Theory of convergent series in $\mathbb{R}$

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Infinite Series of Real Numbers

Proposition 1. ***properties of finite sums***

Let $n \in \mathbb{N}$. Then
1. $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.
2. $\sum_{k=1}^{n}(\lambda a_k) = \lambda \sum_{k=1}^{n} a_k$ *for every* $\lambda \in \mathbb{R}$.

Proof. We prove 1.
Observe that

$$\begin{aligned}\sum_{k=1}^{n}(a_k + b_k) &= (a_1 + b_1) + (a_2 + b_2) + ... + (a_n + b_n) \\ &= (a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n) \\ &= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.\end{aligned}$$

□

Proof. We prove 2.
Let $\lambda \in \mathbb{R}$ be given.
Then

$$\begin{aligned}\sum_{k=1}^{n} \lambda a_k &= \lambda a_1 + \lambda a_2 + ... + \lambda a_n \\ &= \lambda(a_1 + a_2 + ... + a_n) \\ &= \lambda \sum_{k=1}^{n} a_k.\end{aligned}$$

□

Proposition 2. n^{th} ***term of a sequence of partial sums***

Let (a_n) be a sequence of real numbers.
Let (s_n) be a sequence defined by $s_1 = a_1$ and $s_{n+1} = s_n + a_{n+1}$ for all $n \in \mathbb{N}$.
The n^{th} term of the sequence (s_n) is $s_n = a_1 + a_2 + ... + a_n$.

Proof. We prove $s_n = \sum_{i=1}^{n} a_i$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : s_n = \sum_{i=1}^{n} a_i\}$.
Since $s_1 = a_1 = \sum_{i=1}^{1} a_i$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $s_k = \sum_{i=1}^{k} a_i$.
Observe that

$$\begin{aligned} s_{k+1} &= s_k + a_{k+1} \\ &= \sum_{i=1}^{k} a_i + a_{k+1} \\ &= \sum_{i=1}^{k+1} a_i. \end{aligned}$$

Thus, $s_{k+1} = \sum_{i=1}^{k+1} a_i$, so $k+1 \in S$.
Therefore, by PMI, $S = \mathbb{N}$, so $s_n = \sum_{i=1}^{n} a_i$ for all $n \in \mathbb{N}$, as desired. □

Theorem 3. ***uniqueness of a sum of a convergent series***
The sum of a convergent series of real numbers is unique.

Proof. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of real numbers.
Let (s_n) be the sequence of partial sums of the sequence (a_n).
Since the series $\sum_{n=1}^{\infty} a_n$ is convergent, then there exists a real number S such that $\lim_{n\to\infty} s_n = S$.
Since the limit of a convergent sequence is unique and $\lim_{n\to\infty} s_n = S$, then S is unique. □

Theorem 4. n^{th} ***term test for divergence***
If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Let (a_n) be a sequence of real numbers.
Let (s_n) be the sequence of partial sums of (a_n).
Suppose the series $\sum_{n=1}^{\infty} a_n$ is convergent.
Then there exists $S \in \mathbb{R}$ such that $\lim_{n\to\infty} s_n = S$.
Hence, (s_n) is convergent, so $\lim_{n\to\infty} s_{n-1} = \lim_{n\to\infty} s_n = S$.
Observe that

$$\begin{aligned} 0 &= S - S \\ &= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} \\ &= \lim_{n\to\infty} (s_n - s_{n-1}) \\ &= \lim_{n\to\infty} a_n. \end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n = 0$, as desired. □

Proposition 5. ***sum of n terms of a geometric series formula***

Let $r \in \mathbb{R}^*, r \neq 1$.

Then

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1} \text{ for all } n \in \mathbb{Z}^+$$

Proof. We prove the statement $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$ for all $n \in \mathbb{Z}^+$ by induction on n.

Let $S = \{n \in \mathbb{Z}^+ : \sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}\}$.

Basis:

Since $1 \in \mathbb{Z}^+$ and $\sum_{k=0}^{1} r^k = r^0 + r^1 = 1 + r = r + 1 = \frac{(r-1)(r+1)}{r-1} = \frac{r^2-1}{r-1} = \frac{r^{1+1}-1}{r-1}$, then $1 \in S$.

Induction:

Suppose $m \in S$.

Then $m \in \mathbb{Z}^+$ and $\sum_{k=0}^{m} r^k = \frac{r^{m+1}-1}{r-1}$.

Since $m \in \mathbb{Z}^+$, then $m+1 \in \mathbb{Z}^+$.

Observe that

$$\begin{aligned}
\sum_{k=0}^{m+1} r^k &= \sum_{k=0}^{m} r^k + r^{m+1} \\
&= \frac{r^{m+1}-1}{r-1} + r^{m+1} \\
&= \frac{r^{m+1}-1+r^{m+1}(r-1)}{r-1} \\
&= \frac{r^{m+1}-1+r^{m+2}-r^{m+1}}{r-1} \\
&= \frac{r^{m+2}}{r-1} \\
&= \frac{r^{(m+1)+1}}{r-1}.
\end{aligned}$$

Since $m+1 \in \mathbb{Z}^+$ and $\sum_{k=0}^{m+1} r^k = \frac{r^{(m+1)+1}}{r-1}$, then $m+1 \in S$.

Hence, $m \in S$ implies $m+1 \in S$.

Therefore, by PMI, $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$ for all $n \in \mathbb{Z}^+$. ∎

Proposition 6. ***sum of a convergent geometric series***

Let $r \in \mathbb{R}$.

Then $\sum_{n=1}^{\infty} r^n$ is convergent iff $|r| < 1$.

If $|r| < 1$, then $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$.

Proof. Let (r^n) be a geometric sequence of real numbers.

Let (s_n) be the sequence of partial sums of (r^n).

We first prove if $r \neq 1$, then $s_n = \frac{r-r^{n+1}}{1-r}$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.
Then $s_n = r + r^2 + r^3 + ... + r^n$.
Thus, $rs_n = r^2 + r^3 + r^4 + ... + r^{n+1}$.
Subtracting equations we get $s_n - rs_n = r - r^{n+1}$.
Hence, $s_n(1-r) = r - r^{n+1}$.
Since $r \neq 1$, then $1 - r \neq 0$, so $s_n = \frac{r-r^{n+1}}{1-r}$, as desired.
Suppose $\sum_{n=1}^{\infty} r^n$ is convergent.
Then $\lim_{n\to\infty} r^n = 0$.
Since (r^n) is a geometric sequence, then this implies that $|r| < 1$.
Conversely, suppose $|r| < 1$.
Then $-1 < r < 1$, so $r < 1$.
Hence, $r \neq 1$, so $s_n = \frac{r-r^{n+1}}{1-r}$ for all $n \in \mathbb{N}$ and $1 - r \neq 0$.
Observe that

$$\begin{aligned}
\frac{r}{1-r} &= \frac{r}{1-r}(1-0) \\
&= \frac{r}{1-r}[\lim_{n\to\infty} 1 - \lim_{n\to\infty} r^n] \\
&= \frac{r}{1-r}\lim_{n\to\infty}(1-r^n) \\
&= \lim_{n\to\infty}\frac{r}{1-r}(1-r^n) \\
&= \lim_{n\to\infty}\frac{r-r^{n+1}}{1-r} \\
&= \lim_{n\to\infty} s_n.
\end{aligned}$$

Therefore, $\frac{r}{1-r} = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} r^n$, as desired. □

Theorem 7. ***algebraic summation rules for convergent series***

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of real numbers, then
1. Scalar Multiple Rule
$\sum_{n=1}^{\infty}(\lambda a_n) = \lambda \sum_{n=1}^{\infty} a_n$ *for every* $\lambda \in \mathbb{R}$.
2. Sum Rule
$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
3. Difference Rule
$\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

Proof. Let (a_n) and (b_n) be sequences of real numbers.
Let (r_n) be the sequence of partial sums of (a_n).
Then $r_n = \sum_{k=1}^{n} a_k$ for all $n \in \mathbb{N}$.
Let (s_n) be the sequence of partial sums of (b_n).
Then $s_n = \sum_{k=1}^{n} b_k$ for all $n \in \mathbb{N}$.
Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series.

Then the sequence of partial sums of (a_n) is convergent and the sequence of partial sums of (b_n) is convergent.

Thus, there exist real numbers A and B such that $\lim_{n\to\infty} r_n = A$ and $\lim_{n\to\infty} s_n = B$. □

Proof. We prove 1.

Let $\lambda \in \mathbb{R}$ be given.

Let (t_n) be the sequence of partial sums of the sequence (λa_n).

Then $t_n = \sum_{k=1}^{n} (\lambda a_k)$ for all $n \in \mathbb{N}$.

Observe that

$$\begin{aligned} \lambda A &= \lambda \lim_{n\to\infty} r_n \\ &= \lim_{n\to\infty} \lambda r_n \\ &= \lim_{n\to\infty} \lambda \sum_{k=1}^{n} a_k \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} (\lambda a_k) \\ &= \lim_{n\to\infty} t_n. \end{aligned}$$

Therefore, $\lambda A = \lim_{n\to\infty} t_n$, so $\sum_{n=1}^{\infty} (\lambda a_n) = \lambda A$, as desired. □

Proof. We prove 2.

Let (t_n) be the sequence of partial sums of the sequence $(a_n + b_n)$.

Then $t_n = \sum_{k=1}^{n} (a_k + b_k)$ for all $n \in \mathbb{N}$.

Observe that

$$\begin{aligned} A + B &= \lim_{n\to\infty} r_n + \lim_{n\to\infty} s_n \\ &= \lim_{n\to\infty} (r_n + s_n) \\ &= \lim_{n\to\infty} [\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k] \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} (a_k + b_k) \\ &= \lim_{n\to\infty} t_n. \end{aligned}$$

Therefore, $A + B = \lim_{n\to\infty} t_n$, so $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$, as desired. □

Proof. We prove 3.

Since $\sum b_n$ is convergent, then $\sum -b_n = -\sum b_n$, so the series $\sum -b_n$ is convergent.

Since $\sum a_n$ is convergent, then $\sum (a_n - b_n) = \sum [a_n + (-b_n)] = \sum a_n + \sum -b_n$, so the series $\sum (a_n - b_n)$ is convergent, as desired. □

Theorem 8. ***inequality rule for convergent series***

If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent and $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Proof. Suppose the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent and $a_n \leq b_n$ for all $n \in \mathbb{N}$.

Let (s_n) be the sequence of partial sums of the sequence (a_n).

Since the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n$.

Let (t_n) be the sequence of partial sums of the sequence (b_n).

Since the series $\sum_{n=1}^{\infty} b_n$ is convergent, then $\lim_{n\to\infty} t_n = \sum_{n=1}^{\infty} b_n$.

We prove $s_n \leq t_n$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : s_n \leq t_n\}$.

Since $s_1 = a_1 \leq b_1 = t_1$, then $s_1 \leq t_1$, so $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $s_k \leq t_k$.

Since $k \in \mathbb{N}$, then $k+1 \in \mathbb{N}$, so $a_{k+1} \leq b_{k+1}$.

Observe that

$$\begin{aligned} s_{k+1} &= s_k + a_{k+1} \\ &\leq t_k + a_{k+1} \\ &\leq t_k + b_{k+1} \\ &= t_{k+1}. \end{aligned}$$

Thus, $s_{k+1} \leq t_{k+1}$, so $k+1 \in S$.

Therefore, by PMI, $S = \mathbb{N}$, so $s_n \leq t_n$ for all $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n$ and $\lim_{n\to\infty} t_n = \sum_{n=1}^{\infty} b_n$ and $s_n \leq t_n$ for all $n \in \mathbb{N}$, then by the inequality rule for convergent sequences, $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$. ☐

Proof. Suppose the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent and $a_n \leq b_n$ for all $n \in \mathbb{N}$.

Let (s_n) be the sequence of partial sums of the sequence $(b_n - a_n)$.

Since the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then $\sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(b_n - a_n)$, so the series $\sum_{n=1}^{\infty}(b_n - a_n)$ is convergent.

Hence, $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty}(b_n - a_n) = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} a_n$.

We prove $s_n \geq 0$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : s_n \geq 0\}$.

Since $a_1 \leq b_1$, then $s_1 = b_1 - a_1 \geq 0$.

Thus, $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $s_k \geq 0$.

Since $k+1 \in \mathbb{N}$, then $a_{k+1} \leq b_{k+1}$, so $b_{k+1} - a_{k+1} \geq 0$.

Thus, $s_{k+1} = s_k + (b_{k+1} - a_{k+1}) \geq 0$, so $s_{k+1} \geq 0$.

Hence, $k+1 \in S$, so by PMI, $s_n \geq 0$ for all $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} a_n$ and 0 is a lower bound of (s_n), then $0 \leq \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} a_n$.

Therefore, $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$. ☐

Theorem 9. ***tail of a series determines convergence of a series***

Let M be any positive integer.

The series $\sum_{n=1}^{\infty} a_n$ is convergent iff the series $\sum_{n=1}^{\infty} a_{M+n}$ is convergent.

Proof. Suppose the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Then there exists a real number S such that $\sum_{n=1}^{\infty} a_n = S$.

Since $M \geq 1$, then either $M > 1$ or $M = 1$.

We consider these cases separately.

Case 1: Suppose $M = 1$.

Since $\sum_{n=1}^{\infty} a_n = S$, then $a_1 + a_2 + a_3 + ... = S$.

Observe that

$$\begin{aligned} \sum_{n=1}^{\infty} a_{M+n} &= \sum_{n=1}^{\infty} a_{1+n} \\ &= a_2 + a_3 + a_4 + ... \\ &= S - a_1. \end{aligned}$$

Since $S - a_1$ is a real number, then the series $\sum_{n=1}^{\infty} a_{M+n}$ is convergent.

Case 2: Suppose $M > 1$.

Since $\sum_{n=1}^{\infty} a_n = S$, then $(a_1+a_2+...+a_M)+(a_{M+1}+a_{M+2}+a_{M+3}+...) = S$.

Observe that

$$\begin{aligned} \sum_{n=1}^{\infty} a_{M+n} &= a_{M+1} + a_{M+2} + a_{M+3} + ... \\ &= S - (a_1 + a_2 + ... + a_M). \end{aligned}$$

Since $S - (a_1 + a_2 + ... + a_M)$ is a real number, then the series $\sum_{n=1}^{\infty} a_{M+n}$ is convergent.

Therefore, in all cases, the series $\sum_{n=1}^{\infty} a_{M+n}$ is convergent, as desired.

Conversely, suppose the series $\sum_{n=1}^{\infty} a_{M+n}$ is convergent.

Then there exists a real number S such that $\sum_{n=1}^{\infty} a_{M+n} = S$.

Since $M \geq 1$, then either $M > 1$ or $M = 1$.

We consider these cases separately.

Case 1: Suppose $M = 1$.

Then

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= a_1 + (a_2 + a_3 + a_4 + ...) \\ &= a_1 + (a_{M+1} + a_{M+2} + a_{M+3} + ...) \\ &= a_1 + \sum_{n=1}^{\infty} a_{M+n} \\ &= a_1 + S. \end{aligned}$$

Since $a_1 + S$ is a real number, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Case 2: Suppose $M > 1$.
Then

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= (a_1 + a_2 + ... + a_M) + (a_{M+1} + a_{M+2} + a_{M+3} + ...) \\ &= (a_1 + a_2 + ... + a_M) + \sum_{n=1}^{\infty} a_{M+n} \\ &= (a_1 + a_2 + ... + a_M) + S.\end{aligned}$$

Since $(a_1 + a_2 + ... + a_M) + S$ is a real number, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Therefore, in all cases, the series $\sum_{n=1}^{\infty} a_n$ is convergent, as desired. □

Convergence Tests for Series of Real Numbers

Proposition 10. ***Cauchy convergence criterion for series***

The infinite series of real numbers $\sum a_n$ is convergent iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > m > N$, then $|\sum_{k=m+1}^{n} a_k| < \epsilon$.

Proof. Let $\sum a_n$ be an infinite series of real numbers.

Let (s_n) be the sequence of partial sums of the sequence (a_n).

Suppose that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > m > N$, then $|\sum_{k=m+1}^{n} a_k| < \epsilon$.

Let $\epsilon > 0$ be given.

Then there exists $N \in \mathbb{N}$ such that if $n > m > N$, then $|\sum_{k=m+1}^{n} a_k| < \epsilon$.

Let $m, n \in \mathbb{N}$ such that $n > m > N$.

Then $m > N$ and $n > N$ and $|\sum_{k=m+1}^{n} a_k| < \epsilon$.

Observe that

$$\begin{aligned}|s_m - s_n| &= |s_n - s_m| \\ &= |(a_1 + a_2 + ... + a_m + a_{m+1} + a_{m+2} + ... + a_n) - (a_1 + a_2 + ... + a_m)| \\ &= |a_{m+1} + a_{m+2} + ... + a_n| \\ &= |\sum_{k=m+1}^{n} a_k| \\ &< \epsilon.\end{aligned}$$

Hence, (s_n) is a Cauchy sequence of real numbers.

Therefore, the sequence (s_n) is convergent, so the series $\sum a_n$ is convergent, as desired.

Conversely, suppose the series $\sum a_n$ is convergent.

Then the sequence (s_n) of real numbers is convergent, so (s_n) is a Cauchy sequence.

Thus, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m, n > N$, then $|s_m - s_n| < \epsilon$.

Let $\epsilon > 0$ be given.

Then there exists $N \in \mathbb{N}$ such that if $m, n > N$, then $|s_m - s_n| < \epsilon$.
Let $m, n \in \mathbb{N}$ such that $n > m > N$.
Then $m > N$ and $n > N$, so $|s_m - s_n| < \epsilon$.
Observe that

$$
\begin{aligned}
|\sum_{k=m+1}^{n} a_k| &= |a_{m+1} + a_{m+2} + ... + a_n| \\
&= |(a_1 + a_2 + ... + a_m + a_{m+1} + a_{m+2} + ... + a_n) - (a_1 + a_2 + ... + a_m)| \\
&= |s_n - s_m| \\
&= |s_m - s_n| \\
&< \epsilon.
\end{aligned}
$$

□

Theorem 11. ***Boundedness convergence criterion for series of nonnegative terms***

If (a_n) is a sequence of nonnegative terms, then the series $\sum a_n$ is convergent iff the sequence of partial sums of (a_n) is bounded.

Proof. Let (a_n) be a sequence of nonnegative real numbers.
Then $a_n \geq 0$ for all $n \in \mathbb{N}$.
Let (s_n) be the sequence of partial sums of (a_n).
Suppose the series $\sum a_n$ is convergent.
Then the sequence of partial sums (s_n) is convergent.
Hence, (s_n) is bounded.
Conversely, suppose the sequence of partial sums (s_n) is bounded.
Let $n \in \mathbb{N}$ be given.
Then $n + 1 \in \mathbb{N}$, so $s_{n+1} - s_n = a_{n+1} \geq 0$.
Thus, $s_{n+1} - s_n \geq 0$, so $s_{n+1} \geq s_n$.
Hence, $s_n \leq s_{n+1}$, so (s_n) is increasing.
Therefore, (s_n) is monotonic.
Since (s_n) is monotonic and bounded, then by MCT, (s_n) is convergent.
Therefore, the series $\sum a_n$ is convergent. □

Theorem 12. ***direct comparison test***

Let (a_n) and (b_n) be sequences such that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.
If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

Proof. Since $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$, then $0 \leq a_n$ for all $n \in \mathbb{N}$ and $0 \leq b_n$ for all $n \in \mathbb{N}$.

Hence, (a_n) is a sequence of nonnegative terms and (b_n) is a sequence of nonnegative terms.
Let (s_n) be the sequence of partial sums of (a_n).
Let (t_n) be the sequence of partial sums of (b_n).
Suppose the series $\sum b_n$ is convergent.

Since (b_n) is a sequence of nonnegative terms and the series $\sum b_n$ is convergent, then by the boundedness convergence criterion for series of nonnegative terms, the sequence (t_n) is bounded.

Hence, (t_n) is bounded above, so there exists a real number T such that $t_n \leq T$ for all $n \in \mathbb{N}$.

We prove $0 \leq s_n \leq t_n$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : 0 \leq s_n \leq t_n\}$.

Since $0 \leq a_1 \leq b_1$ and $s_1 = a_1$ and $t_1 = b_1$, then $0 \leq s_1 \leq t_1$, so $1 \in S$.

Suppose $m \in S$.

Then $m \in \mathbb{N}$ and $0 \leq s_m \leq t_m$, so $0 \leq s_m$ and $s_m \leq t_m$.

Since $m+1 \in \mathbb{N}$, then $0 \leq a_{m+1} \leq b_{m+1}$, so $0 \leq a_{m+1}$ and $a_{m+1} \leq b_{m+1}$.

Since $s_m \geq 0$ and $a_{m+1} \geq 0$, then $s_{m+1} = s_m + a_{m+1} \geq 0$, so $s_{m+1} \geq 0$.

Observe that

$$\begin{aligned} s_{m+1} &= s_m + a_{m+1} \\ &\leq t_m + a_{m+1} \\ &\leq t_m + b_{m+1} \\ &= t_{m+1}. \end{aligned}$$

Thus, $s_{m+1} \leq t_{m+1}$.

Since $0 \leq s_{m+1}$ and $s_{m+1} \leq t_{m+1}$, then $0 \leq s_{m+1} \leq t_{m+1}$, so $m+1 \in S$.

Therefore, by PMI, $S = \mathbb{N}$, so $0 \leq s_n \leq t_n$ for all $n \in \mathbb{N}$.

Since $t_n \leq T$ for all $n \in \mathbb{N}$, then $0 \leq s_n \leq t_n \leq T$ for all $n \in \mathbb{N}$.

Thus, $0 \leq s_n \leq T$ for all $n \in \mathbb{N}$, so (s_n) is bounded.

Since (a_n) is a sequence of nonnegative terms and (s_n) is bounded, then by the boundedness convergence criterion for series of nonnegative terms, the series $\sum a_n$ is convergent, as desired. □

Theorem 13. ***limit comparison test***

Let (a_n) and (b_n) be sequences of real numbers such that $a_n > 0$ and $b_n > 0$ for all $n \in \mathbb{N}$.

If there exists a positive real number L such that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, then the series $\sum a_n$ is convergent iff the series $\sum b_n$ is convergent.

Proof. Suppose there exists a positive real number L such that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

Suppose the series $\sum b_n$ is convergent.

Since $L > 0$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|\frac{a_n}{b_n} - L| < L$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $b_n > 0$ and $|\frac{a_n}{b_n} - L| < L$.

Observe that

$$\begin{aligned}
|\frac{a_n}{b_n} - L| < L \quad &\Rightarrow \quad -L < \frac{a_n}{b_n} - L < L \\
&\Rightarrow \quad 0 < \frac{a_n}{b_n} < 2L \\
&\Rightarrow \quad 0 < a_n < 2Lb_n \\
&\Rightarrow \quad 0 \leq a_n \leq 2Lb_n.
\end{aligned}$$

Therefore, $0 \leq a_n \leq 2Lb_n$ for all $n \in \mathbb{N}$.

Since $\sum b_n$ is convergent and $\sum 2Lb_n = 2L \sum b_n$, then by the scalar multiple rule, the series $\sum 2Lb_n$ is convergent.

Therefore, by DCT, the series $\sum a_n$ is convergent.

Conversely, suppose the series $\sum a_n$ is convergent.

Since $a_n > 0$ and $b_n > 0$ for all $n \in \mathbb{N}$, then $\frac{a_n}{b_n} > 0$, so $\frac{a_n}{b_n} \neq 0$ for all $n \in \mathbb{N}$.

Since $L > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $\frac{a_n}{b_n} \neq 0$ for all $n \in \mathbb{N}$, then by a previous lemma, $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{L}$.

Since $\frac{1}{L} > 0$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|\frac{b_n}{a_n} - \frac{1}{L}| < \frac{1}{L}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $a_n > 0$ and $|\frac{b_n}{a_n} - \frac{1}{L}| < \frac{1}{L}$.

Observe that

$$\begin{aligned}
|\frac{b_n}{a_n} - \frac{1}{L}| < \frac{1}{L} \quad &\Rightarrow \quad -\frac{1}{L} < \frac{b_n}{a_n} - \frac{1}{L} < \frac{1}{L} \\
&\Rightarrow \quad 0 < \frac{b_n}{a_n} < \frac{2}{L} \\
&\Rightarrow \quad 0 < b_n < \frac{2}{L}a_n \\
&\Rightarrow \quad 0 \leq b_n \leq \frac{2}{L}a_n.
\end{aligned}$$

Therefore, $0 \leq b_n \leq \frac{2}{L}a_n$ for all $n \in \mathbb{N}$.

Since $\sum a_n$ is convergent and $\sum \frac{2}{L}a_n = \frac{2}{L}\sum a_n$, then by the scalar multiple rule, the series $\sum \frac{2}{L}a_n$ is convergent.

Therefore, by DCT, the series $\sum b_n$ is convergent. $\square$

Lemma 14. *Let (a_n) be a sequence in $\mathbb{R}$.*

If there exists a real number L such that $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n-1} = L$, then $\lim_{n\to\infty} a_n = L$.

Proof. Suppose there exists a real number L such that $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n-1} = L$.

Let $\epsilon > 0$ be given.

Since $\lim_{n\to\infty} a_{2n} = L$, then there exists $N_1 \in \mathbb{N}$ such that $|a_{2n} - L| < \epsilon$ whenever $n > N_1$.

Since $\lim_{n\to\infty} a_{2n-1} = L$, then there exists $N_2 \in \mathbb{N}$ such that $|a_{2n-1} - L| < \epsilon$ whenever $n > N_2$.

Let $N = \max\{2N_1, 2N_2 - 1\}$.

Let $n > N$.

Either n is even or n is odd.

We consider these cases separately.

Case 1: Suppose n is even.

Then there exists a natural number m such that $n = 2m$, so $m = \frac{n}{2}$.

Since $n > N \geq 2N_1$, then $n > 2N_1$, so $\frac{n}{2} > N_1$.

Thus, $m > N_1$, so $|a_{2m} - L| = |a_n - L| < \epsilon$.

Case 2: Suppose n is odd.

Then there exists a natural number m such that $n = 2m - 1$, so $m = \frac{n+1}{2}$.

Since $n > N \geq 2N_2 - 1$, then $n > 2N_2 - 1$, so $\frac{n+1}{2} > N_2$.

Thus, $m > N_2$, so $|a_{2m-1} - L| = |a_n - L| < \epsilon$.

Therefore, in either case, $|a_n - L| < \epsilon$, so $\lim_{n\to\infty} a_n = L$, as desired. □

Theorem 15. ***alternating series test***

Let (a_n) be a sequence of positive terms in $\mathbb{R}$.

If (a_n) is monotonic decreasing and $\lim_{n\to\infty} a_n = 0$, then the series $\sum(-1)^n a_n$ is convergent.

Proof. Suppose (a_n) is monotonic decreasing and $\lim_{n\to\infty} a_n = 0$.

Let (s_n) be the sequence of partial sums of the sequence given by $(-1)^n a_n$ for all $n \in \mathbb{N}$.

Then $s_n = \sum_{k=1}^{n}(-1)^k a_k$ for all $n \in \mathbb{N}$ and $s_{n+1} = s_n + (-1)^{n+1}a_{n+1}$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Then

$$\begin{aligned}
s_{2(n+1)} - s_{2n} &= s_{2n+2} - s_{2n} \\
&= \sum_{k=1}^{2n+2}(-1)^k a_k - \sum_{k=1}^{2n}(-1)^k a_k \\
&= \sum_{k=1}^{2n}(-1)^k a_k + (-1)^{2n+1}a_{2n+1} + (-1)^{2n+2}a_{2n+2} - \sum_{k=1}^{2n}(-1)^k a_k \\
&= (-1)^{2n+1}a_{2n+1} + (-1)^{2n+2}a_{2n+2} \\
&= -a_{2n+1} + a_{2n+2}.
\end{aligned}$$

Since (a_n) is decreasing , then $a_{2n+1} \geq a_{2n+2}$, so $0 \geq -a_{2n+1} + a_{2n+2}$.

Thus, $0 \geq s_{2(n+1)} - s_{2n}$, so $s_{2n} \geq s_{2(n+1)}$.

Therefore, the sequence (s_{2n}) is decreasing.

We prove $-a_1 + a_{2n} \leq s_{2n}$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : -a_1 + a_{2n} \leq s_{2n}\}$.

Since $-a_1 + a_2 = s_2$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $-a_1 + a_{2k} \leq s_{2k}$.
Observe that

$$\begin{aligned} -a_1 + a_{2k} - a_{2k+1} + a_{2k+2} &\leq s_{2k} - a_{2k+1} + a_{2k+2} \\ &= s_{2k} + (-1)^{2k+1} a_{2k+1} + a_{2k+2} \\ &= s_{2k+1} + a_{2k+2} \\ &= s_{2k+1} + (-1)^{2k+2} a_{2k+2} \\ &= s_{2k+2}. \end{aligned}$$

Hence, $-a_1 + a_{2k} - a_{2k+1} + a_{2k+2} \leq s_{2k+2}$, so $a_{2k} - a_{2k+1} \leq s_{2k+2} - (-a_1 + a_{2k+2})$.
Since (a_n) is decreasing, then $a_{2k} \geq a_{2k+1}$, so $a_{2k} - a_{2k+1} \geq 0$.
Since $0 \leq a_{2k} - a_{2k+1} \leq s_{2k+2} - (-a_1 + a_{2k+2})$, then $0 \leq s_{2k+2} - (-a_1 + a_{2k+2})$, so $-a_1 + a_{2k+2} \leq s_{2k+2}$.
Hence, $-a_1 + a_{2(k+1)} \leq s_{2(k+1)}$, so $k + 1 \in S$.
Therefore, by PMI, $S = \mathbb{N}$, so $-a_1 + a_{2n} \leq s_{2n}$ for all $n \in \mathbb{N}$.
Let $n \in \mathbb{N}$ be given.
Since (a_n) is a sequence of positive terms, then $a_n > 0$ for all $n \in \mathbb{N}$.
Since $2n \in \mathbb{N}$, then $a_{2n} > 0$, so $-a_1 + a_{2n} > -a_1$.
Thus, $-a_1 < -a_1 + a_{2n} \leq s_{2n}$, so $-a_1 < s_{2n}$.
Therefore, $-a_1 < s_{2n}$ for all $n \in \mathbb{N}$.
Since (s_{2n}) is decreasing, then $-a_1 + a_2 = s_2 \geq s_{2n}$ for all $n \in \mathbb{N}$.
Since $-a_1 < s_{2n}$ for all $n \in \mathbb{N}$ and $s_{2n} \leq -a_1 + a_2$ for all $n \in \mathbb{N}$, then $-a_1 < s_{2n} \leq -a_1 + a_2$ for all $n \in \mathbb{N}$.
Therefore, (s_{2n}) is bounded.
Since (s_{2n}) is decreasing, then (s_{2n}) is monotonic.
Therefore, by MCT, (s_{2n}) is convergent, so there exists a real number L such that $\lim_{n\to\infty} s_{2n} = L$.
Since (a_{2n}) is a subsequence of (a_n) and $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_{2n} = 0$.
Since $s_{2n} = s_{2n-1} + (-1)^{2n} a_{2n} = s_{2n-1} + a_{2n}$, then $s_{2n} = s_{2n-1} + a_{2n}$.
Observe that

$$\begin{aligned} L &= L - 0 \\ &= \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} a_{2n} \\ &= \lim_{n\to\infty} (s_{2n} - a_{2n}) \\ &= \lim_{n\to\infty} s_{2n-1}. \end{aligned}$$

Therefore, $L = \lim_{n\to\infty} s_{2n-1}$.
Since there exists a real number L such that $\lim_{n\to\infty} s_{2n} = L$ and $\lim_{n\to\infty} s_{2n-1} = L$, then by a previous lemma, $\lim_{n\to\infty} s_n = L$.
Therefore, (s_n) is convergent, so $\sum(-1)^n a_n$ is convergent. $\square$

Theorem 16. *absolute convergence implies convergence*
Let (a_n) be a sequence in $\mathbb{R}$.
If the series $\sum |a_n|$ is convergent, then the series $\sum a_n$ is convergent.

Proof. Suppose the series $\sum |a_n|$ is convergent.
Let $n \in \mathbb{N}$ be given.
Then $a_n \in \mathbb{R}$, so $-|a_n| \leq a_n \leq |a_n|$.
Thus, $-|a_n| \leq a_n$ and $a_n \leq |a_n|$.
Since $-|a_n| \leq a_n$, then $0 \leq a_n + |a_n|$.
Since $a_n \leq |a_n|$, then $a_n + |a_n| \leq 2|a_n|$.
Hence, $0 \leq a_n + |a_n| \leq 2|a_n|$.
Since $\sum |a_n|$ is convergent and $\sum 2|a_n| = 2\sum |a_n|$, then the series $\sum 2|a_n|$ is convergent.
Thus, by DCT, the series $\sum(a_n + |a_n|)$ is convergent.
Therefore, $\sum a_n = \sum(a_n + |a_n| - |a_n|) = \sum(a_n + |a_n|) - \sum |a_n|$, so the series $\sum a_n$ is convergent, as desired. □

Proof. Suppose the series $\sum |a_n|$ is convergent.
Then $\sum |a_n|$ satisfies the Cauchy criterion for series.
Let $\epsilon > 0$ be given.
Then there exists $N \in \mathbb{N}$ such that if $n > m > N$, then $|\sum_{k=m+1}^{n} |a_k|| < \epsilon$.
Let $m, n \in \mathbb{N}$ such that $n > m > N$.
Then $|\sum_{k=m+1}^{n} |a_k|| < \epsilon$.
Observe that

$$\begin{aligned}
|\sum_{k=m+1}^{n} a_k| &= |a_{m+1} + a_{m+2} + ... + a_n| \\
&\leq |a_{m+1}| + |a_{m+2}| + ... + |a_n| \\
&= \sum_{k=m+1}^{n} |a_k| \\
&\leq |\sum_{k=m+1}^{n} |a_k|| \\
&< \epsilon.
\end{aligned}$$

Thus, $|\sum_{k=m+1}^{n} a_k| < \epsilon$, so the series $\sum a_n$ satisfies the Cauchy criterion.
Therefore, $\sum a_n$ is convergent, as desired. □

Theorem 17. ***ratio test***
Let (a_n) be a sequence of nonzero real numbers.
a. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
b. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
c. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, then the ratio test is inconclusive.

Proof. Let (a_n) be a sequence of nonzero real numbers.
Then $a_n \neq 0$ for all $n \in \mathbb{N}$.
Since $|a_n| \geq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $|a_n| > 0$ for all $n \in \mathbb{N}$.
Let (b_n) be a convergent sequence defined by $b_n = |\frac{a_{n+1}}{a_n}|$ for all $n \in \mathbb{N}$.
Since (b_n) is convergent, then there exists $L \in \mathbb{R}$ such that $L = \lim_{n\to\infty} b_n = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$. □

Proof. We prove a.

If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Suppose $L < 1$.

Let $n \in \mathbb{N}$ be given.

Since $a_n \neq 0$ and $a_{n+1} \neq 0$, then $\frac{a_{n+1}}{a_n} \neq 0$.

Since $|\frac{a_{n+1}}{a_n}| \geq 0$ and $\frac{a_{n+1}}{a_n} \neq 0$, then $b_n = |\frac{a_{n+1}}{a_n}| > 0$.

Thus, $b_n > 0$ for all $n \in \mathbb{N}$, so 0 is a lower bound of (b_n).

Since $\lim_{n\to\infty} b_n = L$ and 0 is a lower bound of (b_n), then $0 \leq L$.

Hence, $0 \leq L < 1$.

Since L and 1 are real numbers, then by the density of $\mathbb{R}$, there exists $r \in \mathbb{R}$ such that $L < r < 1$.

Thus, $0 \leq L < r < 1$, so $0 < r < 1$ and $L < r$.

Since $r - L > 0$ and $\lim_{n\to\infty} b_n = L$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|b_n - L| < r - L$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|b_n - L| < r - L$.

Observe that

$$\begin{aligned}
|b_n - L| < r - L &\Leftrightarrow -(r - L) < b_n - L < r - L \\
&\Rightarrow b_n - L < r - L \\
&\Leftrightarrow b_n < r \\
&\Leftrightarrow |\frac{a_{n+1}}{a_n}| < r \\
&\Leftrightarrow \frac{|a_{n+1}|}{|a_n|} < r \\
&\Rightarrow |a_{n+1}| < r|a_n|.
\end{aligned}$$

Therefore, $|a_{n+1}| < r|a_n|$ for all natural numbers $n > N$.

We prove $|a_{N+k}| < r^{k-1}|a_{N+1}|$ for all natural numbers $k \geq 2$ by induction on k.

Let $S = \{k \in \mathbb{N} : |a_{N+k}| < r^{k-1}|a_{N+1}|, k \geq 2\}$.

Since $N + 1 \in \mathbb{N}$ and $N + 1 > N$, then $|a_{N+2}| < r|a_{N+1}| = r^{2-1}|a_{N+1}|$.

Thus, $|a_{N+2}| < r^{2-1}|a_{N+1}|$, so $2 \in S$.

Suppose $m \in S$.

Then $m \in \mathbb{N}$ and $m \geq 2$ and $|a_{N+m}| < r^{m-1}|a_{N+1}|$.

Since $m \in \mathbb{N}$, then $m + 1 \in \mathbb{N}$.

Since $m + 1 > m$ and $m \geq 2$, then $m + 1 > 2$.

Since $N + m \in \mathbb{N}$ and $N + m > N$, then $|a_{N+m+1}| < r|a_{N+m}|$.

Since $r > 0$ and $|a_{N+m}| < r^{m-1}|a_{N+1}|$, then $r|a_{N+m}| < r^m|a_{N+1}|$.

Hence, $|a_{N+m+1}| < r|a_{N+m}| < r^m|a_{N+1}|$, so $|a_{N+m+1}| < r^m|a_{N+1}| = r^{m+1-1}|a_{N+1}|$.

Thus, $m + 1 \in S$.

Therefore, by PMI, $|a_{N+k}| < r^{k-1}|a_{N+1}|$ for all natural numbers $k \geq 2$.

Since $|a_n| > 0$ for all $n \in \mathbb{N}$ and $N + k \in \mathbb{N}$ for all natural numbers $k \geq 2$, then $|a_{N+k}| > 0$ for all natural numbers $k \geq 2$.

Thus, $0 < |a_{N+k}| < r^{k-1}|a_{N+1}|$ for all natural numbers $k \geq 2$.

Since $|r| = r < 1$, then the geometric series $\sum_{k=1}^{\infty} r^k$ is convergent and $\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$.

Observe that

$$\begin{aligned}
\sum_{k=2}^{\infty} r^{k-1}|a_{N+1}| &= r|a_{N+1}| + r^2|a_{N+1}| + r^3|a_{N+1}| + ... \\
&= |a_{N+1}|(r + r^2 + r^3 + ...) \\
&= |a_{N+1}| \sum_{k=1}^{\infty} r^k \\
&= \frac{|a_{N+1}|r}{1-r}.
\end{aligned}$$

Thus, the series $\sum_{k=2}^{\infty} r^{k-1}|a_{N+1}|$ is convergent.

Hence, by DCT, the series $\sum_{k=2}^{\infty} |a_{N+k}|$ is convergent.

Observe that

$$\begin{aligned}
\sum_{k=2}^{\infty} |a_{N+k}| &= |a_{N+2}| + |a_{N+3}| + |a_{N+4}| + ... \\
&= |a_{(N+1)+1}| + |a_{(N+1)+2}| + |a_{(N+1)+3}| + ... \\
&= \sum_{n=1}^{\infty} |a_{(N+1)+n}|.
\end{aligned}$$

Thus, the series $\sum_{n=1}^{\infty} |a_{(N+1)+n}|$ is convergent.

Since $N+1$ is a positive integer and the series $\sum_{n=1}^{\infty} |a_{(N+1)+n}|$ is convergent, then the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Since absolute convergence implies convergence, then the series $\sum_{n=1}^{\infty} a_n$ is convergent, as desired. □

Proof. We prove b.

If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Suppose $L > 1$.

Then $L - 1 > 0$.

Since $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $||\frac{a_{n+1}}{a_n}| - L| < L - 1$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $||\frac{a_{n+1}}{a_n}| - L| < L - 1$.

Observe that

$$\begin{aligned}
||\frac{a_{n+1}}{a_n}| - L| < L - 1 &\Leftrightarrow -(L-1) < |\frac{a_{n+1}}{a_n}| - L < L - 1 \\
&\Rightarrow 1 - L < |\frac{a_{n+1}}{a_n}| - L \\
&\Leftrightarrow 1 < |\frac{a_{n+1}}{a_n}| \\
&\Leftrightarrow 1 < \frac{|a_{n+1}|}{|a_n|} \\
&\Rightarrow |a_n| < |a_{n+1}|.
\end{aligned}$$

Therefore, $|a_n| < |a_{n+1}|$ for all natural numbers $n > N$.

We prove $|a_{N+1}| \leq |a_n|$ for all natural numbers $n > N$ by induction on n.
Let $S = \{n \in \mathbb{N} : |a_{N+1}| \leq |a_n|, n > N\}$.
Since $N + 1 \in \mathbb{N}$ and $N + 1 > N$ and $|a_{N+1}| = |a_{N+1}|$, then $N + 1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $k > N$ and $|a_{N+1}| \leq |a_k|$.
Since $k \in \mathbb{N}$, then $k + 1 \in \mathbb{N}$.
Since $k + 1 > k$ and $k > N$, then $k + 1 > N$.
Since $k \in \mathbb{N}$ and $k > N$, then $|a_k| < |a_{k+1}|$.
Since $|a_{N+1}| \leq |a_k|$ and $|a_k| < |a_{k+1}|$, then $|a_{N+1}| < |a_{k+1}|$, so $k + 1 \in S$.
Hence, by PMI, $|a_{N+1}| \leq |a_n|$ for all natural numbers $n > N$.

Suppose $\lim_{n\to\infty} a_n = 0$.
Since $N + 1 \in \mathbb{N}$, then $|a_{N+1}| > 0$.
Hence, there exists $K \in \mathbb{N}$ such that if $n > K$, then $|a_n| < |a_{N+1}|$.
Let $M = \max\{K, N\}$.
Let $n \in \mathbb{N}$ such that $n > M$.
Since $n > M \geq K$, then $n > K$, so $|a_n| < |a_{N+1}|$.
Since $n > M \geq N$, then $n > N$, so $|a_{N+1}| \leq |a_n|$.
Thus, we have $|a_n| < |a_{N+1}|$ and $|a_n| \geq |a_{N+1}|$, a violation of trichotomy.
Hence, $\lim_{n\to\infty} a_n \neq 0$.
Therefore, by the n^{th} term test for divergence, the series $\sum a_n$ is divergent.
□