwards. ☐

Proof. Let $\epsilon > 0$ be given.

Let $M = \max\{\frac{11}{25\epsilon}, 2\}$.

Since $M \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > M$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N > M \geq 2$, then $n > 2$, so $n \geq 3$.

Thus, $n \geq 3 > \frac{27}{11}$, so $n > \frac{27}{11}$.

Hence, $11n > 27$, so $11n - 27 > 0$.

Therefore, $11 - \frac{27}{n} > 0$ and $11 > 11 - \frac{27}{n}$.

Thus, $0 < 11 - \frac{27}{n} < 11$.

Since $n > 0$, then $5 + \frac{2}{n} + \frac{1}{n^2} > 5$.

Hence, $0 < \frac{1}{5+\frac{2}{n}+\frac{1}{n^2}} < \frac{1}{5}$.

Thus, $\frac{11-\frac{27}{n}}{5+\frac{2}{n}+\frac{1}{n^2}} < \frac{11}{5}$.

Multiplying by positive $\frac{1}{5n}$ we obtain $\frac{11-\frac{27}{n}}{5n(5+\frac{2}{n}+\frac{1}{n^2})} < \frac{11}{25n}$.

Since $n > N > M \geq \frac{11}{25\epsilon}$, then $n > \frac{11}{25\epsilon}$, so $\epsilon > \frac{11}{25n}$.

Thus, $\frac{11-\frac{27}{n}}{5n(5+\frac{2}{n}+\frac{1}{n^2})} < \frac{11}{25n} < \epsilon$, so $\frac{11-\frac{27}{n}}{5n(5+\frac{2}{n}+\frac{1}{n^2})} < \epsilon$.

Therefore, $\frac{11n-27}{5n^2(5+\frac{2}{n}+\frac{1}{n^2})} < \epsilon$.

Observe that

$$
\begin{aligned}
|\frac{2n^2+3n-5}{5n^2+2n+1} - \frac{2}{5}| &= |\frac{5(2n^2+3n-5) - 2(5n^2+2n+1)}{5(5n^2+2n+1)}| \\
&= |\frac{10n^2+15n-25-10n^2-4n-2}{5(5n^2+2n+1)}| \\
&= |\frac{11n-27}{5(5n^2+2n+1)}| \\
&= \frac{11n-27}{5(5n^2+2n+1)} \\
&= \frac{11n-27}{5n^2(5+\frac{2}{n}+\frac{1}{n^2})} \\
&< \epsilon.
\end{aligned}
$$

Therefore, $|\frac{2n^2+3n-5}{5n^2+2n+1} - \frac{2}{5}| < \epsilon$, as desired. □

Exercise 35. Let $a_n = \frac{n^2+2}{5n^2+1}$ for all $n \in \mathbb{N}$.
Then $a_n \to \frac{1}{5}$.
Find $N_1 \in \mathbb{N}$ so that if $n > N_1$, then $|a_n - \frac{1}{5}| < 0.02$.
Find $N \in \mathbb{N}$ so that if $n > N_2$, then $|a_n - \frac{1}{5}| < 1.8$.

Solution. We can graph this sequence and observe that it is strictly decreasing and bounded above by $a_1 = 0.5$ and below by $\frac{1}{5}$.
Let $\epsilon_1 = 0.02$.
Let $N_1 = 5$.
Let $n > N_1$.
Then $n > 5$.
Thus,

$$
\begin{aligned}
n > 5 &\Rightarrow n^2 > 25 \\
&\Rightarrow 5n^2 > 125 \\
&\Rightarrow 5n^2 + 1 > 126 \\
&\Rightarrow \frac{1}{5n^2+1} < \frac{1}{126} \\
&\Rightarrow \frac{9}{5(5n^2+1)} < \frac{1}{70}.
\end{aligned}
$$

Hence, $\frac{9}{5(5n^2+1)} < \frac{1}{70}$.

Observe that

$$\begin{aligned}
|a_n - \frac{1}{5}| &= |\frac{n^2+2}{5n^2+1} - \frac{1}{5}| \\
&= |\frac{5(n^2+2)-(5n^2+1)}{5(5n^2+1)}| \\
&= |\frac{9}{5(5n^2+1)}| \\
&= \frac{9}{5(5n^2+1)} \\
&< \frac{1}{70} \\
&< \epsilon.
\end{aligned}$$

□

Proof. To prove $a_n \to \frac{1}{5}$, let $\epsilon > 0$ be given.
Either $\epsilon \geq \frac{9}{5}$ or $\epsilon < \frac{9}{5}$.
We consider these cases separately.
Case 1: Suppose $\epsilon \geq \frac{9}{5}$.
Let $N = 1$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $n > 1$.
Since $n > 0$, then $n^2 > n$, so $n^2 > n > 1$.
Hence, $n^2 > 1$.
Since

$$\begin{aligned}
n^2 > 1 &\Rightarrow 5n^2 > 5 \\
&\Rightarrow 5n^2 + 1 > 6 > 1 \\
&\Rightarrow 5n^2 + 1 > 1 \\
&\Rightarrow 0 < \frac{1}{5n^2+1} < 1 \\
&\Rightarrow \frac{9}{5(5n^2+1)} < \frac{9}{5},
\end{aligned}$$

then $\frac{9}{5(5n^2+1)} < \frac{9}{5}$.

Observe that

$$\begin{aligned} |a_n - \frac{1}{5}| &= |\frac{n^2+2}{5n^2+1} - \frac{1}{5}| \\ &= |\frac{5(n^2+2)-(5n^2+1)}{5(5n^2+1)}| \\ &= |\frac{9}{5(5n^2+1)}| \\ &= \frac{9}{5(5n^2+1)} \\ &< \frac{9}{5} \\ &\leq \epsilon. \end{aligned}$$

Therefore, $|a_n - \frac{1}{5}| < \epsilon$.

Case 2: Suppose $\epsilon < \frac{9}{5}$.

Then $\frac{9}{5} > \epsilon$, so $9 > 5\epsilon$.

Hence, $9 - 5\epsilon > 0$.

Thus, $\frac{9-5\epsilon}{25\epsilon} > 0$, so $\frac{9}{25\epsilon} - \frac{1}{5} > 0$.

Hence, $\sqrt{\frac{9}{25\epsilon} - \frac{1}{5}} > 0$.

Since $\sqrt{\frac{9}{25\epsilon} - \frac{1}{5}} \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > \sqrt{\frac{9}{25\epsilon} - \frac{1}{5}}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N$ and $N > \sqrt{\frac{9}{25\epsilon} - \frac{1}{5}}$, then $n > \sqrt{\frac{9}{25\epsilon} - \frac{1}{5}}$.

Since

$$\begin{aligned} n > \sqrt{\frac{9}{25\epsilon} - \frac{1}{5}} > 0 &\Rightarrow n^2 > \frac{9}{25\epsilon} - \frac{1}{5} \\ &\Rightarrow 25n^2 > \frac{9}{\epsilon} - 5 \\ &\Rightarrow 25n^2 + 5 > \frac{9}{\epsilon} \\ &\Rightarrow \epsilon > \frac{9}{25n^2+5}, \end{aligned}$$

then $\epsilon > \frac{9}{25n^2+5}$.

Observe that

$$\begin{aligned}
|a_n - \frac{1}{5}| &= |\frac{n^2+2}{5n^2+1} - \frac{1}{5}| \\
&= |\frac{5(n^2+2)-(5n^2+1)}{5(5n^2+1)}| \\
&= |\frac{9}{5(5n^2+1)}| \\
&= \frac{9}{5(5n^2+1)} \\
&= \frac{9}{25n^2+5} \\
&< \epsilon.
\end{aligned}$$

Therefore, $|a_n - \frac{1}{5}| < \epsilon$. □

Proof. We prove (a_n) is strictly decreasing.
Let $n \in \mathbb{N}$ be given.
Then

$$\begin{aligned}
a_n - a_{n+1} &= \frac{n^2+2}{5n^2+1} - \frac{(n+1)^2+2}{5(n+1)^2+1} \\
&= \frac{n^2+2}{5n^2+1} - \frac{n^2+2n+3}{5n^2+10n+6} \\
&= \frac{(n^2+2)(5n^2+10n+6)-(5n^2+1)(n^2+2n+3)}{(5n^2+1)(5n^2+10n+6)} \\
&= \frac{(5n^4+10n^3+16n^2+20n+12)-(5n^4+10n^3+16n^2+2n+3)}{(5n^2+1)(5n^2+10n+6)} \\
&= \frac{18n+9}{(5n^2+1)(5n^2+10n+6)} \\
&> 0.
\end{aligned}$$

Hence, $a_n - a_{n+1} > 0$, so $a_n > a_{n+1}$, as desired. □

Proof. We prove $\frac{1}{5} < a_n \leq \frac{1}{2}$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : \frac{1}{5} < a_n \leq \frac{1}{2}\}$.
Since $a_1 = \frac{1}{2}$ and $\frac{1}{5} < \frac{1}{2}$, then $\frac{1}{5} < a_1 \leq \frac{1}{2}$.
Hence, $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $\frac{1}{5} < a_k \leq \frac{1}{2}$.
Hence, $\frac{1}{5} < a_k$ and $a_k \leq \frac{1}{2}$.
Since (a_n) is strictly decreasing, then $a_k > a_{k+1}$.
Since $a_{k+1} < a_k$ and $a_k \leq \frac{1}{2}$, then $a_{k+1} < \frac{1}{2}$.
Suppose $a_{k+1} \leq \frac{1}{5}$.
Then $\frac{(k+1)^2+2}{5(k+1)^2+1} \leq \frac{1}{5}$.

Thus, $\frac{k^2+2k+3}{5k^2+10k+6} \leq \frac{1}{5}$, so $5k^2 + 10k + 15 \leq 5k^2 + 10k + 6$.
Hence, $15 \leq 6$, a contradiction.
Therefore, $a_{k+1} > \frac{1}{5}$.
Thus, $\frac{1}{5} < a_{k+1}$ and $a_{k+1} < \frac{1}{2}$, so $\frac{1}{5} < a_{k+1} < \frac{1}{2}$.
Hence, $k + 1 \in S$.
Therefore, by PMI, $S = \mathbb{N}$, so $\frac{1}{5} < a_n \leq \frac{1}{2}$ for all $n \in \mathbb{N}$.
Thus, (a_n) is bounded above by $\frac{1}{2}$ and below by $\frac{1}{5}$, so (a_n) is bounded.
Since (a_n) is a decreasing convergent sequence, then (a_n) converges to its greatest lower bound, namely, $\frac{1}{2}$. ☐

Exercise 36. Let (a_n) and $(-a_n)$ be sequences of real numbers.
Let $L \in \mathbb{R}$.
Then (a_n) converges to L iff $(-a_n)$ converges to $-L$.
Therefore, $\lim_{n\to\infty} a_n$ exists iff $\lim_{n\to\infty} -a_n$ exists.
Hence, $\lim_{n\to\infty} a_n$ does not exist iff $\lim_{n\to\infty} -a_n$ does not exist.
Therefore, (a_n) diverges iff $(-a_n)$ diverges.

Proof. Observe that

$$\begin{aligned}
\lim_{n\to\infty} a_n = L \quad &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \rightarrow |a_n - L| < \epsilon) \quad &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \rightarrow |-(a_n - L)| < \epsilon) \quad &\Leftrightarrow \\
(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \rightarrow |-a_n - (-L)| < \epsilon) \quad &\Leftrightarrow \\
\lim_{n\to\infty} -a_n = -L.&
\end{aligned}$$

☐

Exercise 37. Show that $\lim_{n\to\infty} 3 - \frac{1}{n} = 3$.

Solution. We prove this statement in multiple ways- using definition of limit or other propositions/theorems. ☐

Proof. Let $\epsilon > 0$ be given.
Since $\epsilon > 0$, then $\frac{1}{\epsilon} \in \mathbb{R}$.
By the archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > \frac{1}{\epsilon}$.
Let $n \in \mathbb{N}$ such that $n > N$.
Since $n > N > \frac{1}{\epsilon}$, then $n > \frac{1}{\epsilon}$.
Hence, $\epsilon > \frac{1}{n}$.
Observe that

$$\begin{aligned}
|(3 - \frac{1}{n}) - 3| &= |-\frac{1}{n}| \\
&= |\frac{1}{n}| \\
&= \frac{1}{n} \\
&< \epsilon \text{ , as desired.}
\end{aligned}$$

□

Proof. Observe that

$$\begin{aligned} 3 &= 3 - 0 \\ &= \lim_{n\to\infty} 3 - \lim_{n\to\infty} \frac{1}{n} \\ &= \lim_{n\to\infty} (3 - \frac{1}{n}). \end{aligned}$$

□

Proof. We prove a_n is bounded above by 3 and below by 2.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1$, so $n > 0$.

Observe that

$$\begin{aligned} n > 0 &\Rightarrow \frac{1}{n} > 0 \\ &\Rightarrow 3 + \frac{1}{n} > 3 \\ &\Rightarrow 3 > 3 - \frac{1}{n} \\ &\Rightarrow 3 - \frac{1}{n} < 3. \end{aligned}$$

Thus, 3 is an upper bound of (a_n).

Observe that

$$\begin{aligned} n \geq 1 &\Rightarrow 1 \geq \frac{1}{n} \\ &\Rightarrow 3 - 2 \geq \frac{1}{n} \\ &\Rightarrow 3 - \frac{1}{n} \geq 2 \\ &\Rightarrow 2 \leq 3 - \frac{1}{n}. \end{aligned}$$

Thus, 2 is a lower bound of (a_n).

Therefore, (a_n) is bounded. □

Exercise 38. Let (s_n) be a sequence of real numbers.

Let $L \in \mathbb{R}$.

Then

$\lim_{n\to\infty} s_n = L$ iff for every natural number m there is a real number N such that $|s_n - L| < \frac{1}{m}$ whenever $n > N$.

Solution. We must prove $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |s_n - L| < \epsilon)$ iff $(\forall m \in \mathbb{N})(\exists N \in \mathbb{R})(\forall n \in \mathbb{N})(n > N \to |s_n - L| < \frac{1}{m})$. □

Proof. We prove if $\lim_{n\to\infty} s_n = L$, then for every natural number m, there exists a real number N such that $|s_n - L| < \frac{1}{m}$ whenever $n > N$.

Suppose $\lim_{n\to\infty} s_n = L$.

Then for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|s_n - L| < \epsilon$ whenever $n > N$.

Let $m \in \mathbb{N}$.

Then $m \geq 1 > 0$, so $m > 0$.

Hence, $\frac{1}{m} > 0$.

Let $\epsilon = \frac{1}{m}$.

Then there exists $N \in \mathbb{N}$ such that $|s_n - L| < \frac{1}{m}$ whenever $n > N$.

Since $N \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{R}$, then $N \in \mathbb{R}$.

Thus, there is a real number N such that $|s_n - L| < \frac{1}{m}$ whenever $n > N$, as desired.

Conversely, we prove if for every natural number m there is a real number N such that $|s_n - L| < \frac{1}{m}$ whenever $n > N$, then $\lim_{n\to\infty} s_n = L$.

Suppose for every natural number m there is a real number N such that $|s_n - L| < \frac{1}{m}$ whenever $n > N$.

To prove $\lim_{n\to\infty} s_n = L$, let $\epsilon > 0$ be given.

We must prove there is a natural number N' such that $|s_n - L| < \epsilon$ whenever $n > N'$.

Since $\epsilon > 0$, then $\frac{1}{\epsilon} > 0$.

Since $\frac{1}{\epsilon} \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists $m \in \mathbb{N}$ such that $m > \frac{1}{\epsilon}$.

Hence, $\epsilon > \frac{1}{m}$.

Since $m \in \mathbb{N}$, then there is a real number N such that $|s_n - L| < \frac{1}{m}$ whenever $n > N$.

Since $N \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists $N' \in \mathbb{N}$ such that $N' > N$.

Let $n \in \mathbb{N}$ such that $n > N'$.

Then $n > N' > N$, so $n > N$.

Hence, $|s_n - L| < \frac{1}{m}$.

Therefore, $|s_n - L| < \frac{1}{m}$ and $\frac{1}{m} < \epsilon$, so $|s_n - L| < \epsilon$, as desired. □

Exercise 39. Show that $\lim_{n\to\infty} \frac{1+2+3+\ldots+n}{n^2} = \frac{1}{2}$.

Proof. Let (s_n) be a sequence defined by $s_n = \frac{1+2+3+\ldots+n}{n^2}$ for all $n \in \mathbb{N}$.

Since the sum of the first n natural numbers is $\frac{n(n+1)}{2}$, then $s_n = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n}$.

Note that this is a strictly decreasing sequence (We could prove this fact formally, but we won't here.)

To prove $\lim_{n\to\infty} s_n = \frac{1}{2}$, let $\epsilon > 0$ be given.

We must prove there is a natural number N such that for each natural number n, if $n > N$, then $|\frac{n+1}{2n} - \frac{1}{2}| < \epsilon$.

Since $\epsilon > 0$, then $\frac{1}{2\epsilon} \in \mathbb{R}$.

By the archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > \frac{1}{2\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.
Since $n > N$ and $N > \frac{1}{2\epsilon}$, then $n > \frac{1}{2\epsilon}$.
Hence, $\epsilon > \frac{1}{2n}$.
Observe that

$$\begin{aligned}
|\frac{n+1}{2n} - \frac{1}{2}| &= |\frac{(n+1)2 - (2n)1}{(2n)2}| \\
&= |\frac{2}{2(2n)}| \\
&= |\frac{1}{2n}| \\
&= \frac{1}{2n} \\
&< \epsilon \text{ , as desired.}
\end{aligned}$$

□

Exercise 40. Show that $\lim_{n\to\infty} \frac{1^2+2^2+3^2+...+n^2}{n^3} = \frac{1}{3}$.

Proof. Let (s_n) be a sequence defined by $s_n = \frac{1^2+2^2+3^2+...+n^2}{n^3}$ for all $n \in \mathbb{N}$.

Since the sum of the squares of the first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$, then $s_n = \frac{n(n+1)(2n+1)}{6n^3} = \frac{(n+1)(2n+1)}{6n^2}$.

Note that this is a strictly decreasing sequence (We could prove this fact formally, but we won't here.)

To prove $\lim_{n\to\infty} s_n = \frac{1}{3}$, let $\epsilon > 0$ be given.

Then $\frac{2}{3\epsilon} \in \mathbb{R}$.

Thus, by the Archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > \frac{2}{3\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N > \frac{2}{3\epsilon}$, then $n > \frac{2}{3\epsilon}$, so $\epsilon > \frac{2}{3n}$.

Since $n \in \mathbb{N}$, then $n \geq 1$.

Since

$$\begin{aligned}
n \geq 1 &\Rightarrow 0 < \frac{1}{n} \leq 1 \\
&\Rightarrow \frac{1}{6n} \leq \frac{1}{6} \\
&\Rightarrow \frac{1}{2} + \frac{1}{6n} \leq \frac{2}{3} \\
&\Rightarrow \frac{1}{n}(\frac{1}{2} + \frac{1}{6n}) \leq \frac{2}{3n},
\end{aligned}$$

then $\frac{1}{n}(\frac{1}{2} + \frac{1}{6n}) \leq \frac{2}{3n}$.

Observe that

$$\begin{aligned}
|\frac{(n+1)(2n+1)}{6n^2} - \frac{1}{3}| &= |\frac{2n^2+3n+1}{6n^2} - \frac{1}{3}| \\
&= |\frac{6n^2+9n+3-6n^2}{18n^2}| \\
&= |\frac{9n+3}{18n^2}| \\
&= |\frac{1}{2n} + \frac{1}{6n^2}| \\
&= \frac{1}{2n} + \frac{1}{6n^2} \\
&= \frac{1}{n}(\frac{1}{2} + \frac{1}{6n}) \\
&\leq \frac{2}{3n} \\
&< \epsilon \text{ , as desired.}
\end{aligned}$$

□

Exercise 41. Let $L \in \mathbb{R}$.

If (s_n) converges to L, then $(2s_n)$ converges to $2L$.

Solution. Suppose $s_n \to L$.

To prove $2s_n \to 2L$, we must prove $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |2s_n - 2L| < \epsilon)$. □

Proof. Suppose $s_n \to L$.

To prove $2s_n \to 2L$, let $\epsilon > 0$ be given.

Then $\frac{\epsilon}{2} > 0$.

Since $s_n \to L$, then there exists $N \in \mathbb{N}$ such that $|s_n - L| < \frac{\epsilon}{2}$ whenever $n > N$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|s_n - L| < \frac{\epsilon}{2}$.

Hence,

$$\begin{aligned}
|2s_n - 2L| &= |2(s_n - L)| \\
&= 2|s_n - L| \\
&< 2(\frac{\epsilon}{2}) \\
&= \epsilon.
\end{aligned}$$

Therefore $|2s_n - 2L| < \epsilon$, as desired. □

Exercise 42. Let (a_n) be a sequence of real numbers.

If there exist $L \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $a_n = L$ for all $n > N$, then $a_n \to L$.

Proof. Suppose there exist $L \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $a_n = L$ for all $n > N$.
To prove $a_n \to L$, let $\epsilon > 0$ be given.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $a_n = L$.
Thus, $|a_n - L| = |L - L| = 0 < \epsilon$.
Therefore, $|a_n - L| < \epsilon$, so $a_n \to L$, as desired. □

Exercise 43. Let (a_n) be a sequence of real numbers.
If $\lim_{n\to\infty} a_n = L$ and $L > 0$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $\frac{L}{2} < a_n < 2L$.

Proof. Suppose $\lim_{n\to\infty} a_n = L$ and $L > 0$.
Since $L > 0$, then $\frac{L}{2} > 0$.
Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n > N$, then $|a_n - L| < \frac{L}{2}$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $|a_n - L| < \frac{L}{2}$, so $-\frac{L}{2} < a_n - L < \frac{L}{2}$.
Hence, $\frac{L}{2} < a_n < \frac{3L}{2}$.
Since $\frac{3}{2} < 2$ and $L > 0$, then $\frac{3L}{2} < 2L$.
Since $\frac{L}{2} < a_n < \frac{3L}{2}$ and $\frac{3L}{2} < 2L$, then $\frac{L}{2} < a_n < \frac{3L}{2} < 2L$, so $\frac{L}{2} < a_n < 2L$. □

Exercise 44. Let (a_n) be a sequence of positive real numbers.
Let $L > 0$.
If $\lim_{n\to\infty} a_n = L$, then there exists $M > 0$ such that $a_n > M$ for all $n \in \mathbb{N}$.

Proof. Suppose $\lim_{n\to\infty} a_n = L$.
We must prove there exists $M > 0$ such that $a_n > M$ for all $n \in \mathbb{N}$.
Since $L > 0$, then $\frac{L}{2} > 0$.
Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < \frac{L}{2}$.
Let $S = \{s_k : 1 \leq k \leq N\}$.
Then $S \subset \mathbb{R}$.
Since $s_1 \in S$, then S is not empty.
Since S contains at most N elements, then S is finite.
Hence, S is a nonempty finite set of real numbers.
Therefore, $\min S$ exists.
Let $M = \min\{\frac{L}{2}, \frac{\min S}{2}\}$.
We prove $M > 0$.
Either $M = \frac{L}{2}$ or $M = \frac{\min S}{2}$.
We consider these cases separately.
Case 1: Suppose $M = \frac{L}{2}$.
Then $M = \frac{L}{2} > 0$, so $M > 0$.
Case 2: Suppose $M = \frac{\min S}{2}$.
Since $\min S \in S$, then there exists $k \in \mathbb{N}$ such that $1 \leq k \leq N$ and $\min S = s_k$.
Since all terms of (a_n) are positive, then $a_n > 0$ for all $n \in \mathbb{N}$.

Hence, $s_k > 0$, so $\min S > 0$.
Thus, $\frac{\min S}{2} > 0$, so $M > 0$.
Let $n \in \mathbb{N}$.
We prove $a_n > M$.
Either $n \leq N$ or $n > N$.
We consider these cases separately.
Case 1: Suppose $n > N$.
Then $|a_n - L| < \frac{L}{2}$.
Observe that

$$\begin{aligned} |a_n - L| < \frac{L}{2} &\Leftrightarrow -\frac{L}{2} < a_n - L < \frac{L}{2} \\ &\Leftrightarrow \frac{L}{2} < a_n < \frac{3L}{2} \\ &\Rightarrow M \leq \frac{L}{2} < a_n < \frac{3L}{2} \\ &\Rightarrow M < a_n. \end{aligned}$$

Therefore, $a_n > M$.
Case 2: Suppose $n \leq N$.
Then $1 \leq n \leq N$, so $a_n \in S$.
Since $M \leq \frac{\min S}{2} < \min S \leq a_n$, then $M < a_n$.
Therefore $a_n > M$. □

Algebraic properties of convergent sequences

Exercise 45. Let (a_n) and (b_n) be sequences of real numbers.
If (a_n) and $(a_n + b_n)$ are convergent, then (b_n) is convergent.

Proof. Suppose (a_n) and $(a_n + b_n)$ are convergent
Since (a_n) is convergent, then there exists $L_1 \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = L_1$.
Since (a_n+b_n) is convergent, then there exists $L_2 \in \mathbb{R}$ such that $\lim_{n\to\infty}(a_n + b_n) = L_2$.
Since $L_1 \in \mathbb{R}$ and $L_2 \in \mathbb{R}$, then $L_2 - L_1 \in \mathbb{R}$.
Thus,

$$\begin{aligned} L_2 - L_1 &= \lim_{n\to\infty}(a_n + b_n) - \lim_{n\to\infty} a_n \\ &= \lim_{n\to\infty}[(a_n + b_n) - a_n] \\ &= \lim_{n\to\infty} b_n. \end{aligned}$$

Therefore, $L_2 - L_1 = \lim_{n\to\infty} b_n$, so (b_n) is convergent, as desired. □

Exercise 46. Let (a_n) and (b_n) be sequences of real numbers.
If (a_n) converges to some nonzero real number and $(a_n b_n)$ is convergent, then (b_n) is convergent.

Proof. Suppose (a_n) converges to some nonzero real number and $(a_n b_n)$ is convergent.

Since (a_n) converges to some nonzero real number, then there exists a real number $L_1 \neq 0$ such that $\lim_{n\to\infty} a_n = L_1$.

Since $(a_n b_n)$ is convergent, then there exists a real number L_2 such that $\lim_{n\to\infty} a_n b_n = L_2$.

Since $L_1 \in \mathbb{R}$ and $L_2 \in \mathbb{R}$ and $L_1 \neq 0$, then $\frac{L_2}{L_1} \in \mathbb{R}$.

Thus,

$$\begin{aligned}
\frac{L_2}{L_1} &= \frac{1}{L_1} \cdot L_2 \\
&= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} a_n} \cdot \lim_{n\to\infty} a_n b_n \\
&= \lim_{n\to\infty} \frac{1}{a_n} \cdot \lim_{n\to\infty} a_n b_n \\
&= \lim_{n\to\infty} (\frac{1}{a_n} \cdot a_n b_n) \\
&= \lim_{n\to\infty} b_n.
\end{aligned}$$

Therefore, $\frac{L_2}{L_1} = \lim_{n\to\infty} b_n$, so (b_n) is convergent, as desired. ☐

Exercise 47. Let (a_n) be a convergent sequence of real numbers.

Then $\lim_{n\to\infty}(a_n)^2 = (\lim_{n\to\infty} a_n)^2$.

Proof. Since (a_n) is convergent, then $\lim_{n\to\infty} a_n$ exists.

Therefore,

$$\begin{aligned}
(\lim_{n\to\infty} a_n)^2 &= (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} a_n) \\
&= \lim_{n\to\infty} (a_n a_n) \\
&= \lim_{n\to\infty} (a_n)^2.
\end{aligned}$$

☐

Proof. Since (a_n) is convergent, then there exists $L \in \mathbb{R}$ such that $L = \lim_{n\to\infty} a_n$.

To prove $\lim_{n\to\infty}(a_n)^2 = L^2$, let $\epsilon > 0$ be given.

Since $0 < \epsilon$, then $0 \leq |L|^2 < \epsilon + |L|^2$, so $0 \leq |L| < \sqrt{\epsilon + |L|^2}$.

Since $\sqrt{\epsilon + |L|^2} - |L| > 0$ and $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that $|a_n - L| < \sqrt{\epsilon + |L|^2} - |L|$ whenever $n > N$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $0 \leq |a_n - L| < \sqrt{\epsilon + |L|^2} - |L|$.

Since $|a_n + L| = |(a_n - L) + 2L| \leq |a_n - L| + 2|L| < (\sqrt{\epsilon + |L|^2} - |L|) + 2|L| = \sqrt{\epsilon + |L|^2} + |L|$, then $0 \leq |a_n + L| < \sqrt{\epsilon + |L|^2} + |L|$.

Observe that

$$\begin{aligned}|(a_n)^2 - L^2| &= |(a_n - L)(a_n + L)| \\ &= |a_n - L||a_n + L| \\ &< (\sqrt{\epsilon + |L|^2} - |L|) \cdot (\sqrt{\epsilon + |L|^2} + |L|) \\ &= (\epsilon + |L|^2) - |L|^2 \\ &= \epsilon.\end{aligned}$$

Therefore, $|(a_n)^2 - L^2| < \epsilon$, as desired. □

Exercise 48. If (a_n) and (b_n) are convergent sequences of real numbers, then $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$.

Find the flaw in the proof below.

Suppose that $\epsilon > 0$.

Choose N_1 such that $|a_n - L| < \frac{\epsilon}{2|M|+1}$ if $n > N_1$.

Choose N_2 such that $|b_n - M| < \frac{\epsilon}{2|a_n|+1}$ if $n > N_2$.

If $n > N = \max\{N_1, N_2\}$, then

$$\begin{aligned}|a_n b_n - LM| &= |a_n b_n - a_n M + a_n M - LM| \\ &= |a_n(b_n - M) + M(a_n - L)| \\ &\leq |a_n(b_n - M)| + |M(a_n - L)| \\ &= |a_n||b_n - M| + |M||a_n - L| \\ &\leq |a_n|\frac{\epsilon}{2|a_n| + 1} + |M|\frac{\epsilon}{2|M| + 1} \\ &< \epsilon.\end{aligned}$$

Solution. The flaw occurs when choosing N_2 such that $|b_n - M| < \frac{\epsilon}{2|a_n|+1}$ if $n > N_2$.

Since b_n converges to M, then each ϵ chosen must be a fixed positive real number.

In this case, the positive real number chosen is $\frac{\epsilon}{2|a_n|+1}$, which depends on a_n and ultimately depends on n, so the chosen positive real number is not constant. □

Exercise 49. Let (a_n) be a sequence of real numbers such that $a_n \to L$.

Let (b_n) be a sequence defined by $b_n = \frac{a_n + a_{n+1}}{2}$ for all $n \in \mathbb{N}$.

Then $b_n \to L$.

Proof. Suppose (a_n) converges to a real number L.

Let $\epsilon > 0$ be given.

Then there exists $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n > N$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|a_n - L| < \epsilon$.

Since $n + 1 > n > N$, then $n + 1 > N$, so $|a_{n+1} - L| < \epsilon$.

Observe that

$$
\begin{aligned}
|b_n - L| &= |\frac{a_n + a_{n+1}}{2} - L| \\
&= |\frac{a_n + a_{n+1} - 2L}{2}| \\
&= |\frac{a_n - L}{2} + \frac{a_{n+1} - L}{2}| \\
&\leq |\frac{a_n - L}{2}| + |\frac{a_{n+1} - L}{2}| \\
&= \frac{1}{2}|a_n - L| + \frac{1}{2}|a_{n+1} - L| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}
$$

Therefore, $|b_n - L| < \epsilon$, so $b_n \to L$, as desired. □

Proof. Since $a_n \to L$, then $\lim_{n\to\infty} a_n = L$, so $\lim_{n\to\infty} a_{n+1} = L$.

Observe that

$$
\begin{aligned}
L &= \frac{1}{2} \cdot 2L \\
&= \frac{1}{2}(L + L) \\
&= \frac{1}{2}(\lim_{n\to\infty} a_n + \lim_{n\to\infty} a_{n+1}) \\
&= \frac{1}{2} \lim_{n\to\infty} (a_n + a_{n+1}) \\
&= \lim_{n\to\infty} \frac{a_n + a_{n+1}}{2} \\
&= \lim_{n\to\infty} b_n.
\end{aligned}
$$

Therefore, $\lim_{n\to\infty} b_n = L$, as desired. □

Exercise 50. limit of a square root equals square root of a limit

Let (a_n) be a sequence of real numbers such that $a_n \geq 0$ for all $n \in \mathbb{N}$.

If $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{\lim_{n\to\infty} a_n}$.

Proof. Suppose $\lim_{n\to\infty} a_n$ exists.

Then there is a real number L such that $\lim_{n\to\infty} a_n = L$.

Since $0 \leq a_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} a_n$, so $0 \leq L$.

Thus, $L \geq 0$, so either $L > 0$ or $L = 0$.

We consider these cases separately.

Case 1: Suppose $L = 0$.

To prove $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{\lim_{n\to\infty} a_n} = \sqrt{L} = \sqrt{0} = 0$, we must prove $\lim_{n\to\infty} \sqrt{a_n} = 0$.

Since $\lim_{n\to\infty} a_n = L$ and $L = 0$, then $\lim_{n\to\infty} a_n = 0$.

Let $\epsilon > 0$ be given.
Then $\epsilon^2 > 0$.
Since $\lim_{n\to\infty} a_n = 0$, then there exists $N \in \mathbb{N}$ such that $|a_n| < \epsilon^2$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Since $n \in \mathbb{N}$, then $a_n \geq 0$.
Since $n > N$, then $|a_n| < \epsilon^2$.
Since $0 \leq a_n = |a_n| < \epsilon^2$, then $0 \leq a_n < \epsilon^2$.
Hence, $0 \leq \sqrt{a_n} < \sqrt{\epsilon^2} = |\epsilon| = \epsilon$.
Thus, $0 \leq \sqrt{a_n} < \epsilon$, so $0 \leq \sqrt{a_n} = |\sqrt{a_n}| < \epsilon$.
Therefore, $|\sqrt{a_n}| < \epsilon$, as desired.
Case 2: Suppose $L > 0$.
To prove $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{\lim_{n\to\infty} a_n}$, we must prove $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.
Let $\epsilon > 0$ be given.
Since $L > 0$, then $\sqrt{L} > 0$, so $\epsilon\sqrt{L} > 0$.
Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon\sqrt{L}$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Since $n \in \mathbb{N}$, then $a_n \geq 0$, so $\sqrt{a_n} \geq 0$.
Since $n > N$, then $|a_n - L| < \epsilon\sqrt{L}$, so $0 \leq |a_n - L| < \epsilon\sqrt{L}$.
Since $0 < \sqrt{L}$ and $0 \leq \sqrt{a_n}$, then $0 < \sqrt{L} \leq \sqrt{a_n} + \sqrt{L}$.
Hence, $0 < \frac{1}{\sqrt{a_n}+\sqrt{L}} \leq \frac{1}{\sqrt{L}}$.
Since $\sqrt{a_n} \geq 0$ and $\sqrt{L} > 0$, then $\sqrt{a_n} + \sqrt{L} > 0$.
Observe that

$$\begin{aligned}
|\sqrt{a_n} - \sqrt{L}| &= |\sqrt{a_n} - \sqrt{L} \cdot \frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}}| \\
&= |\frac{a_n - L}{\sqrt{a_n} + \sqrt{L}}| \\
&= |a_n - L \cdot \frac{1}{\sqrt{a_n} + \sqrt{L}}| \\
&= |a_n - L| \cdot |\frac{1}{\sqrt{a_n} + \sqrt{L}}| \\
&= |a_n - L| \cdot \frac{1}{|\sqrt{a_n} + \sqrt{L}|} \\
&= |a_n - L| \cdot \frac{1}{\sqrt{a_n} + \sqrt{L}} \\
&< \epsilon\sqrt{L} \cdot \frac{1}{\sqrt{L}} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|\sqrt{a_n} - \sqrt{L}| < \epsilon$, as desired. □

Exercise 51. Show that $\lim_{n\to\infty} \frac{2n+4}{3n+1} = \frac{2}{3}$.

Proof. Let $\epsilon > 0$ be given.

Then $\frac{10}{9\epsilon} > 0$, so $\frac{10}{9\epsilon} \in \mathbb{R}$.

Thus, by the Archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > \frac{10}{9\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N > \frac{10}{9\epsilon}$, then $n > \frac{10}{9\epsilon}$, so $\epsilon > \frac{10}{9n}$.

Since $n \in \mathbb{N}$, then $n \geq 1 > 0$, so $n > 0$.

Since $9n + 3 > 9n > 0$, then $0 < \frac{1}{9n+3} < \frac{1}{9n}$, so $\frac{10}{9n+3} < \frac{10}{9n}$.

Observe that

$$\begin{aligned}
|\frac{2n+4}{3n+1} - \frac{2}{3}| &= |\frac{3(2n+4) - 2(3n+1)}{3(3n+1)}| \\
&= |\frac{10}{3(3n+1)}| \\
&= \frac{10}{3(3n+1)} \\
&= \frac{10}{9n+3} \\
&< \frac{10}{9n} \\
&< \epsilon.
\end{aligned}$$

Therefore, $|\frac{2n+4}{3n+1} - \frac{2}{3}| < \epsilon$, as desired. □

Proof. Observe that

$$\begin{aligned}
\frac{2}{3} &= \frac{2 + 4 \cdot 0}{3 + 0} \\
&= \frac{\lim_{n\to\infty} 2 + 4 \lim_{n\to\infty} \frac{1}{n}}{\lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{1}{n}} \\
&= \frac{\lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{4}{n}}{\lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{1}{n}} \\
&= \frac{\lim_{n\to\infty}(2 + \frac{4}{n})}{\lim_{n\to\infty}(3 + \frac{1}{n})} \\
&= \lim_{n\to\infty} \frac{2 + \frac{4}{n}}{3 + \frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{2n+4}{3n+1}.
\end{aligned}$$

□

Exercise 52. Show that $\lim_{n\to\infty} \frac{5-n}{3n+7} = \frac{-1}{3}$.

Proof. Let $\epsilon > 0$ be given.

Then $\frac{4}{\epsilon} > 0$, so $\frac{4}{\epsilon} \in \mathbb{R}$.

Let $M = \max\{3, \frac{4}{\epsilon}\}$.

Then $M \in \mathbb{R}$, so by the Archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > M$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N > M$, then $n > M$.

Since $n > M \geq 3$, then $n > 3$.

Since $n > 3 \Rightarrow 3n > 9 \Rightarrow 3n - 7 > 2 > 0 \Rightarrow 3n - 7 > 0$, then $3n - 7 > 0$.

Since $n > M \geq \frac{4}{\epsilon}$, then $n > \frac{4}{\epsilon}$, so $\epsilon > \frac{4}{n}$.

Since $n > 3 \Rightarrow \frac{1}{n} < \frac{1}{3} \Rightarrow \frac{7}{n} < \frac{7}{3} = 3 - \frac{2}{3}$, then $\frac{7}{n} < 3 - \frac{2}{3}$, so $\frac{2}{3} < 3 - \frac{7}{n}$.

Thus, $0 < \frac{1}{3 - \frac{7}{n}} < \frac{3}{2}$, so $\frac{8}{3n(3 - \frac{7}{n})} < \frac{4}{n}$.

Hence, $\frac{8}{3(3n-7)} < \frac{4}{n}$.

Observe that

$$\begin{aligned}
|\frac{5-n}{3n-7} - \frac{-1}{3}| &= |\frac{5-n}{3n-7} + \frac{1}{3}| \\
&= |\frac{3(5-n) + (3n-7)}{3(3n-7)}| \\
&= |\frac{8}{3(3n-7)}| \\
&= \frac{8}{3(3n-7)} \\
&< \frac{4}{n} \\
&< \epsilon.
\end{aligned}$$

Therefore, $|\frac{5-n}{3n-7} - \frac{-1}{3}| < \epsilon$, as desired. ∎

Exercise 53. Show that $\lim_{n\to\infty} \frac{5n^2+4}{3n^2+4} = \frac{5}{3}$.

Proof. Let $\epsilon > 0$ be given.

Then $\sqrt{\frac{8}{9\epsilon}} > 0$, so $\sqrt{\frac{8}{9\epsilon}} \in \mathbb{R}$.

Thus, by the Archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > \sqrt{\frac{8}{9\epsilon}}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N > \sqrt{\frac{8}{9\epsilon}}$, then $n > \sqrt{\frac{8}{9\epsilon}}$.

Since $n > \sqrt{\frac{8}{9\epsilon}} > 0$, then $n^2 > \frac{8}{9\epsilon}$, so $\epsilon > \frac{8}{9n^2}$.

Since $n \in \mathbb{N}$, then $n \geq 1 > 0$, so $n > 0$.

Thus, $n^2 > 0$, so $3n^2 + 4 > 0$ and $\frac{4}{n^2} > 0$.

Hence, $3 + \frac{4}{n^2} > 3$, so $\frac{1}{3 + \frac{4}{n^2}} < \frac{1}{3}$.

Since $\frac{8}{3n^2} > 0$, then $\frac{8}{3n^2(3 + \frac{4}{n^2})} < \frac{8}{9n^2}$, so $\frac{8}{3(3n^2+4)} < \frac{8}{9n^2}$.

Observe that

$$\begin{aligned}
|\frac{5n^2+4}{3n^2+4} - \frac{5}{3}| &= |\frac{3(5n^2+4) - 5(3n^2+4)}{3(3n^2+4)}| \\
&= |\frac{-8}{3(3n^2+4)}| \\
&= \frac{8}{3(3n^2+4)} \\
&< \frac{8}{9n^2} \\
&< \epsilon.
\end{aligned}$$

Therefore, $|\frac{5n^2+4}{3n^2+4} - \frac{5}{3}| < \epsilon$, as desired. □

Proof. Since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty}(\frac{1}{n} \cdot \frac{1}{n}) = (\lim_{n\to\infty} \frac{1}{n}) \cdot (\lim_{n\to\infty} \frac{1}{n}) = 0 \cdot 0 = 0$, then $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

Observe that

$$\begin{aligned}
\frac{5}{3} &= \frac{5+4\cdot 0}{3+4\cdot 0} \\
&= \frac{\lim_{n\to\infty} 5 + 4\lim_{n\to\infty} \frac{1}{n^2}}{\lim_{n\to\infty} 3 + 4\lim_{n\to\infty} \frac{1}{n^2}} \\
&= \frac{\lim_{n\to\infty} 5 + \lim_{n\to\infty} \frac{4}{n^2}}{\lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{4}{n^2}} \\
&= \frac{\lim_{n\to\infty}(5 + \frac{4}{n^2})}{\lim_{n\to\infty}(3 + \frac{4}{n^2})} \\
&= \lim_{n\to\infty} \frac{5 + \frac{4}{n^2}}{3 + \frac{4}{n^2}} \\
&= \lim_{n\to\infty} \frac{5n^2+4}{3n^2+4}.
\end{aligned}$$

□

Exercise 54. Show that $\lim_{n\to\infty} \frac{2n^2+4n-3}{3n^2+2n+1} = \frac{2}{3}$.

Proof. Let $\epsilon > 0$ be given.

Then $\frac{8}{9\epsilon} > 0$, so $\frac{8}{9\epsilon} \in \mathbb{R}$.

Let $M = \max\{2, \frac{8}{9\epsilon}\}$.

Then $M \in \mathbb{R}$, so by the Archimedean property of $\mathbb{R}$, there exists a natural number N such that $N > M$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N > M \geq 2$, then $n > 2$.

Since $n > 2 > \frac{11}{8} \Rightarrow n > \frac{11}{8} \Rightarrow 8n > 11 \Rightarrow 8n - 11 > 0$, then $8n - 11 > 0$.

Since $n > N > M \geq \frac{8}{9\epsilon}$, then $n > \frac{8}{9\epsilon}$, so $\epsilon > \frac{8}{9n}$.

Since $n \in \mathbb{N}$, then $n \geq 1 > 0$, so $n > 0$.

Hence, $n^2 > 0$, so $3n^2 + 2n + 1 > 0$.
Since $3 + \frac{2}{n} + \frac{1}{n^2} > 3$, then $0 < \frac{1}{3+\frac{2}{n}+\frac{1}{n^2}} < \frac{1}{3}$.
Since $0 < 8 - \frac{11}{n} < 8$, then we have $\frac{8-\frac{11}{n}}{3+\frac{2}{n}+\frac{1}{n^2}} < \frac{8}{3}$.
Since $\frac{1}{3n} > 0$, then $\frac{8-\frac{11}{n}}{3n(3+\frac{2}{n}+\frac{1}{n^2})} < \frac{8}{9n}$.
Thus, $\frac{8n-11}{3n^2(3+\frac{2}{n}+\frac{1}{n^2})} < \frac{8}{9n}$, so $\frac{8n-11}{3(3n^2+2n+1)} < \frac{8}{9n}$.
Observe that

$$\begin{aligned} \left|\frac{2n^2+4n-3}{3n^2+2n+1} - \frac{2}{3}\right| &= \left|\frac{3(2n^2+4n-3) - 2(3n^2+2n+1)}{3(3n^2+2n+1)}\right| \\ &= \left|\frac{8n-11}{3(3n^2+2n+1)}\right| \\ &= \frac{8n-11}{3(3n^2+2n+1)} \\ &< \frac{8}{9n} \\ &< \epsilon. \end{aligned}$$

Therefore, $|\frac{2n^2+4n-3}{3n^2+2n+1} - \frac{2}{3}| < \epsilon$, as desired. □

Proof. Let $n \in \mathbb{N}$.
Since $n > 0$, then $n^2 > 0$, so $3 + \frac{2}{n} + \frac{1}{n^2} > 0$.
Thus, $3 + \frac{2}{n} + \frac{1}{n^2} \neq 0$.
Therefore, $3 + \frac{2}{n} + \frac{1}{n^2} \neq 0$ for all $n \in \mathbb{N}$.
Observe that

$$\begin{aligned} \frac{2}{3} &= \frac{2 + 4\cdot 0 - 3\cdot 0\cdot 0}{3 + 2\cdot 0 + 0\cdot 0} \\ &= \frac{\lim_{n\to\infty} 2 + 4\cdot\lim_{n\to\infty}\frac{1}{n} - 3\cdot\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty} 3 + 2\cdot\lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n}} \\ &= \frac{\lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{4}{n} - 3\cdot\lim_{n\to\infty}\frac{1}{n^2}}{\lim_{n\to\infty} 3 + \lim_{n\to\infty}\frac{2}{n} + \lim_{n\to\infty}\frac{1}{n^2}} \\ &= \frac{\lim_{n\to\infty}(2 + \frac{4}{n}) - \lim_{n\to\infty}\frac{3}{n^2}}{\lim_{n\to\infty}(3 + \frac{2}{n} + \frac{1}{n^2})} \\ &= \frac{\lim_{n\to\infty}(2 + \frac{4}{n} - \frac{3}{n^2})}{\lim_{n\to\infty}(3 + \frac{2}{n} + \frac{1}{n^2})} \\ &= \lim_{n\to\infty}\frac{2 + \frac{4}{n} - \frac{3}{n^2}}{3 + \frac{2}{n} + \frac{1}{n^2}} \\ &= \lim_{n\to\infty}\frac{2n^2+4n-3}{3n^2+2n+1}. \end{aligned}$$

□

Proposition 55. *Let r be a real number.*
Then there is a sequence of rational numbers that converges to r.

Proof. Let $(r, r + \frac{1}{n})$ be a sequence of intervals for each $n \in \mathbb{N}$.
Let $n \in \mathbb{N}$ be arbitrary.
Then $(r, r + \frac{1}{n})$ is an interval and $r < r + \frac{1}{n}$.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $q \in \mathbb{Q}$ such that $r < q < r + \frac{1}{n}$.
Hence, for every $n \in \mathbb{N}$, there exists $q \in \mathbb{Q}$ such that $r < q < r + \frac{1}{n}$.
Thus, there exists a sequence (a_n) in $\mathbb{Q}$ such that $r < a_n < r + \frac{1}{n}$ for each $n \in \mathbb{N}$.

Observe that $\lim_{n\to\infty}(r+\frac{1}{n}) = \lim_{n\to\infty} r+\lim_{n\to\infty} \frac{1}{n} = r+0 = r = \lim_{n\to\infty} r$.
Since $r < a_n < r + \frac{1}{n}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} r = r = \lim_{n\to\infty}(r + \frac{1}{n})$, then by the squeeze rule for sequences, $\lim_{n\to\infty} a_n = r$.
Therefore, there exists a sequence in $\mathbb{Q}$ that converges to r. ☐

Proposition 56. *Let r be a real number.*
Then there is a sequence of irrational numbers that converges to r.

Proof. Let $(r, r + \frac{1}{n})$ be a sequence of intervals for each $n \in \mathbb{N}$.
Let $n \in \mathbb{N}$ be arbitrary.
Then $(r, r + \frac{1}{n})$ is an interval and $r < r + \frac{1}{n}$.
Since $\mathbb{R}-\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $t \in \mathbb{R}-\mathbb{Q}$ such that $r < t < r+\frac{1}{n}$.
Hence, for every $n \in \mathbb{N}$, there exists $t \in \mathbb{R} - \mathbb{Q}$ such that $r < t < r + \frac{1}{n}$.
Thus, there exists a sequence (a_n) in $\mathbb{R} - \mathbb{Q}$ such that $r < a_n < r + \frac{1}{n}$ for each $n \in \mathbb{N}$.

Observe that $\lim_{n\to\infty}(r+\frac{1}{n}) = \lim_{n\to\infty} r+\lim_{n\to\infty} \frac{1}{n} = r+0 = r = \lim_{n\to\infty} r$.
Since $r < a_n < r + \frac{1}{n}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} r = r = \lim_{n\to\infty}(r + \frac{1}{n})$, then by the squeeze rule for sequences, $\lim_{n\to\infty} a_n = r$.
Therefore, there exists a sequence in $\mathbb{R} - \mathbb{Q}$ that converges to r. ☐

Exercise 57. The sequence (a_n) defined by $a_n = n$ diverges.
Therefore, $\lim_{n\to\infty} n$ does not exist.

Solution. Clearly, the sequence (a_n) defined by $a_n = n$ does not converge.
To rigorously prove this fact, we must prove there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $N > n$ and $|a_N - L| \geq \epsilon$. ☐

Proof. Let $L \in \mathbb{R}$.
Let $\epsilon = 1$.
To prove $\lim_{n\to\infty} n$ does not exist, we must prove $(\forall n \in \mathbb{N})(\exists N \in \mathbb{N})(N > n \wedge |a_N - L| \geq 1)$.
Let $n \in \mathbb{N}$.
Since $L + 1$ is a real number, then by the Archimedean property of $\mathbb{R}$, there exists $M \in \mathbb{N}$ such that $M > L + 1$.
Either $n < M$ or $n \geq M$.

We consider these cases separately.
Case 1: Suppose $n < M$.
Let $N = M$.
Then $N > n$ and $a_N = N = M > L + 1$.
Since $a_N > L + 1$, then $a_N - L > 1$, so $a_N - L \geq 1$.
Since $|a_N - L| \geq 1$ iff either $a_N - L \geq 1$ or $a_N - L \leq -1$, then $|a_N - L| \geq 1$.
Case 2: Suppose $n \geq M$.
Let $N = n + 1$.
Since $n + 1 > n$, then $N > n$.
Observe that $a_N = N = n + 1 > n \geq M > L + 1$, so $a_N > L + 1$.
Hence, $a_N - L > 1$, so $a_N - L \geq 1$.
Since $|a_N - L| \geq 1$ iff either $a_N - L \geq 1$ or $a_N - L \leq -1$, then $|a_N - L| \geq 1$.
Therefore, in all cases, $N > n$ and $|a_N - L| \geq 1$, as desired. ☐

Exercise 58. Let (a_n) be a sequence in $\mathbb{R}$.
Let $A, B \in \mathbb{R}$.
If $\lim_{n\to\infty} a_n = A$ and B is an accumulation point of the set $\{a_n : n \in \mathbb{N}\}$, then $A = B$.

Proof. Suppose $\lim_{n\to\infty} a_n = A$ and B is an accumulation point of the set $\{a_n : n \in \mathbb{N}\}$.
Let $S = \{a_n : n \in \mathbb{N}\}$.
Then B is an accumulation point of S, so by a previous proposition, there is a sequence (b_n) of points in $S - \{B\}$ such that $\lim_{n\to\infty} b_n = B$.
We prove (b_n) is a subsequence of (a_n).
Since $\lim_{n\to\infty} a_n = A$ and (b_n) is a subsequence of (a_n), then $B = \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = A$, so $B = A$.
Therefore, $A = B$.
Since B is an accumulation point of S, then for every $\epsilon > 0$, there exists $s \in S$ such that $s \in N'(B; \epsilon)$.
Let $\epsilon = \frac{1}{n}$ for each $n \in \mathbb{N}$.
Then for each $n \in \mathbb{N}$, there exists $s \in S$ such that $s \in N'(B; \frac{1}{n})$.
Thus, there exists a sequence (b_n) such that $b_n \in S$ and $b_n \in N'(B; \frac{1}{n})$.
Let $n \in \mathbb{N}$ be given.
Then $b_n \in S$ and $b_n \in N'(B; \frac{1}{n})$.
Since $b_n \in S$, then $b_n = a_m$ for some $m \in \mathbb{N}$.
Since $b_n \in N'(B; \frac{1}{n})$, then $|b_n - B| < \frac{1}{n}$, so $|a_m - B| < \frac{1}{n}$.
Therefore, for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $|a_m - B| < \frac{1}{n}$.
☐

Exercise 59. Prove $\lim_{n\to\infty} \frac{n^2+1}{n+1} = \infty$.

Proof. Let $M > 0$.
We must prove there exists $N \in \mathbb{N}$ such that $\frac{n^2+1}{n+1} > M$ whenever $n > N$.
Since $M+1 \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > M + 1$.
Let $n \in \mathbb{N}$ such that $n > N$.

Then $n > N > M + 1 > M > M - 1$, so $n > M - 1$.
Hence, $1 > \frac{M-1}{n}$, so $M + 1 > M + \frac{M-1}{n}$.
Thus, $n > M + 1 > M + \frac{M-1}{n}$, so $n > M + \frac{M-1}{n}$.
Consequently, $n^2 > Mn + M - 1$, so $n^2 + 1 > M(n + 1)$.
Therefore, $\frac{n^2+1}{n+1} > M$, as desired. □

Lemma 60. *Let $a, b \in \mathbb{R}$.*
If $a > 1$ and $b > 1$, then $ab > 1$.

Proof. Suppose $a > 1$ and $b > 1$.
Then there exists $a' > 0$ such that $1 + a' = a$ and there exists $b' > 0$ such that $1 + b' = b$.
Hence, $ab = (1 + a')(1 + b') = 1 + b' + a' + a'b'$, so $ab - 1 = b' + a' + a'b'$.
Since $b' > 0$ and $a' > 0$, then $a'b' > 0$, so $b' + a' + a'b' > 0$.
Therefore, $ab - 1 > 0$, so $ab > 1$, as desired. □

Exercise 61. Prove $\lim_{n\to\infty} \frac{n^3+1}{n^2+1} = \infty$.

Proof. Let $M > 0$.
We must prove there exists $N \in \mathbb{N}$ such that $\frac{n^3+1}{n^2+1} > M$ whenever $n > N$.
Since $M+1 \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > M + 1$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $n > N > M + 1$.
Since $N - M > 1$, then $N - M > 0$, so $1 + (N - M) > 0$.
Hence, $-1 - (N - M) < 0$.
Since $n > N \geq 1$, then $n > 1$.
Since $N - M > 1$, then $n(N - M) > 1$.
Since $n > 0$, then $n^2(N - M) > n$, so $n^2(N - M) - n > 0$.
Thus, $-1-(N-M) < 0 < n^2(N-M)-n$, so $-1-(N-M) < n^2(N-M)-n$.
Hence, $n - 1 < n^2(N - M) + (N - M)$, so $n - 1 < (n^2 + 1)(N - M)$.
Since $n^2 + 1 > 0$, then $\frac{n-1}{n^2+1} < N - M$.
Therefore, we have the inequality $\frac{1-n}{n^2+1} > M - N$.
Adding the inequality $n > N$, we obtain $n + \frac{1-n}{n^2+1} > M$.
Thus, $\frac{n^3+1}{n^2+1} > M$, as desired. □

Exercise 62. Let (a_n) be a sequence of real numbers.
If $a_n \to \infty$, then $(a_n)^2 \to \infty$.

Proof. Suppose $a_n \to \infty$.
To prove $(a_n)^2 \to \infty$, let $M > 0$ be given.
We must prove there exists $N \in \mathbb{N}$ such that $(a_n)^2 > M$ whenever $n > N$.
Since $M+1 > 0$ and $a_n \to \infty$, then there exists $N \in \mathbb{N}$ such that $a_n > M+1$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $a_n > M + 1$.
Thus, $a_n > M + 1 > M > 0$.

Since $a_n > M + 1 > 0$, then $(a_n)^2 > (M + 1)^2$.
Since $M + 1 > 1$ and $M + 1 > 0$, then $(M + 1)^2 > M + 1$.
Thus, $(a_n)^2 > (M + 1)^2 > M + 1 > M$, so $(a_n)^2 > M$, as desired. □

Exercise 63. Let (a_n) be a sequence of real numbers.
If $a_n \to \infty$ then $\frac{1}{a_n} \to 0$.

Proof. Suppose $a_n \to \infty$.
To prove $\frac{1}{a_n} \to 0$, let $\epsilon > 0$ be given.
Then $\frac{1}{\epsilon} > 0$.
Since $a_n \to \infty$, then there exists $N \in \mathbb{N}$ such that $a_n > \frac{1}{\epsilon}$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $a_n > \frac{1}{\epsilon}$.
Since $a_n > \frac{1}{\epsilon} > 0$, then $a_n > 0$.
Hence, $\epsilon > \frac{1}{a_n} > 0$.
Therefore, $|\frac{1}{a_n}| = \frac{1}{a_n} < \epsilon$, as desired. □

Exercise 64. Let (a_n) be a sequence of real numbers.
If $a_n \to 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $|\frac{1}{a_n}| \to \infty$.

Proof. Suppose $a_n \to 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$.
To prove $|\frac{1}{a_n}| \to \infty$, let $M > 0$ be given.
Then $\frac{1}{M} > 0$.
Since $a_n \to 0$, then there exists $N \in \mathbb{N}$ such that $|a_n| < \frac{1}{M}$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $|a_n| < \frac{1}{M}$.
Since $n \in \mathbb{N}$, then $a_n \neq 0$, so $|a_n| \neq 0$.
Thus, $|a_n| > 0$.
Hence, $M < \frac{1}{|a_n|} = |\frac{1}{a_n}|$, so $M < |\frac{1}{a_n}|$.
Therefore, $|\frac{1}{a_n}| > M$, as desired. □

Exercise 65. Let (a_n) be a sequence of positive real numbers.
Then $a_n \to 0$ iff $\frac{1}{a_n} \to \infty$.

Proof. We prove if $a_n \to 0$ then $\frac{1}{a_n} \to \infty$.
Suppose $a_n \to 0$.
To prove $\frac{1}{a_n} \to \infty$, let $M > 0$ be given.
Then $\frac{1}{M} > 0$.
Since $a_n \to 0$, then there exists $N \in \mathbb{N}$ such that $|a_n| < \frac{1}{M}$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $|a_n| < \frac{1}{M}$.
Since all terms of (a_n) are positive, then $a_n > 0$ for all $n \in \mathbb{N}$.
In particular, $a_n > 0$, so $a_n < \frac{1}{M}$.
Since $M > 0$, then $M < \frac{1}{a_n}$.
Therefore, $\frac{1}{a_n} > M$, as desired.
Conversely, we prove if $\frac{1}{a_n} \to \infty$, then $a_n \to 0$.

Suppose $\frac{1}{a_n} \to \infty$.
To prove $a_n \to 0$, let $\epsilon > 0$ be given.
Then $\frac{1}{\epsilon} > 0$.
Since $\frac{1}{a_n} \to \infty$, then there exists $N \in \mathbb{N}$ such that $\frac{1}{a_n} > \frac{1}{\epsilon}$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $\frac{1}{a_n} > \frac{1}{\epsilon}$.
Since $\epsilon > 0$, then $\frac{\epsilon}{a_n} > 1$.
Since all terms of (a_n) are positive, then $a_n > 0$ for all $n \in \mathbb{N}$.
In particular, $a_n > 0$, so $\epsilon > a_n$.
Therefore, $a_n < \epsilon$, so $|a_n| < \epsilon$, as desired. □

Exercise 66. Let (x_n) be a sequence of real numbers.
If (x_n) diverges to ∞, then $\frac{x_n}{x_n+1}$ converges.
The converse if $\frac{x_n}{x_n+1}$ converges, then (x_n) diverges to ∞ is false.

Proof. We first prove if (x_n) diverges to ∞, then $\frac{x_n}{x_n+1}$ converges.
Suppose (x_n) diverges to ∞.
To prove $\frac{x_n}{x_n+1}$ converges, we prove $\frac{x_n}{x_n+1}$ converges to 1.
Let $\epsilon > 0$ be given.
Then $\frac{1}{\epsilon} > 0$.
We must prove there exists $N \in \mathbb{N}$ such that $|\frac{x_n}{x_n+1} - 1| < \epsilon$ whenever $n > N$.
Since $x_n \to \infty$, then there exists $N \in \mathbb{N}$ such that $x_n > \frac{1}{\epsilon}$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $x_n > \frac{1}{\epsilon} > 0$.
Since $\frac{1}{\epsilon} < x_n$ and $x_n < x_n + 1$, then $\frac{1}{\epsilon} < x_n + 1$.
Since $x_n > 0$, then $x_n + 1 > 0$, so $\frac{1}{x_n+1} < \epsilon$.
Observe that

$$\begin{aligned} |\frac{x_n}{x_n+1} - 1| &= |\frac{x_n - (x_n+1)}{x_n+1}| \\ &= |\frac{-1}{x_n+1}| \\ &= \frac{1}{x_n+1} \\ &< \epsilon. \end{aligned}$$

Therefore, $|\frac{x_n}{x_n+1} - 1| < \epsilon$, so $\frac{x_n}{x_n+1}$ converges to 1.

To disprove the converse, we must prove there exists a sequence (x_n) such that $\frac{x_n}{x_n+1}$ converges and (x_n) does not diverge to ∞.
Let (x_n) be a sequence defined by $x_n = \frac{1}{n}$.
To prove $\frac{x_n}{x_n+1}$ converges, we prove $\frac{x_n}{x_n+1}$ converges to zero.
Let $\epsilon > 0$.
Then $\frac{1}{\epsilon} > 0$. Since $\frac{1}{\epsilon} \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $n > N > \frac{1}{\epsilon}$, so $n > \frac{1}{\epsilon}$.

Hence, $\epsilon > \frac{1}{n}$.
Thus,

$$\begin{aligned}
|\frac{x_n}{x_n+1}| &= |\frac{\frac{1}{n}}{\frac{1}{n}+1}| \\
&= \frac{\frac{1}{n}}{\frac{1}{n}+1} \\
&= \frac{1}{1+n} \\
&< \frac{1}{n} \\
&< \epsilon.
\end{aligned}$$

Therefore, $\frac{x_n}{x_n+1}$ converges to zero, so $\frac{x_n}{x_n+1}$ converges.
We prove (x_n) does not diverge to ∞.
Since $x_n = \frac{1}{n}$ and $\frac{1}{n} \to 0$, then x_n converges to zero.
Hence, (x_n) converges, so (x_n) does not diverge.
In particular, (x_n) does not diverge to ∞. $\square$

Exercise 67. Let (s_n) be a sequence of positive real numbers.
Let $\alpha > 1$.
If $s_{n+1} > \alpha s_n$ for all $n \in \mathbb{N}$, then $s_n \to \infty$.

Proof. To prove $s_n \to \infty$, let $M > 0$ be given.
We must prove there exists $N' \in \mathbb{N}$ such that if $n > N'$, then $s_n > M$.
We first prove the sequence α^n diverges to ∞.
Let $b > 0$.
We must prove there exists $N \in \mathbb{N}$ such that if $n > N$, then $\alpha^n > b$.
Since $\alpha > 0$ and $b > 0$, let $x = \log_\alpha b$.
Then $\alpha^x = b$.
Since $x \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > x$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $n > N > x$, so $n > x$.
Hence, $n - x > 0$.
Since $\alpha > 1$ and $n - x > 0$, then $\alpha^{n-x} > 1$.
Since $\alpha^x > 0$, then $\alpha^n > \alpha^x$, so $\alpha^n > b$, as desired.
Therefore, α^n diverges to ∞.
Hence, for every $b > 0$, there exists $N \in \mathbb{N}$ such that $\alpha^n > b$ if $n > N$.
Since $M > 0$ and $\alpha > 0$ and $s_1 > 0$, then $\frac{M\alpha}{s_1} > 0$.
Thus, there exists $N \in \mathbb{N}$ such that $\alpha^n > \frac{M\alpha}{s_1}$ if $n > N$.
We next prove $s_n > \alpha^{n-1}s_1$ for all $n \geq 2$ by induction on n.
Let $p(n) : s_n > \alpha^{n-1}s_1$ be a predicate defined over $\mathbb{N}$.
Basis:
Since $s_2 > \alpha s_1$, then the statement $p(2)$ is true.
Induction:

Let $n \geq 2$ such that $p(n)$ is true.
Then $s_n > \alpha^{n-1} s_1$.
To prove $p(n+1)$ is true, we must prove $s_{n+1} > \alpha^n s_1$.
Observe that $s_{n+1} > \alpha s_n$ and $\alpha s_n > \alpha^n s_1$.
Therefore, $s_{n+1} > \alpha^n s_1$, as desired.
Hence, $p(n)$ implies $p(n+1)$ for all $n \geq 2$.
Since $p(2)$ is true and $p(n)$ implies $p(n+1)$ for all $n \geq 2$, then by induction, $p(n)$ is true for all $n \geq 2$.
Therefore, $s_n > \alpha^{n-1} s_1$ for all $n \geq 2$.
Let $N' = \max\{2, N\}$.
Then $N' \in \mathbb{N}$ and $2 \leq N'$ and $N \leq N'$.
Let $n \in \mathbb{N}$ such that $n > N'$.
Since $n > N' \geq N$, then $n > N$.
Hence, $\alpha^n > \frac{M\alpha}{s_1}$.
Thus, $\alpha^n s_1 > M\alpha$, so $\alpha^{n-1} s_1 > M$.
Since $n > N' \geq 2$, then $n > 2$.
Hence, $s_n > \alpha^{n-1} s_1$.
Thus, $s_n > \alpha^{n-1} s_1$ and $\alpha^{n-1} s_1 > M$, so $s_n > M$, as desired. □

Exercise 68. Let (s_n) be a sequence of real numbers.
If (s_n) is unbounded, then either $s_n \to \infty$ or $s_n \to -\infty$.

Solution. This is a false statement.
Here is a counterexample. Let

$$s_n = \begin{cases} n & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even} \end{cases}$$

We prove (s_n) is unbounded.
Let $M > 0$ be given.
Since $\mathbb{N}$ is unbounded above in $\mathbb{R}$, then there is a natural number $n > M$.
Let $n \in \mathbb{N}$ such that $n > M$.
Either n is even or n is odd.
We consider these cases separately.
Case 1: Suppose n is even.
Then $n+1$ is odd.
Therefore, $|s_{n+1}| = |n+1| = n+1 > n > M$.
Hence, there exists $n+1 \in \mathbb{N}$ such that $|s_{n+1}| > M$.
Case 2: Suppose n is odd.
Then $|s_n| = |n| = n > M$.
Therefore, there exists $n \in \mathbb{N}$ such that $|s_n| > M$.
Therefore, (s_n) is unbounded.
To prove (s_n) does not diverge to ∞, we must prove $(\exists M > 0)(\forall n \in \mathbb{N})(\exists N \in \mathbb{N})(N > n \land s_N \leq M)$.
Let $M = 1$.
Let $n \in \mathbb{N}$.

Then $n \geq 1$, so $1 \geq \frac{1}{n}$.
Either n is even or n is odd.
We consider these cases separately.
Case 1: Suppose n is even.
Then $n+2$ is even.
Let $N = n+2$.
Since $n+2 > n$, then $N > n$.
Hence, $s_N = s_{n+2} = \frac{1}{n+2} < \frac{1}{n} \leq 1$, so $s_N < 1$.
Therefore, $s_N \leq 1$.
Case 2: Suppose n is odd.
Then $n+1$ is even.
Let $N = n+1$.
Since $n+1 > n$, then $N > n$.
Hence, $s_N = s_{n+1} = \frac{1}{n+1} < \frac{1}{n} \leq 1$, so $s_N < 1$.
Therefore, $s_N \leq 1$.
Thus, (s_n) does not diverge to ∞.
We prove (s_n) does not diverge to $-\infty$.
Suppose that (s_n) diverges to $-\infty$.
Then $(-s_n)$ diverges to ∞.
Hence, for every $M > 0$, there exists $N \in \mathbb{N}$ such that $-s_n > M$ whenever $n > N$.
Let $M = 1$.
Then there exists $N \in \mathbb{N}$ such that $-s_n > 1$ whenever $n > N$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $-s_n > 1$.
Since $-s_n > 1 > 0$, then $-s_n > 0$.
Hence, $s_n < 0$.
Thus, there exists $n \in \mathbb{N}$ such that $s_n < 0$, contradicting the fact that $s_n > 0$ for all $n \in \mathbb{N}$.
Therefore, (s_n) does not diverge to $-\infty$. $\square$

Exercise 69. Let (s_n) be a sequence of real numbers.
If (s_n) is unbounded, then $|s_n| \to \infty$.

Solution. This is a false statement.
Here is a counterexample. Let

$$s_n = \begin{cases} n & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even} \end{cases}$$

We prove (s_n) is unbounded.
Let $M > 0$ be given.
Since $\mathbb{N}$ is unbounded above in $\mathbb{R}$, then there is a natural number $n > M$.
Let $n \in \mathbb{N}$ such that $n > M$.
Either n is even or n is odd.
We consider these cases separately.

Case 1: Suppose n is even.
Then $n+1$ is odd.
Therefore, $|s_{n+1}| = |n+1| = n+1 > n > M$.
Hence, there exists $n+1 \in \mathbb{N}$ such that $|s_{n+1}| > M$.
Case 2: Suppose n is odd.
Then $|s_n| = |n| = n > M$.
Therefore, there exists $n \in \mathbb{N}$ such that $|s_n| > M$.
Therefore, (s_n) is unbounded.
To prove $|s_n|$ does not diverge to ∞, we must prove $(\exists M > 0)(\forall n \in \mathbb{N})(\exists N \in \mathbb{N})(N > n \wedge |s_N| \leq M)$.
Let $M = 1$.
Let $n \in \mathbb{N}$.
Then $n \geq 1$, so $1 \geq \frac{1}{n}$.
Either n is even or n is odd.
We consider these cases separately.
Case 1: Suppose n is even.
Then $n+2$ is even.
Let $N = n+2$.
Since $n+2 > n$, then $N > n$.
Hence, $|s_N| = |s_{n+2}| = |\frac{1}{n+2}| = \frac{1}{n+2} < \frac{1}{n} \leq 1$, so $|s_N| < 1$.
Therefore, $|s_N| \leq 1$.
Case 2: Suppose n is odd.
Then $n+1$ is even.
Let $N = n+1$.
Since $n+1 > n$, then $N > n$.
Hence, $|s_N| = |s_{n+1}| = |\frac{1}{n+1}| = \frac{1}{n+1} < \frac{1}{n} \leq 1$, so $|s_N| < 1$.
Therefore, $|s_N| \leq 1$.
Thus, $|s_n|$ does not diverge to ∞. ☐

Exercise 70. Let (a_n) and (b_n) be unbounded sequences of real numbers.
Then $(a_n + b_n)$ is unbounded.

Solution. This is a false statement.
Here is a counterexample.
Let $a_n = n^2$ and $b_n = -a_n$ for all $n \in \mathbb{N}$.
Then $a_n \to \infty$ and $b_n \to -\infty$.
Thus, (a_n) is unbounded and (b_n) is unbounded.
Since $a_n + b_n = a_n - a_n = 0$ and 0 is constant, then $(a_n + b_n)$ is convergent.
Therefore, $(a_n + b_n)$ is bounded. ☐

Exercise 71. Let (a_n) and (b_n) be sequences of real numbers.
If (a_n) and (b_n) are unbounded, then $(a_n b_n)$ is unbounded.

Solution. This is a false statement.
Here is a counterexample. Let

$$a_n = \begin{cases} n & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even} \end{cases}$$

Let

$$b_n = \begin{cases} n & \text{if } n \text{ is even} \\ \frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$$

Then (a_n) and (b_n) are unbounded sequences.

But, $a_n b_n = 1$ for all $n \in \mathbb{N}$, so $(a_n b_n)$ is the constant sequence which converges.

Hence, $(a_n b_n)$ is bounded. □

Exercise 72. Let (s_n) be a sequence of real numbers.

If (s_n) is bounded, then $(\frac{1}{s_n})$ is bounded.

Solution. This is a false statement.

Here is a counterexample.

Let $s_n = \frac{1}{n}$.

Then (s_n) converges to zero, so (s_n) is bounded.

But, $(\frac{1}{s_n}) = n$ diverges to ∞, so $(\frac{1}{s_n})$ is unbounded. □

Exercise 73. Let (s_n) be a sequence of real numbers.

If (s_n) is unbounded, then $(\frac{1}{s_n})$ is bounded.

Solution. This is a false statement.

Here is a counterexample.

Let

$$s_n = \begin{cases} n & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even} \end{cases}$$

We proved previously that (s_n) is unbounded.

Observe that

$$\frac{1}{s_n} = \begin{cases} n & \text{if } n \text{ is even} \\ \frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$$

Thus, $(\frac{1}{s_n})$ is unbounded.

Therefore, (s_n) is unbounded and $(\frac{1}{s_n})$ is unbounded. □

Exercise 74. Let (a_n) be a bounded sequence of real numbers.

Then $(\frac{a_n}{n})$ is convergent.

Proof. Since the sequence given by $\frac{1}{n}$ converges to 0 and (a_n) is bounded, then the sequence given by $\frac{1}{n} \cdot a_n = \frac{a_n}{n}$ converges to 0.

Therefore, the sequence $(\frac{a_n}{n})$ is convergent. □

Monotone Convergence Theorem

Exercise 75. Let (x_n) be a sequence of real numbers defined by $x_1 = \sqrt{2}$ and $x_n = \sqrt{2 + x_{n-1}}$ for $n > 1$.

Show that $x_n < 2$ for all $n \in \mathbb{N}$ and $x_n < x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = 2$.

Solution. We compute several terms of the sequence and observe that the terms are all positive and less than 2.

We observe that the sequence appears to be strictly increasing and the terms get close to 2.

We must prove these observations of this sequence are true. □

Proof. We prove $x_n < 2$ for all $n \in \mathbb{N}$ by induction on n.

Let $p(n) : x_n < 2$ be a predicate defined over $\mathbb{N}$.

Basis:

Since $0 < 2 < 4$, then $0 < \sqrt{2} < 2$, so $\sqrt{2} < 2$.

Therefore, $x_1 < 2$, so $p(1)$ is true.

Induction:

Let $k \in \mathbb{N}$ such that $p(k)$ is true.

Then $x_k < 2$.

Suppose $x_{k+1} \geq 2$.

Then $\sqrt{2 + x_k} \geq 2$, so $2 + x_k \geq 4$.

Hence, $x_k \geq 2$.

But, $x_k < 2$ by hypothesis.

Therefore, $x_{k+1} < 2$, so $p(k+1)$ is true.

Hence, $p(k)$ implies $p(k+1)$ for all $k \in \mathbb{N}$.

Since $p(1)$ is true and $p(k)$ implies $p(k+1)$ for all $k \in \mathbb{N}$, then by induction, $p(n)$ is true for all $n \in \mathbb{N}$.

Therefore, $x_n < 2$ for all $n \in \mathbb{N}$. □

Proof. We prove $x_n > 0$ for all $n \in \mathbb{N}$ by induction on n.

Let $p(n) : x_n > 0$ be a predicate defined over $\mathbb{N}$.

Basis:

Since $2 > 0$, then $\sqrt{2} > 0$, so $x_1 > 0$.

Therefore, $p(1)$ is true.

Induction:

Let $k \in \mathbb{N}$ such that $p(k)$ is true.

Then $x_k > 0$.

Hence, $2 + x_k > 2$, so $\sqrt{2 + x_k} > \sqrt{2}$.

Thus, $x_{k+1} > \sqrt{2}$.

Since $\sqrt{2} > 0$, then $x_{k+1} > 0$, so $p(k+1)$ is true.

Hence, $p(k)$ implies $p(k+1)$ for all $k \in \mathbb{N}$.

Since $p(1)$ is true and $p(k)$ implies $p(k+1)$ for all $k \in \mathbb{N}$, then by induction, $p(n)$ is true for all $n \in \mathbb{N}$.

Therefore, $x_n > 0$ for all $n \in \mathbb{N}$. □

Proof. We prove $x_n < x_{n+1}$ for all $n \in \mathbb{N}$ by induction on n.

Let $p(n) : x_n < x_{n+1}$ be a predicate defined over $\mathbb{N}$.

Basis:

Since $0 < \sqrt{2}$, then $2 < 2 + \sqrt{2}$.

Since $0 < 2 < 2 + \sqrt{2}$, then $0 < \sqrt{2} < \sqrt{2 + \sqrt{2}}$.

Hence, $\sqrt{2} < \sqrt{2 + \sqrt{2}}$, so $x_1 < \sqrt{2 + x_1}$.

Therefore, $x_1 < x_2$, so $p(1)$ is true.

Induction:

Let $k \in \mathbb{N}$ such that $p(k)$ is true.

Then $x_k < x_{k+1}$.

Suppose $x_{k+1} \geq x_{k+2}$.

Then $\sqrt{2 + x_k} \geq \sqrt{2 + x_{k+1}}$.

Since $x_n > 0$ for all $n \in \mathbb{N}$ and $k + 1 \in \mathbb{N}$, then $x_{k+1} > 0$.

Hence, $2 + x_{k+1} > 0$, so $\sqrt{2 + x_{k+1}} > 0$.

Thus, $\sqrt{2 + x_k} \geq \sqrt{2 + x_{k+1}} > 0$, so $2 + x_k \geq 2 + x_{k+1} > 0$.

Hence, $2 + x_k \geq 2 + x_{k+1}$, so $x_k \geq x_{k+1}$.

But, $x_k < x_{k+1}$, by the induction hypothesis.

Therefore, $x_{k+1} < x_{k+2}$, so $p(k + 1)$ is true.

Hence, $p(k)$ implies $p(k + 1)$ for all $k \in \mathbb{N}$.

Since $p(1)$ is true and $p(k)$ implies $p(k + 1)$ for all $k \in \mathbb{N}$, then by induction, $p(n)$ is true for all $n \in \mathbb{N}$.

Therefore, $x_n < x_{n+1}$ for all $n \in \mathbb{N}$. $\square$

Proof. We prove $\lim_{n\to\infty} x_n = 2$.

Since $x_n < x_{n+1}$ for all $n \in \mathbb{N}$, then (x_n) is strictly increasing, so (x_n) is monotonic.

Since $x_n > 0$ for all $n \in \mathbb{N}$ and $x_n < 2$ for all $n \in \mathbb{N}$, then $0 < x_n < 2$ for all $n \in \mathbb{N}$, so (x_n) is bounded.

Since (x_n) is monotonic and bounded, then by MCT, (x_n) is convergent.

Thus, there is a real number L such that $\lim_{n\to\infty} x_n = L$.

We must prove $L = 2$.

Since $0 < x_n < 2$ for all $n \in \mathbb{N}$, then $0 \leq x_n \leq 2$ for all $n \in \mathbb{N}$, so $0 \leq \lim_{n\to\infty} x_n \leq 2$.

Hence, $0 \leq L \leq 2$, so $0 \leq L$ and $L \leq 2$.

Since $\lim_{n\to\infty} x_n = L$, then $\lim_{n\to\infty} x_{n+1} = L$.

Thus, $L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n}$.

Observe that

$$\begin{aligned}
2 + L &= \lim_{n\to\infty} 2 + \lim_{n\to\infty} x_n \\
&= \lim_{n\to\infty} (2 + x_n) \\
&= \lim_{n\to\infty} (\sqrt{2 + x_n})(\sqrt{2 + x_n}) \\
&= \lim_{n\to\infty} (\sqrt{2 + x_n}) \cdot \lim_{n\to\infty} (\sqrt{2 + x_n}) \\
&= L \cdot L.
\end{aligned}$$

Thus, $2 + L = L^2$, so $0 = L^2 - L - 2 = (L-2)(L+1)$.
Hence, either $L - 2 = 0$ or $L + 1 = 0$, so either $L = 2$ or $L = -1$.
Since $L \geq 0 > -1$, then $L > -1$, so $L \neq -1$.
Therefore, $L = 2$, as desired. □

Exercise 76. Let (x_n) be a sequence of real numbers defined by $x_1 = 6$ and $x_n = \sqrt{6 + x_{n-1}}$ for $n > 1$.

Then $x_n > 0$ for all $n \in \mathbb{N}$ and $x_n > x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = 3$.

Solution. We compute several terms of the sequence and observe that the terms are all positive and less than or equal to 6.

We observe that the sequence appears to be strictly decreasing and the terms get close to 3.

We must prove these observations of this sequence are true. □

Proof. We prove $x_n > 0$ for all $n \in \mathbb{N}$ by induction on n.
Let $p(n) : x_n > 0$ be a predicate defined over $\mathbb{N}$.
Basis:
Since $0 < 6 = x_1$, then $0 < x_1$, so $x_1 > 0$.
Therefore, $p(1)$ is true.
Induction:
Let $k \in \mathbb{N}$ such that $p(k)$ is true.
Then $x_k > 0$.
Hence, $6 + x_k > 6 > 0$, so $6 + x_k > 0$.
Thus, $\sqrt{6 + x_k} > 0$, so $x_{k+1} > 0$.
Hence, $p(k + 1)$ is true, so $p(k)$ implies $p(k + 1)$ for all $k \in \mathbb{N}$.
Since $p(1)$ is true and $p(k)$ implies $p(k + 1)$ for all $k \in \mathbb{N}$, then by induction, $p(n)$ is true for all $n \in \mathbb{N}$.
Therefore, $x_n > 0$ for all $n \in \mathbb{N}$. □

Proof. We prove $x_n > x_{n+1}$ for all $n \in \mathbb{N}$ by induction on n.
Let $p(n) : x_n > x_{n+1}$ be a predicate defined over $\mathbb{N}$.
Basis:
Since $36 > 12$, then $6 > \sqrt{12}$.
Thus, $x_1 = 6 > \sqrt{12} = \sqrt{6 + 6} = \sqrt{6 + x_1} = x_2$, so $x_1 > x_2$.
Hence, $p(1)$ is true.
Induction:
Let $k \in \mathbb{N}$ such that $p(k)$ is true.
Then $x_k > x_{k+1}$.
Hence, $x_k > \sqrt{6 + x_k}$, so $6 + x_k > 6 + \sqrt{6 + x_k}$.
Since $x_n > 0$ for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$, then $x_k > 0$, so $6 + x_k > 0$.
Thus, $\sqrt{6 + x_k} > 0$, so $6 + \sqrt{6 + x_k} > 0$.
Therefore, $6 + x_k > 6 + \sqrt{6 + x_k} > 0$, so $\sqrt{6 + x_k} > \sqrt{6 + \sqrt{6 + x_k}}$.
Hence, $x_{k+1} > \sqrt{6 + x_{k+1}}$, so $x_{k+1} > x_{k+2}$.
Thus, $p(k + 1)$ is true.
Hence, $p(k)$ implies $p(k + 1)$ for all $k \in \mathbb{N}$.

Since $p(1)$ is true and $p(k)$ implies $p(k+1)$ for all $k \in \mathbb{N}$, then by induction, $p(n)$ is true for all $n \in \mathbb{N}$.

Therefore, $x_n > x_{n+1}$ for all $n \in \mathbb{N}$. □

Proof. We prove $\lim_{n\to\infty} x_n = 3$.

Since $x_n > x_{n+1}$ for all $n \in \mathbb{N}$, then (x_n) is strictly decreasing, so (x_n) is decreasing.

Since $x_n > 0$ for all $n \in \mathbb{N}$, then 0 is a lower bound of (x_n), so (x_n) is bounded below.

Since (x_n) is decreasing and bounded below, then by MCT, (x_n) is convergent.

Thus, there is a real number L such that $\lim_{n\to\infty} x_n = L$.

We must prove $L = 3$.

Since (x_n) is convergent and 0 is a lower bound of (x_n), then $0 \leq L$, so $L \geq 0$.

Since $\lim_{n\to\infty} x_n = L$, then $\lim_{n\to\infty} x_{n+1} = L$.

Thus, $L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{6 + x_n}$.

Observe that

$$\begin{aligned} 6 + L &= \lim_{n\to\infty} 6 + \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} (6 + x_n) \\ &= \lim_{n\to\infty} (\sqrt{6 + x_n})(\sqrt{6 + x_n}) \\ &= \lim_{n\to\infty} (\sqrt{6 + x_n}) \cdot \lim_{n\to\infty} (\sqrt{6 + x_n}) \\ &= L \cdot L. \end{aligned}$$

Thus, $6 + L = L^2$, so $0 = L^2 - L - 6 = (L - 3)(L + 2)$.

Hence, either $L - 3 = 0$ or $L + 2 = 0$.

Since $L \geq 0$, then $L + 2 \geq 2 > 0$, so $L + 2 > 0$.

Thus, $L + 2 \neq 0$, so $L - 3 = 0$.

Therefore, $L = 3$, as desired. □

Exercise 77. Let (a_n) be a sequence of real numbers bounded above in $\mathbb{R}$.

Let $b_n = \max\{a_1, a_2, ..., a_n\}$ for all $n \in \mathbb{N}$.

Show that the sequence (b_n) is increasing.

Show that if M is an upper bound of (a_n), then M is an upper bound of (b_n).

Show that $\sup(a_n)$ exists.

Proof. Since $b_n = \max\{a_1, a_2, ..., a_n\}$ for all $n \in \mathbb{N}$, then (b_n) is a sequence of real numbers.

Let $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$.

Since (a_n) is bounded above in $\mathbb{R}$, then A has an upper bound in $\mathbb{R}$.

Let $M \in \mathbb{R}$ be an arbitrary upper bound of A.

Let $n \in \mathbb{N}$ be given.

Let $S = \{a_1, a_2, ..., a_n\}$.

Since $b_n = \max\{a_1, a_2, ..., a_n\}$, then $b_n = \max S$, so $b_n \in S$.
Let $S' = S \cup \{a_{n+1}\}$.
Since $b_{n+1} = \max\{a_1, a_2, ..., a_n, a_{n+1}\}$, then $b_{n+1} = \max S'$.
Since $S \subset S \cup \{a_{n+1}\} = S'$, then $S \subset S'$.
Since $b_n \in S$ and $S \subset S'$, then $b_n \in S'$.
Since $b_{n+1} = \max S'$, then b_{n+1} is an upper bound of S', so $b_n \leq b_{n+1}$.
Thus, (b_n) is increasing.
Since $b_n \in S$ and $S \subset A$, then $b_n \in A$.
Since M is an upper bound of A, then $b_n \leq M$.
Therefore, M is an upper bound of B, so (b_n) is bounded above in $\mathbb{R}$.
Since (b_n) is increasing and bounded above, then by MCT, $\sup B$ exists and $\lim_{n\to\infty} b_n = \sup B$.
Since $\sup B$ is the least upper bound of B and M is an upper bound of B, then $\sup B \leq M$.
Since $a_n \in S$ and $b_n = \max S$ and $b_n \in B$, then $a_n \leq \max S = b_n \leq \sup B$, so $a_n \leq \sup B$.
Hence, $\sup B$ is an upper bound of A.
Since $\sup B \leq M$ and M is an arbitrary upper bound of A, then $\sup B$ is the least upper bound of A.
Therefore, $\sup B = \sup A$.
Thus, if (a_n) is a sequence of real numbers bounded above in $\mathbb{R}$, then $\sup(a_n)$ exists. □

Exercise 78. Let $r \in \mathbb{R}$.
Let (r^n) be a geometric sequence.
If $0 < r < 1$, then $r^n \to 0$.

Proof. Suppose $0 < r < 1$.
Then $0 < r$ and $r < 1$.
Since $r > 0$, then $r^n > 0$ for all $n \in \mathbb{N}$.

We prove $r^n < 1$ by induction on n.
Let $S = \{n \in \mathbb{N} : r^n < 1\}$.
Since $r^1 = r < 1$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $r^k < 1$.
Since $r^k < 1$ and $r > 0$, then $r^{k+1} = r^k r < 1 \cdot r = r < 1$.
Thus, $r^{k+1} < 1$, so $k + 1 \in S$.
Therefore, by PMI, $S = \mathbb{N}$, so $r^n < 1$ for all $n \in \mathbb{N}$.
Since $r^n > 0$ for all $n \in \mathbb{N}$ and $r^n < 1$ for all $n \in \mathbb{N}$, then $0 < r^n < 1$ for all $n \in \mathbb{N}$.
Thus, (r^n) is bounded.

We prove (r^n) is strictly decreasing.
Let $n \in \mathbb{N}$ be given.
Since $r^n > 0$ and $1 > r$, then $r^n > r^n r = r^{n+1}$.
Therefore, $r^n > r^{n+1}$, so (r^n) is strictly decreasing.
Thus, (r^n) is monotonic.
Hence, by MCT, (r^n) is convergent.
Therefore, there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} r^n = L$.
Observe that

$$\begin{aligned} L &= \lim_{n\to\infty} r^n \\ &= \lim_{n\to\infty} r^{n+1} \\ &= \lim_{n\to\infty} (r^n r) \\ &= (\lim_{n\to\infty} r^n)(\lim_{n\to\infty} r) \\ &= Lr. \end{aligned}$$

Thus, $L = Lr$, so $Lr - L = 0$.
Hence, $L(r-1) = 0$, so either $L = 0$ or $r - 1 = 0$.
Suppose $r - 1 = 0$.
Then $r = 1$.
But, this contradicts the assumption that $r < 1$.
Thus, $r - 1 \neq 0$, so $L = 0$.
Therefore, $\lim_{n\to\infty} r^n = 0$, as desired. $\square$

Exercise 79. Let (a_n) be a sequence defined by $a_1 = 3$ and $a_2 = \frac{3}{2}$ and $a_n = \frac{a_{n-1}+a_{n-2}}{2}$ for all natural numbers $n > 2$.
Then $a_n = 2 + (\frac{-1}{2})^{n-1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 2$.

Proof. We first prove $a_n = 2 + (\frac{-1}{2})^{n-1}$ for all $n \in \mathbb{N}$ by strong induction on n.
Let $S = \{n \in \mathbb{N} : a_n = 2 + (\frac{-1}{2})^{n-1}\}$.
Since $1 \in \mathbb{N}$ and $a_1 = 2 + (\frac{-1}{2})^{1-1} = 2 + 1 = 3$, then $1 \in S$.
Since $2 \in \mathbb{N}$ and $a_2 = 2 + (\frac{-1}{2})^{2-1} = 2 - \frac{1}{2} = \frac{3}{2}$, then $2 \in S$.
Let $k \in \mathbb{N}$ with $n \geq 2$.
Suppose $m \in S$ for all $m \in \mathbb{N}$ such that $1 \leq m \leq k$.
To prove $k+1 \in S$, we must prove $k+1 \in \mathbb{N}$ and $a_{k+1} = 2 + (\frac{-1}{2})^k$.
Since $k \in \mathbb{N}$, then $k+1 \in \mathbb{N}$.
Since $k \in \mathbb{N}$ and $1 < 2 \leq k$, then $1 < k$, so $k \in S$.
Hence, $a_k = 2 + (\frac{-1}{2})^{k-1}$.
Since $2 \leq k < k+1$, then $1 \leq k-1 < k$, so $k-1 \in S$.
Hence, $a_{k-1} = 2 + (\frac{-1}{2})^{k-2}$.

Observe that

$$\begin{aligned}
a_{k+1} &= \frac{a_k + a_{k-1}}{2} \\
&= \frac{2 + (\frac{-1}{2})^{k-1} + 2 + (\frac{-1}{2})^{k-2}}{2} \\
&= \frac{4 + (\frac{-1}{2})^{k-2}(\frac{-1}{2} + 1)}{2} \\
&= \frac{4 + (\frac{-1}{2})^{k-2}(\frac{1}{2})}{2} \\
&= 2 + (\frac{-1}{2})^{k-2}(\frac{1}{2})^2 \\
&= 2 + (\frac{-1}{2})^{k-2}(\frac{-1}{2})^2 \\
&= 2 + (\frac{-1}{2})^k.
\end{aligned}$$

Since $k+1 \in \mathbb{N}$ and $a_{k+1} = 2 + (\frac{-1}{2})^k$, then $k+1 \in S$.

Thus, $k \in S$ implies $k+1 \in S$, so by the principle of mathematical induction, $a_n = 2 + (\frac{-1}{2})^{n-1}$ for all $n \in \mathbb{N}$, as desired. □

Proof. We prove $\lim_{n\to\infty} a_n = 2$.

Thus,

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} [2 + (\frac{-1}{2})^{n-1}] \\
&= \lim_{n\to\infty} 2 + \lim_{n\to\infty} (\frac{-1}{2})^{n-1} \\
&= 2 + 0 \\
&= 2.
\end{aligned}$$

□

Exercise 80. limit of a monotonic convergent sequence is a bound of the sequence

Let (a_n) be a convergent sequence in $\mathbb{R}$.

1. If (a_n) is increasing, then $\lim_{n\to\infty} a_n$ is an upper bound of (a_n).
2. If (a_n) is decreasing, then $\lim_{n\to\infty} a_n$ is a lower bound of (a_n).

Proof. We prove 1.

Since (a_n) is convergent, then there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = L$.

Suppose (a_n) is increasing.

We must prove L is an upper bound of (a_n).

Suppose L is not an upper bound of (a_n).

Then there exists $k \in \mathbb{N}$ such that $a_k > L$.

Thus, $a_k - L > 0$.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < a_k - L$.

Let $m = \max\{k, N\}$.

Then $m \geq k$ and $m \geq N$.

Since $m + 1 > m$ and $m \geq N$, then $m + 1 > N$, so $|a_{m+1} - L| < a_k - L$.

Since $k \leq m$ and $m < m + 1$, then $k < m + 1$.

Since (a_n) is increasing and $k < m + 1$, then $a_k \leq a_{m+1}$.

Since $L < a_k$ and $a_k \leq a_{m+1}$, then $L < a_{m+1}$, so $a_{m+1} - L > 0$.

Thus, $a_{m+1} - L = |a_{m+1} - L| < a_k - L$, so $a_{m+1} - L < a_k - L$.

Hence, $a_{m+1} < a_k$, so $a_k > a_{m+1}$.

But, this contradicts the fact that $a_k \leq a_{m+1}$.

Hence, L is an upper bound of (a_n). ☐

Proof. We prove 2.

Since (a_n) is convergent, then there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = L$.

Suppose (a_n) is decreasing.

We must prove L is a lower bound of (a_n).

Suppose L is not a lower bound of (a_n).

Then there exists $k \in \mathbb{N}$ such that $a_k < L$.

Thus, $L - a_k > 0$.

Since $\lim_{n\to\infty} a_n = L$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < L - a_k$.

Let $m = \max\{k, N\}$.

Then $m \geq k$ and $m \geq N$.

Since $m + 1 > m$ and $m \geq N$, then $m + 1 > N$, so $|a_{m+1} - L| < L - a_k$.

Since $k \leq m$ and $m < m + 1$, then $k < m + 1$.

Since (a_n) is decreasing and $k \leq m + 1$, then $a_k \geq a_{m+1}$.

Since $a_{m+1} \leq a_k$ and $a_k < L$, then $a_{m+1} < L$, so $a_{m+1} - L < 0$.

Thus, $L - a_{m+1} = |a_{m+1} - L| < L - a_k$, so $L - a_{m+1} < L - a_k$.

Hence, $-a_{m+1} < -a_k$, so $a_{m+1} > a_k$.

But, this contradicts the fact that $a_{m+1} \leq a_k$.

Hence, L is a lower bound of (a_n). ☐

Let (a_n) be an increasing sequence such that $\lim_{n\to\infty} a_n = L$.

Then L is an upper bound of (a_n).

Let M be any upper bound of (a_n).

Since $\lim_{n\to\infty} a_n = L$ and M is an upper bound of (a_n), then $L \leq M$.

Hence, L is the least upper bound of (a_n).

Therefore, an increasing convergent sequence converges to the least upper bound of the sequence.

Thus, if (a_n) is increasing and convergent, then $\lim_{n\to\infty} a_n = \sup(a_n)$.

Let (a_n) be a decreasing sequence such that $\lim_{n\to\infty} a_n = L$.

Then L is a lower bound of (a_n).

Let m be any lower bound of (a_n).

Since $\lim_{n\to\infty} a_n = L$ and m is a lower bound of (a_n), then $m \leq L$.

Hence, L is the greatest lower bound of (a_n).

Therefore, a decreasing convergent sequence converges to the greatest lower bound of the sequence.

Thus, if (a_n) is decreasing and convergent, then $\lim_{n\to\infty} a_n = \inf(a_n)$.

Exercise 81. Let (x_n) be a sequence of real numbers defined by $x_{n+1} = \frac{1}{2+x_n}$ for all $n \in \mathbb{N}$ and $x_1 = \frac{1}{2}$.

Then $\lim_{n\to\infty} x_n = \sqrt{2} - 1$.

Proof. We prove (x_n) is convergent by proving its even and odd subsequences converge to the same value. We prove the even subsequence (x_{2n}) is convergent using MCT by proving it is increasing and bounded above. We prove the odd subsequence (x_{2n-1}) is convergent using MCT by proving it is decreasing and bounded below.

We first prove $x_n > 0$ for all $n \in \mathbb{Z}^+$. To do this, we must prove $x_{2n} > 0$ for all $n \in \mathbb{Z}^+$ and $x_{2n-1} > 0$ for all $n \in \mathbb{Z}^+$. $\square$

Proof. We prove $x_{2n} > 0$ for all $n \in \mathbb{Z}^+$ by induction on n.

Let $S = \{n \in \mathbb{Z}^+ : x_{2n} > 0\}$.

Since $1 \in \mathbb{Z}^+$ and $x_2 = \frac{1}{2+x_1} = \frac{1}{2+\frac{1}{2}} = \frac{2}{5} > 0$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{Z}^+$ and $x_{2k} > 0$.

Since $k \in \mathbb{Z}^+$, then $k+1 \in \mathbb{Z}^+$ and $2k \in \mathbb{Z}^+$ and $2k+1 \in \mathbb{Z}^+$.

Observe that

$$\begin{aligned} x_{2(k+1)} &= x_{2k+2} \\ &= x_{(2k+1)+1} \\ &= \frac{1}{2 + x_{2k+1}} \\ &= \frac{1}{2 + \frac{1}{2+x_{2k}}} \\ &= \frac{x_{2k}+2}{2x_{2k}+5}. \end{aligned}$$

Since $x_{2(k+1)} = \frac{x_{2k}+2}{2x_{2k}+5}$ and $x_{2k} > 0$, then $x_{2k} + 2 > 0$ and $2x_{2k} + 5 > 0$, so $x_{2(k+1)} > 0$.

Since $k+1 \in \mathbb{Z}^+$ and $x_{2(k+1)} > 0$, then $k+1 \in S$.

Thus, $k \in S$ implies $k+1 \in S$.

By the principle of mathematical induction, $x_{2n} > 0$ for all $n \in \mathbb{Z}^+$. $\square$

Proof. We prove $x_{2n-1} > 0$ for all $n \in \mathbb{Z}^+$ by induction on n.

Let $S = \{n \in \mathbb{Z}^+ : x_{2n-1} > 0\}$.

Since $1 \in \mathbb{Z}^+$ and $x_1 = \frac{1}{2} > 0$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{Z}^+$ and $x_{2k-1} > 0$.
Since $k \in \mathbb{Z}^+$, then $k \in \mathbb{Z}$ and $k+1 \in \mathbb{Z}^+$ and $2k \in \mathbb{Z}^+$.
Since $k \in \mathbb{Z}^+$, then $k \geq 1 > \frac{1}{2}$, so $k > \frac{1}{2}$.
Hence, $2k > 1$, so $2k - 1 > 0$.
Since $k \in \mathbb{Z}$, then $2k - 1 \in \mathbb{Z}$.
Since $2k - 1 \in \mathbb{Z}$ and $2k - 1 > 0$, then $2k - 1 \in \mathbb{Z}^+$.
Observe that

$$\begin{aligned} x_{2(k+1)-1} &= x_{2k+1} \\ &= \frac{1}{2 + x_{2k}} \\ &= \frac{1}{2 + x_{(2k-1)+1}} \\ &= \frac{1}{2 + \frac{1}{2 + x_{2k-1}}} \\ &= \frac{x_{2k-1} + 2}{2x_{2k-1} + 5}. \end{aligned}$$

Since $x_{2(k+1)-1} = \frac{x_{2k-1}+2}{2x_{2k-1}+5}$ and $x_{2k-1} > 0$, then $x_{2k-1} + 2 > 0$ and $2x_{2k-1} + 5 > 0$, so $x_{2(k+1)-1} > 0$.
Since $k + 1 \in \mathbb{Z}^+$ and $x_{2(k+1)-1} > 0$, then $k + 1 \in S$.
Thus, $k \in S$ implies $k + 1 \in S$.
By the principle of mathematical induction, $x_{2n-1} > 0$ for all $n \in \mathbb{Z}^+$.
Therefore, 0 is a lower bound for (x_{2n-1}), so (x_{2n-1}) is bounded below. □

Proof. We prove $x_n > 0$ for all $n \in \mathbb{Z}^+$.
Let $n \in \mathbb{Z}^+$.
Then either n is even or n is odd.
We consider these cases separately.
Case 1: Suppose n is even.
Then $n = 2k$ for some positive integer k.
Since $x_{2n} > 0$ for all $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$, then $x_n = x_{2k} > 0$.
Case 2: Suppose n is odd.
Then $n = 2k - 1$ for some positive integer k.
Since $x_{2n-1} > 0$ for all $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$, then $x_n = x_{2k-1} > 0$.
Hence, in all cases, $x_n > 0$.
Therefore, $x_n > 0$ for all $n \in \mathbb{Z}^+$, as desired. □

Proof. To prove (x_{2n}) is bounded above, we prove $x_{2n} < \frac{1}{2}$ for all $n \in \mathbb{Z}^+$ by induction on n.
Let $S = \{n \in \mathbb{Z}^+ : x_{2n} < \frac{1}{2}\}$.
Since $1 \in \mathbb{Z}^+$ and $x_2 = \frac{2}{5} < \frac{1}{2}$, then $1 \in S$.
Suppose $k \in S$.

Then $k \in \mathbb{Z}^+$ and $x_{2k} < \frac{1}{2}$.
Since $k \in \mathbb{Z}^+$, then $k+1 \in \mathbb{Z}^+$ and $2k \in \mathbb{Z}^+$.
Since $2k \in \mathbb{Z}^+$, then $2k+1 \in \mathbb{Z}^+$.
Hence, $x_{2k+1} > 0$, so $2 + x_{2k+1} > 2 > 0$.
Thus, $\frac{1}{2} > \frac{1}{2+x_{2k+1}} = x_{2k+2} = x_{2(k+1)}$.
Since $k+1 \in \mathbb{Z}^+$ and $x_{2(k+1)} < \frac{1}{2}$, then $k+1 \in S$.
Thus, $k \in S$ implies $k+1 \in S$.
By the principle of mathematical induction, $x_{2n} < \frac{1}{2}$ for all $n \in \mathbb{Z}^+$.
Therefore, $\frac{1}{2}$ is an upper bound for (x_{2n}), so (x_{2n}) is bounded above, as desired. ☐

Proof. To prove (x_{2n}) is strictly increasing, we prove $x_{2n} < x_{2n+2}$ for all $n \in \mathbb{Z}^+$ by induction on n.
Let $S = \{n \in \mathbb{Z}^+ : x_{2n} < x_{2n+2}\}$.
Since $1 \in \mathbb{Z}^+$, then $x_2 = \frac{1}{2+x_1} = \frac{1}{2+\frac{1}{2}} = \frac{2}{5}$ and $x_3 = \frac{1}{2+x_2} = \frac{1}{2+\frac{2}{5}} = \frac{5}{12}$ and $x_4 = \frac{1}{2+x_3} = \frac{1}{2+\frac{5}{12}} = \frac{12}{29}$.
Since $1 \in \mathbb{Z}^+$ and $x_2 = \frac{2}{5} < \frac{12}{29} = x_4$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{Z}^+$ and $x_{2k} < x_{2k+2}$.
Since $k \in \mathbb{Z}^+$, then $k+1 \in \mathbb{Z}^+$ and $2k \in \mathbb{Z}^+$ and $2k+1 \in \mathbb{Z}^+$ and $2k+2 \in \mathbb{Z}^+$ and $2k+3 \in \mathbb{Z}^+$.
Since $2k \in \mathbb{Z}^+$, then $x_{2k} > 0$, so $2 + x_{2k} > 0$.
Since $2k+1 \in \mathbb{Z}^+$, then $x_{2k+1} > 0$, so $2 + x_{2k+1} > 0$.
Since $2k+2 \in \mathbb{Z}^+$, then $x_{2k+2} > 0$, so $2 + x_{2k+2} > 0$.
Since $2k+3 \in \mathbb{Z}^+$, then $x_{2k+3} > 0$, so $2 + x_{2k+3} > 0$.
Observe that

$$\begin{aligned}
x_{2k} < x_{2k+2} &\Leftrightarrow 2 + x_{2k} < 2 + x_{2k+2} \\
&\Rightarrow \frac{1}{2+x_{2k+2}} < \frac{1}{2+x_{2k}} \\
&\Rightarrow x_{2k+3} < x_{2k+1} \\
&\Leftrightarrow 2 + x_{2k+3} < 2 + x_{2k+1} \\
&\Rightarrow \frac{1}{2+x_{2k+1}} < \frac{1}{2+x_{2k+3}} \\
&\Rightarrow x_{2k+2} < x_{2k+4} \\
&\Leftrightarrow x_{2(k+1)} < x_{2(k+1)+2}.
\end{aligned}$$

Since $x_{2k} < x_{2k+2}$, then $x_{2(k+1)} < x_{2(k+1)+2}$.
Since $k+1 \in \mathbb{Z}^+$ and $x_{2(k+1)} < x_{2(k+1)+2}$, then $k+1 \in S$.
Thus, $k \in S$ implies $k+1 \in S$.
By the principle of mathematical induction, $x_{2n} < x_{2n+2}$ for all $n \in \mathbb{Z}^+$.
Therefore, (x_{2n}) is strictly increasing, as desired. ☐

Proof. To prove (x_{2n-1}) is strictly decreasing, we prove $x_{2n-1} > x_{2n+1}$ for all $n \in \mathbb{Z}^+$ by induction on n.

Let $S = \{n \in \mathbb{Z}^+ : x_{2n-1} > x_{2n+1}\}$.
Since $1 \in \mathbb{Z}^+$ and $x_1 = \frac{1}{2} > \frac{5}{12} = x_3$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{Z}^+$ and $x_{2k-1} > x_{2k+1}$.
Since $k \in \mathbb{Z}^+$, then $k \in \mathbb{Z}$ and $k \geq 1$ and $k+1 \in \mathbb{Z}^+$ and $2k \in \mathbb{Z}^+$ and $2k+1 \in \mathbb{Z}^+$ and $2k+2 \in \mathbb{Z}^+$.
Since $k \in \mathbb{Z}$, then $2k-1 \in \mathbb{Z}$.
Since $k \geq 1$, then $2k \geq 2$, so $2k-1 \geq 1 > 0$.
Hence, $2k-1 > 0$.
Since $2k-1 \in \mathbb{Z}$ and $2k-1 > 0$, then $2k-1 \in \mathbb{Z}^+$
Thus, $x_{2k-1} > 0$, so $2 + x_{2k-1} > 0$.
Since $2k \in \mathbb{Z}^+$, then $x_{2k} > 0$, so $2 + x_{2k} > 0$.
Since $2k+1 \in \mathbb{Z}^+$, then $x_{2k+1} > 0$, so $2 + x_{2k+1} > 0$.
Since $2k+2 \in \mathbb{Z}^+$, then $x_{2k+2} > 0$, so $2 + x_{2k+2} > 0$.
Observe that

$$\begin{aligned}
x_{2k-1} > x_{2k+1} \quad &\Leftrightarrow \quad 2 + x_{2k-1} > 2 + x_{2k+1} \\
&\Rightarrow \quad \frac{1}{2 + x_{2k+1}} > \frac{1}{2 + x_{2k-1}} \\
&\Rightarrow \quad x_{2k+2} > x_{2k} \\
&\Leftrightarrow \quad 2 + x_{2k+2} > 2 + x_{2k} \\
&\Rightarrow \quad \frac{1}{2 + x_{2k}} > \frac{1}{2 + x_{2k+2}} \\
&\Rightarrow \quad x_{2k+1} > x_{2k+3} \\
&\Leftrightarrow \quad x_{2(k+1)-1} > x_{2(k+1)+1}.
\end{aligned}$$

Since $x_{2k-1} > x_{2k+1}$, then $x_{2(k+1)-1} > x_{2(k+1)+1}$.
Since $k+1 \in \mathbb{Z}^+$ and $x_{2(k+1)-1} > x_{2(k+1)+1}$, then $k+1 \in S$.
Thus, $k \in S$ implies $k+1 \in S$.
By the principle of mathematical induction, $x_{2n-1} > x_{2n+1}$ for all $n \in \mathbb{Z}^+$.
Therefore, (x_{2n-1}) is strictly decreasing, as desired. □

Proof. Since (x_{2n}) is strictly increasing and bounded above, then by MCT, $\lim_{n\to\infty} x_{2n} = \sup(x_{2n})$.
Since (x_{2n-1}) is strictly decreasing and bounded below, then by MCT, $\lim_{n\to\infty} x_{2n-1} = \inf(x_{2n-1})$.
To prove $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n-1}$, we must prove $\sup(x_{2n}) = \inf(x_{2n-1})$. □

Proof. We prove $x_2 < x_{2n-1}$ for all $n \in \mathbb{Z}^+$ by induction on n.
Let $S = \{n \in \mathbb{Z}^+ : x_2 < x_{2n-1}\}$.
Since $1 \in \mathbb{Z}^+$ and $x_2 = \frac{2}{5} < \frac{1}{2} = x_1$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{Z}^+$ and $x_2 < x_{2k-1}$.
Since $k \in \mathbb{Z}^+$, then $k+1 \in \mathbb{Z}^+$ and $2k \in \mathbb{Z}^+$.
Since $x_{2n} < \frac{1}{2}$ for all $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$, then $x_{2k} < \frac{1}{2} = x_1$.

Since $2k \in \mathbb{Z}^+$, then $x_{2k} > 0$, so $2 + x_{2k} > 0$.
Thus, $0 < 2 + x_{2k} < 2 + x_1$, so $\frac{1}{2+x_1} < \frac{1}{2+x_{2k}}$.
Hence, $x_2 < x_{2k+1} = x_{2(k+1)-1}$.
Since $k + 1 \in \mathbb{Z}^+$ and $x_2 < x_{2(k+1)-1}$, then $k + 1 \in S$.
Thus, $k \in S$ implies $k + 1 \in S$.
By the principle of mathematical induction, $x_2 < x_{2n-1}$ for all $n \in \mathbb{Z}^+$. □

Proof. We next prove $x_{2n} < x_{2m-1}$ for all $m, n \in \mathbb{Z}^+$.
Let $m \in \mathbb{Z}^+$ be given.
We must prove $x_{2n} < x_{2m-1}$ for all $n \in \mathbb{Z}^+$.
Either $m = 1$ or $m > 1$.
We consider these cases separately.
Case 1: Suppose $m = 1$.
Let $n \in \mathbb{Z}^+$ be given.
Since $x_{2n} < \frac{1}{2}$ for all $n \in \mathbb{Z}^+$, then $x_{2n} < \frac{1}{2} = x_1 = x_{2m-1}$.
Therefore, $x_{2n} < x_{2m-1}$ for all $n \in \mathbb{Z}^+$.
Case 2: Suppose $m > 1$.
Since $m \in \mathbb{Z}^+$ and $m > 1$, then $m \geq 2$.
Let $T = \{n \in \mathbb{Z}^+ : x_{2n} < x_{2m-1}\}$.
Since $x_2 < x_{2m-1}$ for all $m \in \mathbb{Z}^+$, then $x_2 < x_{2m-1}$.
Since $1 \in \mathbb{Z}^+$ and $x_2 < x_{2m-1}$, then $1 \in T$.
Suppose $k \in T$.
Then $k \in \mathbb{Z}^+$ and $x_{2k} < x_{2m-1}$.
Since $k \in \mathbb{Z}^+$, then $k \in \mathbb{Z}$ and $k + 1 \in \mathbb{Z}^+$ and $2k \in \mathbb{Z}^+$.
Since $2k \in \mathbb{Z}^+$, then $2k + 1 \in \mathbb{Z}^+$, so $x_{2k+1} > 0$.
Since $m \in \mathbb{Z}^+$, then $m \in \mathbb{Z}$, so $2m - 3 \in \mathbb{Z}$.
Since $m \geq 2$, then $2m \geq 4$, so $2m - 3 \geq 1 > 0$.
Hence, $2m - 3 > 0$.
Since $2m - 3 \in \mathbb{Z}$ and $2m - 3 > 0$, then $2m - 3 \in \mathbb{Z}^+$, so $2m - 2 \in \mathbb{Z}^+$ and $2m - 1 \in \mathbb{Z}^+$.
Since (x_{2n-1}) is strictly decreasing and $2m - 3 \in \mathbb{Z}^+$ and $2m - 1 \in \mathbb{Z}^+$ and $2m - 3 < 2m - 1$, then $x_{2m-3} > x_{2m-1}$.
Since $2k \in \mathbb{Z}^+$, then $x_{2k} > 0$, so $2 + x_{2k} > 0$.
Since $2m - 3 \in \mathbb{Z}^+$, then $x_{2m-3} > 0$, so $2 + x_{2m-3} > 0$.
Since $x_{2k+1} > 0$, then $2 + x_{2k+1} > 0$.
Since $2m - 2 \in \mathbb{Z}^+$, then $x_{2m-2} > 0$, so $2 + x_{2m-2} > 0$.
Since $x_{2k} < x_{2m-1}$ and $x_{2m-1} < x_{2m-3}$, then $x_{2k} < x_{2m-3}$.

Observe that

$$\begin{aligned}
x_{2k} < x_{2m-3} &\Leftrightarrow 2 + x_{2k} < 2 + x_{2m-3} \\
&\Rightarrow \frac{1}{2 + x_{2m-3}} < \frac{1}{2 + x_{2k}} \\
&\Rightarrow x_{2m-2} < x_{2k+1} \\
&\Leftrightarrow 2 + x_{2m-2} < 2 + x_{2k+1} \\
&\Rightarrow \frac{1}{2 + x_{2k+1}} < \frac{1}{2 + x_{2m-2}} \\
&\Rightarrow x_{2k+2} < x_{2m-1} \\
&\Leftrightarrow x_{2(k+1)} < x_{2m-1}.
\end{aligned}$$

Thus, $x_{2(k+1)} < x_{2m-1}$.
Since $k + 1 \in \mathbb{Z}^+$ and $x_{2(k+1)} < x_{2m-1}$, then $k + 1 \in T$.
Thus, $k \in T$ implies $k + 1 \in T$.
By the principle of mathematical induction, $x_{2n} < x_{2m-1}$ for all $n \in \mathbb{Z}^+$.
Hence, in all cases, $x_{2n} < x_{2m-1}$ for all $n \in \mathbb{Z}^+$.
Since m is arbitrary, then $x_{2n} < x_{2m-1}$ for all $n \in \mathbb{Z}^+$ for all $m \in \mathbb{Z}^+$. □

Proof. Let $n \in \mathbb{Z}^+$.
Then $x_{2n} < x_{2m-1}$ for all $m \in \mathbb{Z}^+$, so x_{2n} is a lower bound of (x_{2n-1}).
Since $\inf(x_{2n-1})$ is the greatest lower bound of (x_{2n-1}), then $x_{2n} \leq \inf(x_{2n-1})$.
Since n is arbitrary, then $x_{2n} \leq \inf(x_{2n-1})$ for all $n \in \mathbb{Z}^+$.
Hence, $\inf(x_{2n-1})$ is an upper bound of (x_{2n}).
We now prove $\inf(x_{2n-1})$ is truly the least upper bound of (x_{2n}).
Since $\sup(x_{2n})$ is the least upper bound of (x_{2n}), then $\sup(x_{2n}) \leq \inf(x_{2n-1})$.
Now, we must prove $\sup(x_{2n}) = \inf(x_{2n-1})$.
Could we argue by contradiction that it must be impossible for $\sup(x_{2n}) < \inf(x_{2n-1})$? □

Proof. To prove $\inf(x_{2n-1})$ is the least upper bound of (x_{2n}), we must prove $\inf(x_{2n-1})$ is an upper bound of (x_{2n}) and that it is the least such upper bound.

Since $\lim_{n\to\infty} x_{2n} = L = \lim_{n\to\infty} x_{2n-1}$, then by a lemma proved in real series notes, we have $\lim_{n\to\infty} x_n = L$. Since $\lim_{n\to\infty} x_n = L$ and $x_{n+1} = \frac{1}{2+x_n}$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}
L &= \lim_{n\to\infty} x_n \\
&= \lim_{n\to\infty} x_{n+1} \\
&= \lim_{n\to\infty} \frac{1}{2 + x_n} \\
&= \frac{1}{2 + \lim_{n\to\infty} x_n} \\
&= \frac{1}{2 + L}.
\end{aligned}$$

Thus, $L = \frac{1}{2+L}$, so $2L + L^2 = 1$.

Hence, $(L+1)^2 = L^2 + 2L + 1 = 1 + 1 = 2$, so $L + 1 = \pm\sqrt{2}$.
Therefore, either $L = -1 + \sqrt{2}$ or $L = -1 - \sqrt{2}$.
Since $L > 0 > -(1+\sqrt{2}) = -1 - \sqrt{2}$, then $L > -1 - \sqrt{2}$, so $L \neq -1 - \sqrt{2}$.
Thus, $L = -1 + \sqrt{2} = \sqrt{2} - 1$, so $\lim_{n\to\infty} x_n = \sqrt{2} - 1$, as desired. □

Bolzano-Weierstrass theorem

Exercise 82. Let (a_n) and (b_n) be sequences in $\mathbb{R}$.
Let (I_n) be a sequence of intervals defined by $I_n = [a_n, b_n]$ for all $n \in \mathbb{N}$.
If $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$ and $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, then $\alpha \in \cap_{n=1}^{\infty} I_n$ is unique.

Proof. Suppose $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$ and $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$.
Since (I_n) is a sequence of nonempty closed bounded intervals in $\mathbb{R}$ and $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$, then by the Nested intervals theorem, $\sup(a_n) \in \cap_{n=1}^{\infty} I_n$.
Let $\alpha = \sup(a_n)$.
Then $\alpha \in \cap_{n=1}^{\infty} I_n$, so $\alpha \in I_n$ for all $n \in \mathbb{N}$.
To prove α is unique, let $\beta \in \cap_{n=1}^{\infty} I_n$.
Then $\beta \in \mathbb{R}$ and $\beta \in I_n$ for all $n \in \mathbb{N}$.
We must prove $\beta = \alpha$.
Since $\beta \in I_n$ for all $n \in \mathbb{N}$, then $\beta \in [a_n, b_n]$ for all $n \in \mathbb{N}$, so $a_n \leq \beta \leq b_n$ for all $n \in \mathbb{N}$.
Thus, $a_n \leq \beta$ and $\beta \leq b_n$ for all $n \in \mathbb{N}$, so $a_n \leq \beta$ for all $n \in \mathbb{N}$.
Hence, β is an upper bound of (a_n).
Since α is the least upper bound of (a_n) and β is an upper bound of (a_n), then $\alpha \leq \beta$, so $\beta - \alpha \geq 0$.
Let $S = \{b_n - a_n : n \in \mathbb{N}\}$.
Then $\inf S = 0$.
Since 0 is the greatest lower bound of S, then for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $b_k - a_k < \epsilon$.
Let $\epsilon > 0$ be given.
Then there exists $k \in \mathbb{N}$ such that $b_k - a_k < \epsilon$.
Since $k \in \mathbb{N}$, then $I_k = [a_k, b_k]$.
Since $k \in \mathbb{N}$ and $\alpha \in I_n$ for all $n \in \mathbb{N}$, then $\alpha \in I_k$.
Since $k \in \mathbb{N}$ and $\beta \in I_n$ for all $n \in \mathbb{N}$, then $\beta \in I_k$.
Since $\alpha \in I_k$ and $\beta \in I_k$ and $I_k = [a_k, b_k]$, then the distance between α and β is less than the length of the interval I_k.
Thus, $|\beta - \alpha| = \beta - \alpha < b_k - a_k < \epsilon$, so $|\beta - \alpha| < \epsilon$.
Hence, $|\beta - \alpha| < \epsilon$ for every $\epsilon > 0$, so $\beta - \alpha = 0$.
Therefore, $\beta = \alpha$, as desired. □

Cauchy sequences

Exercise 83. The sequence $(\frac{1}{n})$ is a Cauchy sequence.

Proof. Let $\epsilon > 0$ be given.
Then $\epsilon \neq 0$, so $\frac{1}{\epsilon} \in \mathbb{R}$.
By the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$, so $\epsilon > \frac{1}{N}$.
Let $m, n \in \mathbb{Z}^+$ such that $m > N$ and $n > N$.
Since $N \in \mathbb{N}$, then $N > 0$.
Since $m > N > 0$, then $0 < \frac{1}{m} < \frac{1}{N}$.
Since $n > N > 0$, then $0 < \frac{1}{n} < \frac{1}{N}$.
Since the distance between any two points in the interval $(0, \frac{1}{N})$ is less than $\frac{1}{N}$, then $|\frac{1}{m} - \frac{1}{n}| < \frac{1}{N} < \epsilon$.
Therefore, $(\frac{1}{n})$ is a Cauchy sequence. □

Exercise 84. If (a_n) and (b_n) are Cauchy sequences, then $(a_n + b_n)$ is a Cauchy sequence.

Proof. Suppose (a_n) and (b_n) are Cauchy sequences.
To prove $(a_n + b_n)$ is a Cauchy sequence, let $\epsilon > 0$ be given.
Then $\frac{\epsilon}{2} > 0$.
Since (a_n) is Cauchy, then there exists $N_1 \in \mathbb{N}$ such that if $m, n > N_1$, then $|a_m - a_n| < \frac{\epsilon}{2}$.
Since (b_n) is Cauchy, then there exists $N_2 \in \mathbb{N}$ such that if $m, n > N_2$, then $|b_m - b_n| < \frac{\epsilon}{2}$.
Let $N = \max\{N_1, N_2\}$.
Let $m, n \in \mathbb{N}$ such that $m > N$ and $n > N$.
Since $m > N \geq N_1$, then $m > N_1$.
Since $n > N \geq N_1$, then $n > N_1$.
Thus, $|a_m - a_n| < \frac{\epsilon}{2}$.
Since $m > N \geq N_2$, then $m > N_2$.
Since $n > N \geq N_2$, then $n > N_2$.
Thus, $|b_m - b_n| < \frac{\epsilon}{2}$.
Therefore,

$$\begin{aligned} |(a_m + b_m) - (a_n + b_n)| &= |(a_m - a_n) + (b_m - b_n)| \\ &\leq |a_m - a_n| + |b_m - b_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, $|(a_m + b_m) - (a_n + b_n)| < \epsilon$, as desired. □

Proof. Suppose (a_n) and (b_n) are Cauchy sequences of real numbers.
Then (a_n) and (b_n) are convergent.
Hence, the sum $(a_n + b_n)$ is convergent, so $(a_n + b_n)$ is Cauchy. □