Sequences in $\mathbb{R}$ Notes

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Sequences of Real Numbers

A sequence is an ordered list of objects.

Definition 1. infinite sequence
An **infinite sequence** (a_n) is a function whose domain is $\mathbb{N}$.

The functional values $a_1, a_2, ..., a_n, ...$ are called the **terms** of the sequence and a_n is the n^{th} term of the sequence.
Since an infinite sequence is a function whose domain is $\mathbb{N}$, then an infinite sequence maps n to a_n for all $n \in \mathbb{N}$.
The term a_n is the n^{th} element in the list which means that a_n is in the n^{th} position in the list.

Definition 2. sequence of real numbers
An **infinite sequence of real numbers** is a function $f : \mathbb{N} \to \mathbb{R}$.

Let (a_n) be an infinite sequence of real numbers.
Then there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$.
The terms of the sequence are $a_1, a_2, a_3, ..., a_n, ...$
The n^{th} term of the sequence is a_n and $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

Conversely, let $f : \mathbb{N} \to \mathbb{R}$ be a function.
Then there exists a sequence (a_n) such that $a_n = f(n)$ for all $n \in \mathbb{N}$.

Definition 3. constant sequence in $\mathbb{R}$
Let k be a real number.
Let (a_n) be a sequence of real numbers defined by $a_n = k$ for all $n \in \mathbb{N}$.
Then (a_n) is called a **constant sequence**.

Let $k \in \mathbb{R}$.
Let (a_n) be a constant sequence given by $a_n = k$ for all $n \in \mathbb{N}$.
Then $a_1 = a_2 = a_3 = ... = k$ and $a_{n+1} = k = a_n$.
The first few terms are $k, k, k, ...$
The graph of (a_n) is the constant linear function $f(n) = k$ with slope 0.
The distance between any terms is zero because all of the terms are equal.

Definition 4. sequence defined recursively

Let f be a function.

Let (a_n) be a sequence defined by $a_{n+1} = f(a_n)$ and $a_n \in domf$ for all positive integers n.

Then $f(n+1) = a_{n+1} = f(a_n)$ for all $n \in \mathbb{Z}^+$, so the sequence (a_n) is defined recursively.

Observe that

$f(1) = a_1$

$f(2) = a_2 = f(a_1)$

$f(3) = a_3 = f(a_2) = f(f(a_1)) = (f \circ f)(a_1)$

$f(4) = a_4 = f(a_3) = f((f \circ f)(a_1)) = (f \circ (f \circ f))(a_1) = (f \circ f \circ f)(a_1)$

...

$f(n) = a_n = (f \circ f ... \circ f)(a_1)$

Therefore, a_n is the value obtained by applying the function f $n-1$ times to a_1.

Definition 5. arithmetic sequence

Let $d \in \mathbb{R}$.

Let (a_n) be a sequence of real numbers defined by $a_{n+1} = a_n + d$ for all positive integers n.

Then (a_n) is called an **arithmetic sequence with common difference** d.

Let (a_n) be an arithmetic sequence with common difference d.

Then there is a function f such that $f(a_n) = a_{n+1} = a_n + d$ for all positive integers n.

Proposition 6. n^{th} ***term of an arithmetic sequence***

Let $d \in \mathbb{R}$.

The n^{th} term of an arithmetic sequence with common difference d and initial value a_1 is $a_n = a_1 + (n-1)d$.

Let (a_n) be an arithmetic sequence with common difference d and initial value a_1.

Then $a_{n+1} = a_n + d$ for all positive integers n and there is a function f such that $f(n) = a_n = a_1 + (n-1)d$ for all positive integers n.

The first few terms are $a_1, a_1 + d, a_1 + 2d, a_1 + 3d, ...$

The graph of (a_n) is the linear function $f(n) = a_1 + (n-1)d$ with slope d.

The difference between consecutive terms is the constant $d = a_{n+1} - a_n$.

Therefore, the distance between any consecutive terms is constant, so the terms are equally spaced.

If $d = 0$, then $a_n = a_1$, so (a_n) is the constant sequence.

Proposition 7. *Let (a_n) be an arithmetic sequence of real numbers with common difference d.*

Then $a_n = \frac{a_{n-1}+a_{n+1}}{2}$ for all integers $n > 1$.

Therefore, the n^{th} term of an arithmetic sequence is the average (arithmetic mean) of its neighboring terms.

Definition 8. geometric sequence

Let $r \in \mathbb{R}$.

Let (a_n) be a sequence of real numbers defined by $a_{n+1} = a_n r$ for all positive integers n.

Then (a_n) is called a **geometric sequence with common ratio** r.

Let (a_n) be a geometric sequence with common ratio r.

Then there is a function f such that $f(a_n) = a_{n+1} = a_n r$ for all positive integers n.

Proposition 9. ***n^{th} term of a geometric sequence***

Let $r \in \mathbb{R}, r \neq 0$.

The n^{th} term of a geometric sequence with common ratio r and initial value a_1 is $a_n = a_1 r^{n-1}$.

Let (a_n) be a geometric sequence with common ratio r and initial value a_1.

Then $a_{n+1} = a_n r$ for all positive integers n and there is a function f such that $f(n) = a_n = a_1 r^{n-1}$ for all positive integers n.

The first few terms are $a_1, a_1 r, a_1 r^2, a_1 r^3, ...$

The graph of (a_n) is the exponential function $f(n) = a_1 r^{n-1}$.

The ratio between consecutive terms is the constant $r = \frac{a_{n+1}}{a_n}$.

The distance between any consecutive terms increases or decreases, based on the value of r.

Let $a_1 > 0$.

If $r > 1$, then the terms increase by a factor of r.

If $r = 1$, then $a_n = a_1(1)^{n-1} = a_1(1) = a_1$, so (a_n) is the constant sequence.

If $0 < r < 1$, then the terms decrease by a factor of r.

Proposition 10. *Let (a_n) be a geometric sequence of positive real numbers with common ratio positive r.*

Then $a_n = \sqrt{a_{n-1}a_{n+1}}$ for all integers $n > 1$.

Therefore, the n^{th} term of a geometric sequence of positive terms is the geometric mean of its neighboring terms.

Example 11. Fibonacci sequence

Let (f_n) be a sequence of natural numbers defined by $f_1 = 1$, $f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all natural numbers $n > 2$.

The sequence (f_n) is called the **Fibonacci sequence**.

The first few terms are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$

Sequences as Functions

Definition 12. bounded sequence in $\mathbb{R}$

A sequence (a_n) of real numbers is said to be

i. **bounded above** iff its range is bounded above in $\mathbb{R}$.
ii. **bounded below** iff its range is bounded below in $\mathbb{R}$.
iii. **bounded** iff its range is a bounded set in $\mathbb{R}$.
iv. **unbounded** iff (a_n) is not bounded.

Let S be the range of (a_n).
Then S is the set of all the terms of the sequence.
Hence, $S = \{a_n : n \in \mathbb{N}\}$ and $S \subset \mathbb{R}$.
S is bounded above in $\mathbb{R}$ iff there exists $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in S$.
Hence, (a_n) is bounded above iff there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$.
Therefore, (a_n) is bounded above iff $(\exists M \in \mathbb{R})(\forall n \in \mathbb{N})(a_n \leq M)$.
Similarly, (a_n) is bounded below iff $(\exists m \in \mathbb{R})(\forall n \in \mathbb{N})(m \leq a_n)$.
S is bounded in $\mathbb{R}$ iff S is bounded above and below in $\mathbb{R}$.
Hence, S is bounded in $\mathbb{R}$ iff there exist $m, M \in \mathbb{R}$ such that $m \leq x \leq M$ for all $x \in S$.
Thus, (a_n) is bounded in $\mathbb{R}$ iff there exist $m, M \in \mathbb{R}$ such that $m \leq a_n \leq M$ for all $n \in \mathbb{N}$.
Therefore, (a_n) is bounded in $\mathbb{R}$ iff $(\exists m, M \in \mathbb{R})(\forall n \in \mathbb{N})(m \leq a_n \leq M)$.
Equivalently, S is bounded in $\mathbb{R}$ iff $(\exists M \in \mathbb{R})(\forall x \in S)(|x| \leq M)$.
Therefore, (a_n) is bounded in $\mathbb{R}$ iff $(\exists M \in \mathbb{R})(\forall n \in \mathbb{N})(|a_n| \leq M)$.
Therefore, (a_n) is unbounded in $\mathbb{R}$ iff $(\forall M \in \mathbb{R})(\exists n \in \mathbb{N})(|a_n| > M)$.

Suppose (a_n) is a bounded sequence of real numbers.
Then there exists $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.
Hence, there exists $M > 0$ such that $|a_n| < M$ for all $n \in \mathbb{N}$.

Example 13. sequence of natural numbers $\mathbb{N}$
Let (a_n) be a sequence defined by $a_n = n$ for all $n \in \mathbb{N}$.
Then (a_n) is the sequence of natural numbers and is an arithmetic sequence with common difference 1 and initial value 1.
The first few terms are $1, 2, 3, 4, ...$
Since $\mathbb{N}$ is unbounded above in $\mathbb{R}$, then (a_n) is unbounded above, so (a_n) is an unbounded sequence.
Since all natural numbers are positive, then $a_n = n > 0$ for all $n \in \mathbb{N}$.
Hence, 0 is a lower bound of (a_n), so (a_n) is bounded below.

Example 14. The sequence defined by $a_n = n^2$ for all $n \in \mathbb{N}$ is bounded below by 0, but is not bounded above.

Example 15. The sequence defined by $a_n = -n$ for all $n \in \mathbb{N}$ is bounded above by 0, but is not bounded below.

Example 16. The sequence defined by $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ is bounded above by 1 and is bounded below by 0.

Example 17. The sequence defined by $a_n = \cos(n)$ for all $n \in \mathbb{N}$ is bounded above by 1 and is bounded below by -1.

Proposition 18. ***sum and product of bounded sequences is bounded***

Let (a_n) and (b_n) be bounded sequences of real numbers. Then
1. $(a_n + b_n)$ is bounded.
2. $(a_n b_n)$ is bounded.

Definition 19. monotonic sequence in $\mathbb{R}$

A sequence (a_n) of real numbers is said to be
i. **strictly increasing** iff $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.
ii. **(monotonic) increasing** iff $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.
iii. **strictly decreasing** iff $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.
iv. **(monotonic) decreasing** iff $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.
v. **monotonic** iff (a_n) is either monotonic increasing or monotonic decreasing.

Therefore, (a_n) is
strictly increasing iff $a_1 < a_2 < a_3 < ... < a_n < ...$
strictly decreasing iff $a_1 > a_2 > a_3 > ... > a_n > ...$
monotonic increasing iff $a_1 \leq a_2 \leq a_3 \leq ... \leq a_n \leq ...$
monotonic decreasing iff $a_1 \geq a_2 \geq a_3 \geq ... \geq a_n \geq ...$.

Example 20. The sequence given by $a_n = n^2$ is strictly increasing.

Example 21. The sequence given by $a_n = 3 + \frac{(-1)^n}{n}$ is neither increasing nor decreasing.

Example 22. constant sequence is increasing and decreasing.

Let k be a real number.

The sequence given by $a_n = k$ is monotonic increasing and decreasing, but is neither strictly increasing nor strictly decreasing.

Example 23. every strictly increasing sequence is (monotonic) increasing.

Let (a_n) be a strictly increasing sequence of real numbers.
Then (a_n) is monotonic increasing.

Therefore, a strictly increasing sequence is monotonic.

Example 24. every strictly decreasing sequence is (monotonic) decreasing.

Let (a_n) be a strictly decreasing sequence of real numbers.
Then (a_n) is monotonic decreasing.

Therefore, a strictly decreasing sequence is monotonic.

Proposition 25. ***necessary and sufficient conditions for a monotonic sequence***

Let (a_n) be a sequence of real numbers. Then

1. (a_n) is strictly increasing iff $m < n$ implies $a_m < a_n$ for all $m, n \in \mathbb{N}$.

2. (a_n) is (monotonic) increasing iff $m < n$ implies $a_m \leq a_n$ for all $m, n \in \mathbb{N}$.

3. (a_n) is strictly decreasing iff $m < n$ implies $a_m > a_n$ for all $m, n \in \mathbb{N}$.

4. (a_n) is (monotonic) decreasing iff $m < n$ implies $a_m \geq a_n$ for all $m, n \in \mathbb{N}$.

Let (a_n) be a sequence.

A subsequence is a sequence that contains only some of terms of (a_n) in the same order.

Definition 26. subsequence in $\mathbb{R}$

Let (a_n) be a sequence of real numbers defined by the function $f : \mathbb{N} \to \mathbb{R}$.

A sequence (b_n) of real numbers is a **subsequence of** (a_n) iff there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $b_n = (f \circ g)(n)$ for all $n \in \mathbb{N}$.

Let (a_n) be a sequence of real numbers.

Then there exists a function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = a_n$ for all $n \in \mathbb{N}$.

Let (b_n) be a sequence of real numbers.

Suppose there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $b_n = (f \circ g)(n)$ for all $n \in \mathbb{N}$.

Then (b_n) is a subsequence of (a_n) and $b_n = (f \circ g)(n) = f(g(n)) = a_{g(n)}$ for all $n \in \mathbb{N}$.

Example 27. The sequence defined by $b_n = (2n+1)^2$ is a subsequence of the sequence defined by $a_n = n^2$ for all $n \in \mathbb{N}$.

Example 28. a sequence is a subsequence of itself

If (a_n) is a sequence, then (a_n) is a subsequence of (a_n).

Proposition 29. *If $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, then $f(n) \geq n$ for all $n \in \mathbb{N}$.*

Proposition 30. ***subsequence preserves monotonicity and boundedness***

1. Every subsequence of an increasing sequence is increasing.

2. Every subsequence of a decreasing sequence is decreasing.

3. Every subsequence of a bounded sequence is bounded.

Definition 31. M tail of a sequence

Let (a_n) be a sequence in $\mathbb{R}$.

Let $M \in \mathbb{N}$.

The M **tail** of (a_n) is the sequence (b_n) defined by $b_n = a_{M+n}$ for all $n \in \mathbb{N}$.

Proposition 32. ***M tail of a sequence is a subsequence of the sequence***

Let (a_n) be a sequence in $\mathbb{R}$.

If (b_n) is an M tail of (a_n), then (b_n) is a subsequence of (a_n).

Convergent Sequences in $\mathbb{R}$

A convergent sequence is a sequence whose terms eventually get arbitrarily close to some number.

A sequence (a_n) has a limit L if the terms of the sequence get arbitrarily close to some number L when n is sufficiently large.

Definition 33. limit of a sequence in $\mathbb{R}$

Let (a_n) be a sequence of real numbers.

A real number L **is a limit of** (a_n), denoted $\lim_{n\to\infty} a_n = L$ or $a_n \to L$, iff for every positive real ϵ, there exists a natural number N such that $|a_n - L| < \epsilon$ whenever $n > N$.

Therefore, $\lim_{n\to\infty} a_n = L$ iff

$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |a_n - L| < \epsilon)$.

If the limit of a sequence is L, we say that **the sequence converges to** L.

Definition 34. convergent sequence in $\mathbb{R}$

A sequence (a_n) is said to be **convergent** iff there exists a real number L such that $\lim_{n\to\infty} a_n = L$.

A sequence that is not convergent is said to be **divergent**.

Therefore, a sequence is either convergent or divergent.

Let (a_n) be a sequence in $\mathbb{R}$.

Suppose (a_n) is convergent.

Then there exists a real number L such that $\lim_{n\to\infty} a_n = L$.

Therefore, $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |a_n - L| < \epsilon)$.

Observe that $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |a_n - L| < \epsilon)$ iff

$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to a_n \in N(L;\epsilon))$.

Therefore, $\lim_{n\to\infty} a_n = L$ implies that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n > \mathbb{N}$, then a_n is in the ϵ neighborhood of L.

Suppose a real number L is not the limit of (a_n).

Then $\lim_{n\to\infty} a_n \neq L$.

Therefore, $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |a_n - L| < \epsilon)$ is false.

Observe that

$$\begin{aligned}
\neg(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |a_n - L| < \epsilon) &\Leftrightarrow \\
\neg(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to a_n \in N(L;\epsilon)) &\Leftrightarrow \\
(\exists \epsilon > 0)(\forall N \in \mathbb{N})(\exists n \in \mathbb{N})(n > N \land a_n \notin N(L;\epsilon)) &\Leftrightarrow \\
(\exists \epsilon > 0)(\forall N \in \mathbb{N})(\exists n \in \mathbb{N}, n > N)(a_n \notin N(L;\epsilon)).&
\end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n \neq L$ implies that there exists $\epsilon > 0$ such that for every $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $n > N$ and there exists an ϵ neighborhood of L such that $a_n \notin N(L;\epsilon)$.

Suppose (a_n) is divergent.

Then (a_n) is not convergent, so there does not exist $L \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = L$.

Thus, $\neg(\exists L \in \mathbb{R})(\lim_{n\to\infty} a_n = L)$.

Example 35. The sequence $(\frac{1}{n})$ converges to zero in $\mathbb{R}$.

Therefore $\lim_{n\to\infty} \frac{1}{n} = 0$.

Equivalently, $\frac{1}{n} \to 0$.

Example 36. limit of a constant sequence

For all $k \in \mathbb{R}$, $\lim_{n\to\infty} k = k$. (limit of a constant k is k)

Theorem 37. ***uniqueness of a limit of a convergent sequence***

The limit of a convergent sequence of real numbers is unique.

Let (a_n) be a convergent sequence of real numbers.

Then there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = L$ and L is unique.

Therefore, a convergent sequence cannot have more than one limit.

Proposition 38. ***a difference in a finite number of initial terms does not affect the convergence of a sequence***

Let (a_n) and (b_n) be sequences of real numbers.

If there exists $K \in \mathbb{N}$ such that $b_n = a_n$ for all $n > K$ and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} b_n = L$.

Proposition 39. *Let $L \in \mathbb{R}$.*

Let (a_n) and $(a_n - L)$ be sequences in $\mathbb{R}$.

Then $\lim_{n\to\infty} a_n = L$ iff $\lim_{n\to\infty}(a_n - L) = 0$.

Theorem 40. ***every subsequence of a convergent sequence is convergent***

Let (a_n) be a convergent sequence of real numbers.

If (b_n) is a subsequence of (a_n), then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

Thus, if a sequence is convergent, then every subsequence is convergent.

Hence, if there is some subsequence that is not convergent, then the sequence is not convergent.

Therefore, if there is some subsequence that is divergent, then the sequence is divergent.

Example 41. If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n+1} = L$.

Corollary 42. *Let (a_n) be a sequence of real numbers.*

If (b_n) and (c_n) are convergent subsequences of (a_n) such that $\lim_{n\to\infty} b_n \neq \lim_{n\to\infty} c_n$, then (a_n) is divergent.

Example 43. bounded divergent sequence that oscillates

Let (a_n) be a sequence of real numbers defined by $a_n = (-1)^n$ for all $n \in \mathbb{N}$.

Then (a_n) consists of even terms all equal to 1 and of odd terms all equal to -1.

Hence, the even subsequence given by $b_n = a_{2n}$ converges to 1 and the odd subsequence given by $c_n = a_{2n-1}$ converges to -1.

Therefore, (a_n) is divergent.

Proposition 44. ***M tail of a sequence is convergent iff the sequence is convergent***

Let (a_n) be a sequence of real numbers.

Let $M \in \mathbb{N}$.

If (a_n) is convergent, then $\lim_{n\to\infty} a_{M+n} = \lim_{n\to\infty} a_n$.

If (a_{M+n}) is convergent, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{M+n}$.

Let (a_n) be a sequence of real numbers.

Let $M \in \mathbb{N}$.

Suppose (a_n) is convergent.

Then $\lim_{n\to\infty} a_{M+n} = \lim_{n\to\infty} a_n$, so (a_{M+n}) is convergent.

Hence, if (a_n) is convergent, then (a_{M+n}) is convergent.

Suppose (a_{M+n}) is convergent.

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{M+n}$, so (a_n) is convergent.

Hence, if (a_{M+n}) is convergent then (a_n) is convergent.

Thus, (a_{M+n}) is convergent iff (a_n) is convergent.

Therefore, the M tail of a sequence is convergent iff the sequence is convergent.

Example 45. If (a_n) is a convergent sequence of real numbers, then $\lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty} a_n$.

Algebraic properties of convergent sequences

Theorem 46. ***convergence implies boundedness***

Every convergent sequence of real numbers is bounded.

Therefore, if a sequence is convergent, then it is bounded.

Hence, if a sequence is not bounded, then it is not convergent.

Thus, if a sequence is unbounded, then it is divergent.

Therefore, an unbounded sequence is divergent.

Let (a_n) be a convergent sequence of real numbers.

Then (a_n) is bounded.

Therefore, there exist $m, M \in \mathbb{R}$ such that $m \leq a_n \leq M$ for all $n \in \mathbb{N}$.

Equivalently, there exists $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Example 47. sequence of natural numbers is divergent

Let $a_n = n$ for all $n \in \mathbb{N}$.

Since the set $\{a_n : n \in \mathbb{N}\} = \{n : n \in \mathbb{N}\} = \mathbb{N}$ is unbounded, then the sequence (a_n) is unbounded.

Therefore, (a_n) is divergent.

Example 48. bounded sequence does not imply convergence

Let (a_n) be a sequence of real numbers defined by $a_n = (-1)^n$ for all $n \in \mathbb{N}$.

Let $S = \{a_n : n \in \mathbb{N}\} = \{(-1)^n : n \in \mathbb{N}\}$.

Since S is bounded above by 1 and below by -1, then the sequence (a_n) is bounded.

Since the terms oscillate between 1 and -1, then the sequence (a_n) does not converge.

Therefore, (a_n) is bounded, but is not convergent.

Example 49. bounded divergent sequence that oscillates

The sequence (a_n) defined by $a_n = (-1)^n$ diverges.

Therefore, $\lim_{n\to\infty}(-1)^n$ does not exist.

Moreover, (a_n) is bounded.

Therefore, (a_n) is a bounded divergent sequence.

Proposition 50. *If* $\lim_{n\to\infty} a_n = 0$ *and* (b_n) *is bounded, then* $\lim_{n\to\infty} a_n b_n = 0$.

Example 51. a. $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

b. $\lim_{n\to\infty} \frac{1}{n^k} = 0$ for every $k \in \mathbb{N}$.

c. $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$.

Lemma 52. *Let* (a_n) *be a sequence of real numbers.*

If there exists $L \neq 0$ *such that* $\lim_{n\to\infty} a_n = L$*, then there is a natural number* N *such that* $|a_n| > \frac{|L|}{2}$ *for all* $n > N$.

Lemma 53. *Let* (a_n) *be a sequence of real numbers.*

If there exists $L \neq 0$ *such that* $\lim_{n\to\infty} a_n = L$ *and* $a_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$.

Theorem 54. ***algebraic limit rules for convergent sequences***

If (a_n) *and* (b_n) *are convergent sequences of real numbers, then*

1. Scalar Multiple Rule

$\lim_{n\to\infty}(\lambda a_n) = \lambda \lim_{n\to\infty} a_n$ *for every* $\lambda \in \mathbb{R}$.

2. Sum Rule (limit of sum equals sum of limits)

$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

3. Difference Rule (limit of difference equals difference of limits)

$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$.

4. Product Rule (limit of product equals product of limits)

$\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$.

5. Quotient Rule (limit of quotient equals quotient of limits)

If $\lim_{n\to\infty} b_n \neq 0$, *then*

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

If $a_n \to L$ and $b_n \to M$, then

1. Scalar Multiple Rule

$\lambda a_n \to \lambda L$ for every $\lambda \in \mathbb{R}$.

2. Sum Rule

$a_n + b_n \to L + M$.
3. Difference Rule
$a_n - b_n \to L - M$.
4. Product Rule
$a_n b_n \to LM$.
5. Quotient Rule
If $M \neq 0$, then $\frac{a_n}{b_n} \to \frac{L}{M}$.

Theorem 55. ***a limit preserves a non strict inequality***
Let (a_n) and (b_n) be convergent sequences of real numbers.
If there exists $K > 0$ such that $a_n \leq b_n$ for all $n > K$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Corollary 56. *Let (a_n) and (b_n) be convergent sequences of real numbers.*
If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Corollary 57. *Let (a_n) be a convergent sequence in $\mathbb{R}$.*
1. If M is an upper bound of (a_n), then $\lim_{n\to\infty} a_n \leq M$.
2. If m is a lower bound of (a_n), then $m \leq \lim_{n\to\infty} a_n$.

Corollary 58. ***limit of a convergent sequence is between any upper and lower bound of the sequence***
Let (a_n) be a convergent sequence in $\mathbb{R}$.
If there exist real numbers m and M such that $m \leq a_n \leq M$ for all $n \in \mathbb{N}$, then $m \leq \lim_{n\to\infty} a_n \leq M$.

Corollary 59. *Let (a_n) be a convergent sequence in $\mathbb{R}$.*
If there exist $K \in \mathbb{N}$ and real numbers m and M such that $m \leq a_n \leq M$ for all $n > K$, then $m \leq \lim_{n\to\infty} a_n \leq M$.

Example 60. If $\lim_{n\to\infty} a_n = L$ and $a_n \geq 0$ for all $n \in \mathbb{N}$, then $L \geq 0$.
However, $a_n > 0$ for all $n \in \mathbb{N}$ does not imply $L > 0$.

Proof. Suppose $\lim_{n\to\infty} a_n = L$ and $a_n \geq 0$ for all $n \in \mathbb{N}$.
Since $a_n \geq 0$ for all $n \in \mathbb{N}$, then 0 is a lower bound of (a_n).
Since $\lim_{n\to\infty} a_n = L$ and 0 is a lower bound of (a_n), then $0 \leq L$, so $L \geq 0$, as desired.
Here is a counterexample:
Let $a_n = \frac{1}{n}$.
Then $\lim_{n\to\infty} a_n = 0$ and $a_n = \frac{1}{n} > 0$ for all $n \in \mathbb{N}$, but $0 \not> 0$. □

Theorem 61. ***squeeze rule for convergent sequences***
Let $(a_n), (b_n)$, and (c_n) be sequences of real numbers.
If there exists $K \in \mathbb{N}$ such that $a_n \leq c_n \leq b_n$ for all $n > K$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, then $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Corollary 62. *Let $(a_n), (b_n)$, and (c_n) be sequences of real numbers.*
If $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, then $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Proposition 63. ***limit of an absolute value equals absolute value of a limit***

Let (a_n) *be a convergent sequence.*

Then the sequence $(|a_n|)$ *is convergent and* $\lim_{n\to\infty} |a_n| = |\lim_{n\to\infty} a_n|$.

Lemma 64. *Let* $a, b, c, d \in \mathbb{R}$.

If $0 \leq a < b$ *and* $0 < c < d$, *then* $ac < bd$.

Lemma 65. ***sequence converging to a positive real number eventually has positive terms***

Let (a_n) *be a sequence of real numbers.*

If $\lim_{n\to\infty} a_n$ *exists and is positive, then there exists* $N \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$, *if* $n > N$, *then* $a_n > 0$.

Proposition 66. ***limit of a square root equals square root of a limit***

Let (a_n) *be a sequence of real numbers.*

If $\lim_{n\to\infty} a_n$ *exists and is positive, then* $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{\lim_{n\to\infty} a_n}$.

Divergent Sequences

A sequence that diverges to ∞ consists of terms that eventually get arbitrarily large.

Definition 67. divergent sequence to ∞

Let (a_n) be a sequence of real numbers.

The sequence (a_n) **diverges to** ∞, denoted $\lim_{n\to\infty} a_n = \infty$, iff for every positive real M, there exists a natural number N such that $a_n > M$ whenever $n > N$.

In symbols, $\lim_{n\to\infty} a_n = \infty$ iff $(\forall M > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to a_n > M)$.

A sequence (a_n) **diverges to** $-\infty$, denoted $\lim_{n\to\infty} a_n = -\infty$, iff $(-a_n)$ diverges to ∞.

Therefore, $\lim_{n\to\infty} a_n = -\infty$ iff $\lim_{n\to\infty} -a_n = \infty$.

Example 68. Let (a_n) be a sequence of real numbers.

Then $\lim_{n\to\infty} a_n = \infty$ iff $\lim_{n\to\infty} -a_n = -\infty$.

Proof. Observe that

$$\begin{aligned} \lim_{n\to\infty} a_n = \infty \quad &\Leftrightarrow \quad \lim_{n\to\infty} -(-a_n) = \infty \\ &\Leftrightarrow \quad \lim_{n\to\infty} -a_n = -\infty. \end{aligned}$$

□

Proposition 69. ***divergence to ∞ implies divergence***

A sequence that diverges to ∞ *is divergent.*

Let (a_n) be a sequence.
If (a_n) diverges to $-\infty$, then $(-a_n)$ diverges to ∞.
Hence, $(-a_n)$ is divergent, so (a_n) is divergent.
Therefore, a sequence that diverges to $-\infty$ is divergent.

Example 70. unbounded divergent sequence
The sequence (a_n) defined by $a_n = n^2$ diverges to infinity.
Hence, $\lim_{n\to\infty} n^2 = \infty$, so (a_n) is divergent.
Moreover, (a_n) is unbounded.
Therefore, (a_n) is an unbounded divergent sequence.

Proposition 71. ***sequences that diverge to infinity are unbounded***
Let (a_n) be a sequence of real numbers.
1. If $\lim_{n\to\infty} a_n = \infty$, then (a_n) is unbounded above.
2. If $\lim_{n\to\infty} a_n = -\infty$, then (a_n) is unbounded below.

If $\lim_{n\to\infty} a_n = \infty$, then (a_n) is unbounded above, so (a_n) is unbounded.
If $\lim_{n\to\infty} a_n = -\infty$, then (a_n) is unbounded below, so (a_n) is unbounded.
Therefore, if either $\lim_{n\to\infty} a_n = \infty$ or $\lim_{n\to\infty} a_n = -\infty$, then (a_n) is unbounded.

Monotone Convergence Theorem

Theorem 72. ***Monotone convergence theorem***
Let (a_n) be a sequence of real numbers.
1. If (a_n) is increasing and bounded above, then $\lim_{n\to\infty} a_n = \sup(a_n)$.
2. If (a_n) is increasing and unbounded above, then $\lim_{n\to\infty} a_n = \infty$.
3. If (a_n) is decreasing and bounded below, then $\lim_{n\to\infty} a_n = \inf(a_n)$.
4. If (a_n) is decreasing and unbounded below, then $\lim_{n\to\infty} a_n = -\infty$.

Let (a_n) be a monotonic sequence of real numbers.
Suppose (a_n) is bounded.
Since (a_n) is monotonic, then either (a_n) is increasing or (a_n) is decreasing.
We consider these cases separately.
Case 1: Suppose (a_n) is increasing.
Since (a_n) is bounded, then (a_n) is bounded above.
Thus, $\lim_{n\to\infty} a_n = \sup(a_n)$, so (a_n) is convergent.
Case 2: Suppose (a_n) is decreasing.
Since (a_n) is bounded, then (a_n) is bounded below.
Thus, $\lim_{n\to\infty} a_n = \inf(a_n)$, so (a_n) is convergent.
Hence, in all cases, (a_n) is convergent.
Thus, if (a_n) is a monotonic sequence, then if (a_n) is bounded, then (a_n) is convergent.
Hence, if (a_n) is a bounded monotonic sequence, then (a_n) is convergent.
Therefore, every bounded monotonic sequence of real numbers is convergent.

Suppose (a_n) is a monotonic sequence of real numbers.

If (a_n) is convergent, then (a_n) is bounded, since every convergent sequence of real numbers is bounded.

If (a_n) is bounded, then (a_n) is convergent, since every bounded monotonic sequence of real numbers is convergent.

Thus, (a_n) is convergent iff (a_n) is bounded.

Hence, if (a_n) is a monotonic sequence of real numbers, then (a_n) is convergent iff (a_n) is bounded.

Therefore, a monotonic sequence of real numbers is convergent iff it is bounded.

Since every bounded monotonic sequence of real numbers is convergent, then if (a_n) is bounded and monotonic, then (a_n) is convergent.

Hence, if (a_n) is not convergent, then either (a_n) is not bounded or (a_n) is not monotonic.

Thus, if (a_n) is divergent, then either (a_n) is unbounded or (a_n) is not monotonic.

Therefore, a divergent sequence of real numbers is either unbounded or not monotonic.

Example 73. sequence of rational numbers that converges to an irrational number

Let (x_n) be a sequence of rational numbers defined recursively by $x_1 = 2$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for all $n \in \mathbb{N}$.

Then $\lim_{n\to\infty} x_n = \sqrt{2}$.

Lemma 74. *Let $r \in \mathbb{R}$.*

1. *If $r > 0$, then $r^n > 0$ for all $n \in \mathbb{N}$.*
2. *If $r > 1$, then $r^n \geq (r-1)n + 1$ for all $n \in \mathbb{N}$.*

Proposition 75. ***convergence behavior of a geometric sequence***

Let $r \in \mathbb{R}$.

Let (r^n) be a geometric sequence.

1. *If $r > 1$, then $\lim_{n\to\infty} r^n = \infty$.*
2. *If $r = 1$, then $\lim_{n\to\infty} r^n = 1$.*
3. *If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$.*
4. *If $r = -1$, then (r^n) is divergent (oscillates).*
5. *If $r < -1$, then (r^n) is divergent.*

Suppose $|r| > 1$.

Then either $r > 1$ or $r < -1$.

We consider these cases separately.

Case 1: Suppose $r > 1$.

Then $\lim_{n\to\infty} r^n = \infty$, so (r^n) is divergent.

Case 2: Suppose $r < -1$.

Then (r^n) is divergent.

Thus, in either case (r^n) is divergent.

Therefore, if $|r| > 1$, then (r^n) is divergent.

Bolzano-Weierstrass theorem

Theorem 76. ***Nested intervals theorem***

Let (I_n) be a sequence of nonempty closed, bounded intervals in $\mathbb{R}$ such that $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$. Then there exists $\alpha \in \mathbb{R}$ such that $\alpha \in I_n$ for all $n \in \mathbb{N}$.

Let (I_n) be a sequence of nonempty closed, bounded intervals in $\mathbb{R}$ such that $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$.

Since (I_n) is a sequence of nonempty closed bounded intervals in $\mathbb{R}$, then there exist $a_n, b_n \in \mathbb{R}$ such that $I_n = [a_n, b_n]$ for all $n \in \mathbb{N}$.

Since I_n is not empty, then there exists $x \in \mathbb{R}$ such that $x \in I_n$, so $a_n \leq x \leq b_n$.

Thus, $a_n \leq b_n$, so $a_n \leq b_n$ for all $n \in \mathbb{N}$.

Since $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, then (a_n) is a sequence in $\mathbb{R}$.

Since $b_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, then (b_n) is a sequence in $\mathbb{R}$.

By the Nested interval theorem, (a_n) is increasing and (b_n) is decreasing and there exists $\alpha \in \mathbb{R}$ such that $\alpha \in I_n$ for all $n \in \mathbb{N}$ and $\alpha = \sup(a_n)$ and $\inf(b_n)$ exists and $\sup(a_n) \leq \inf(b_n)$.

Since $\sup(a_n)$ is an upper bound of (a_n), then (a_n) is bounded above.

Since (a_n) is increasing and bounded above, then by MCT, $\lim_{n\to\infty} a_n = \sup(a_n)$.

Since $\alpha \in I_n$ for all $n \in \mathbb{N}$, then $\alpha \in \cap_{n=1}^{\infty} I_n$, so $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Since $\alpha \in I_n$ for all $n \in \mathbb{N}$, then $\alpha \in [a_n, b_n]$ for all $n \in \mathbb{N}$, so $a_n \leq \alpha \leq b_n$ for all $n \in \mathbb{N}$.

Hence, $a_n \leq \alpha$ and $\alpha \leq b_n$ for all $n \in \mathbb{N}$, so $\alpha \leq b_n$ for all $n \in \mathbb{N}$.

Therefore, α is a lower bound of (b_n), so (b_n) is bounded below.

Since (b_n) is decreasing and bounded below, then by MCT, $\lim_{n\to\infty} b_n = \inf(b_n)$.

Theorem 77. ***Bolzano-Weierstrass theorem***

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Cauchy sequences

A Cauchy sequence consists of terms that are eventually arbitrarily close together.

Definition 78. Cauchy sequence

A sequence (a_n) is a **Cauchy sequence** iff, to every positive real ϵ, there corresponds a natural number N such that $|a_m - a_n| < \epsilon$ for all $m, n > N$.

In symbols, (a_n) is Cauchy iff $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \in \mathbb{Z}^+)(m > N \land n > N \to |a_m - a_n| < \epsilon)$.

If the terms of a convergent sequence eventually get arbitrarily close to some real number, then the terms eventually must get close together.

Lemma 79. *Every convergent sequence in $\mathbb{R}$ is a Cauchy sequence.*

Lemma 80. *Every Cauchy sequence in $\mathbb{R}$ is bounded.*

Theorem 81. ***Cauchy convergence criterion for sequences***
A sequence in $\mathbb{R}$ is convergent iff it is a Cauchy sequence.

Let (a_n) be a sequence of real numbers.
Then (a_n) is convergent iff (a_n) is Cauchy.
Thus, every sequence of real numbers is convergent iff it is Cauchy.
Therefore, every convergent sequence of real numbers is Cauchy and every Cauchy sequence of real numbers is convergent.