Series in $\mathbb{R}$ Examples

Jason Sass

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Infinite Series of Real Numbers

Example 1. telescoping series

Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Solution. Let (a_n) be a sequence of real numbers defined by $a_n = \frac{1}{n(n+1)}$ for all $n \in \mathbb{N}$.

Then $a_n = \frac{1}{n} - \frac{1}{n+1}$.

Let (s_n) be the sequence of partial sums of (a_n).

We first prove $s_n = \frac{n}{n+1}$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : s_n = \frac{n}{n+1}\}$.

Since $s_1 = a_1 = \frac{1}{2} = \frac{1}{1+1}$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $s_k = \frac{k}{k+1}$.

Observe that

$$\begin{aligned} s_{k+1} &= s_k + a_{k+1} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2} \\ &= \frac{k+1}{[(k+1)+1]}. \end{aligned}$$

Thus, $k+1 \in S$.

Hence, by PMI, $S = \mathbb{N}$, so $s_n = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

Thus, $1 = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} s_n$, so $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$, as desired. $\square$

Example 2. harmonic series is divergent

Let (a_n) be a sequence defined by $a_n = \frac{1}{n}$.

Let (s_n) be the sequence of partial sums defined by $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$ for all $n \in \mathbb{N}$.

Then (s_n) is divergent, so the sequence of terms (a_n) is not summable.

Therefore, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$ diverges.

Proof. To prove (s_n) diverges, we prove (s_n) is unbounded.

We prove (s_n) is unbounded above.

Hence, we prove $(\forall M \in \mathbb{R})(\exists n \in \mathbb{N})(s_n > M)$.

Let $M \in \mathbb{R}$ be given.

Either $M \leq 1$ or $M > 1$.

We consider these cases separately.

Case 1: Suppose $M \leq 1$.

Let $n = 2$. Then $s_2 = 1 + \frac{1}{2} > 1 \geq M$, so $s_2 > M$.

Case 2: Suppose $M > 1$.

Since $2M - 2 \in \mathbb{R}$, then by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > 2M - 2$. Hence, $\frac{2+N}{2} > M$, so $1 + \frac{N}{2} > M$.

Let $n = 2^N$. Then $n \in \mathbb{N}$.

We first prove $s_{2^k} \geq 1 + \frac{k}{2}$ for all $k \in \mathbb{N}$ by induction on k.

Since $N \in \mathbb{N}$ and $s_{2^k} \geq 1 + \frac{k}{2}$ for all $k \in \mathbb{N}$, then $s_n = s_{2^N} \geq 1 + \frac{N}{2}$. Thus, $s_n \geq 1 + \frac{N}{2}$ and $1 + \frac{N}{2} > M$, so $s_n > M$.

Hence, (s_n) is unbounded above, so (s_n) is unbounded.

Therefore, (s_n) is divergent. $\square$

Example 3. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Proof. Let (a_n) be a sequence of real numbers defined by $a_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

Let (s_n) be the sequence of partial sums of (a_n).

We first prove (s_n) is strictly increasing.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1$, so $n + 1 \geq 2 > 0$.

Thus, $n + 1 > 0$, so $(n + 1)^2 > 0$.

Hence, $\frac{1}{(n+1)^2} > 0$.

Observe that $s_{n+1} - s_n = a_{n+1} = \frac{1}{(n+1)^2} > 0$, so $s_{n+1} - s_n > 0$.

Thus, $s_{n+1} > s_n$, so $s_n < s_{n+1}$.

Therefore, (s_n) is strictly increasing, so (s_n) is monotonic and $s_1 = a_1 = 1$ is a lower bound of (s_n).

We next prove $s_n \leq 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : s_n \leq 2 - \frac{1}{n}\}$.

Since $s_1 = a_1 = 1 = 2 - \frac{1}{1}$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $s_k \leq 2 - \frac{1}{k}$.

Since $k \in \mathbb{N}$, then $k \geq 1$.

Since $k \geq 1 > 0$, then $k > 0$.

Since $k \geq 1$ and $k \geq 1 \Rightarrow k+1 \geq 2 > 0 \Rightarrow k+1 > 0 \Rightarrow (k+1)^2 > 0$, then $(k+1)^2 > 0$.

Observe that

$$\begin{aligned} k(k+2) = k^2 + 2k < k^2 + 2k + 1 = (k+1)^2 \quad &\Rightarrow \quad k(k+2) < (k+1)^2 \\ &\Rightarrow \quad \frac{k+2}{(k+1)^2} < \frac{1}{k} \\ &\Rightarrow \quad \frac{1+(k+1)}{(k+1)^2} < \frac{1}{k} \\ &\Rightarrow \quad \frac{1}{(k+1)^2} + \frac{1}{k+1} < \frac{1}{k} \\ &\Rightarrow \quad s_k + \frac{1}{(k+1)^2} + \frac{1}{k+1} < s_k + \frac{1}{k} \leq 2 \\ &\Rightarrow \quad s_k + \frac{1}{(k+1)^2} + \frac{1}{k+1} < 2 \\ &\Rightarrow \quad s_k + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1} \\ &\Rightarrow \quad s_k + a_{k+1} < 2 - \frac{1}{k+1} \\ &\Rightarrow \quad s_{k+1} < 2 - \frac{1}{k+1} \\ &\Rightarrow \quad k+1 \in S. \end{aligned}$$

Thus, $k \in S$ implies $k+1 \in S$.

Hence, by PMI, $S = \mathbb{N}$, so $s_n \leq 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

We next prove $2 - \frac{1}{n} < 2$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1 > 0$, so $n > 0$.

Hence, $\frac{1}{n} > 0$, so $\frac{1}{n} > 2 - 2$.

Therefore, $2 > 2 - \frac{1}{n}$, so $2 - \frac{1}{n} < 2$ for all $n \in \mathbb{N}$.

Since $s_n \leq 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$ and $2 - \frac{1}{n} < 2$ for all $n \in \mathbb{N}$, then $s_n \leq 2 - \frac{1}{n} < 2$ for all $n \in \mathbb{N}$, so $s_n < 2$ for all $n \in \mathbb{N}$.

Since 1 is a lower bound of (s_n), then $1 \leq s_n$ for all $n \in \mathbb{N}$.

Thus, $1 \leq s_n < 2$ for all $n \in \mathbb{N}$, so (s_n) is bounded.

Since (s_n) is bounded and monotonic, then by MCT, (s_n) is convergent.

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. $\square$

Example 4. Show that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Proof. Let (a_n) be a sequence of real numbers defined by $a_n = \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

Then $a_n = (\frac{1}{2})^n$ for all $n \in \mathbb{N}$, so (a_n) is a geometric sequence with common ratio $r = \frac{1}{2}$.

Since $|r| = |\frac{1}{2}| = \frac{1}{2} < 1$, then $a_n \to 0$, so $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

Let (s_n) be the sequence of partial sums of (a_n).

We prove $s_n = 1 - \frac{1}{2^n}$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : s_n = 1 - \frac{1}{2^n}\}$.
Since $s_1 = a_1 = \frac{1}{2} = 1 - \frac{1}{2^1}$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $s_k = 1 - \frac{1}{2^k}$.
Observe that

$$\begin{aligned} s_{k+1} &= s_k + a_{k+1} \\ &= (1 - \frac{1}{2^k}) + \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^k} + \frac{1}{2}\frac{1}{2^k} \\ &= 1 - \frac{1}{2}\frac{1}{2^k} \\ &= 1 - \frac{1}{2^{k+1}}. \end{aligned}$$

Hence, $s_{k+1} = 1 - \frac{1}{2^{k+1}}$, so $k+1 \in S$.
Thus, by PMI, $s_n = 1 - \frac{1}{2^n}$ for all $n \in \mathbb{N}$.
Observe that

$$\begin{aligned} 1 &= 1 - 0 \\ &= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{2^n} \\ &= \lim_{n\to\infty} (1 - \frac{1}{2^n}) \\ &= \lim_{n\to\infty} s_n. \end{aligned}$$

Therefore, $1 = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$, so $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, as desired. $\square$

Convergence Tests for Series of Real Numbers

Example 5. applying the direct comparison test
a. The series $\sum \frac{1}{\sqrt{n}}$ and $\sum \frac{n+1}{n^2+1}$ are divergent.
b. The series $\sum \frac{1}{n^2+1}$ and $\sum \frac{1}{n^3}$ are convergent.

Solution. a1. We prove the series $\sum \frac{1}{\sqrt{n}}$ is divergent.
Let $n \in \mathbb{N}$ be given.
Then $n \geq 1 > 0$.

Observe that

$$\begin{aligned} n \geq 1 \wedge n > 0 \quad &\Rightarrow \quad n^2 \geq n > 0 \\ &\Rightarrow \quad \sqrt{n^2} \geq \sqrt{n} > 0 \\ &\Rightarrow \quad |n| \geq \sqrt{n} > 0 \\ &\Rightarrow \quad n \geq \sqrt{n} > 0 \\ &\Rightarrow \quad \frac{1}{\sqrt{n}} \geq \frac{1}{n} > 0 \\ &\Rightarrow \quad 0 < \frac{1}{n} \leq \frac{1}{\sqrt{n}} \\ &\Rightarrow \quad 0 \leq \frac{1}{n} \leq \frac{1}{\sqrt{n}}. \end{aligned}$$

Hence, $0 \leq \frac{1}{n} \leq \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$.

Since the series $\sum \frac{1}{n}$ is divergent, then by DCT, the series $\sum \frac{1}{\sqrt{n}}$ is divergent. □

Solution. a2. We prove the series $\sum \frac{n+1}{n^2+1}$ is divergent.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1 > 0$.

Observe that

$$\begin{aligned} 0 < 1 \leq n \quad &\Rightarrow \quad 0 < n^2 + 1 \leq n^2 + n = n(n+1) \\ &\Rightarrow \quad 0 < n^2 + 1 \leq n(n+1) \\ &\Rightarrow \quad 0 < \frac{1}{n} \leq \frac{n+1}{n^2+1} \\ &\Rightarrow \quad 0 \leq \frac{1}{n} \leq \frac{n+1}{n^2+1}. \end{aligned}$$

Since the series $\sum \frac{1}{n}$ is divergent, then by DCT, the series $\sum \frac{n+1}{n^2+1}$ is divergent. □

Solution. b1. We prove the series $\sum \frac{1}{n^2+1}$ is convergent.

Let $n \in \mathbb{N}$ be given.

Clearly, $n^2 + 1 > n^2 > 0$, so $0 < \frac{1}{n^2+1} < \frac{1}{n^2}$.

Hence, $0 \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$, so $0 \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

Since the series $\sum \frac{1}{n^2}$ is convergent, then by DCT, the series $\sum \frac{1}{n^2+1}$ is convergent. □

Solution. b2. We prove the series $\sum \frac{1}{n^3}$ is convergent.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1 > 0$, so $n > 0$.

Hence, $n^2 > 0$.

Since $n \geq 1$ and $n^2 > 0$, then $n^3 \geq n^2 > 0$, so $0 < \frac{1}{n^3} \leq \frac{1}{n^2}$.

Thus, $0 \leq \frac{1}{n^3} \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

Since the series $\sum \frac{1}{n^2}$ is convergent, then by DCT, the series $\sum \frac{1}{n^3}$ is convergent. □

Example 6. applying the limit comparison test
The series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ is convergent.

Solution. Let $n \in \mathbb{N}$ such that $n > 1$.
Then $n^2 > 1$, so $n^2 - 1 > 0$.
Hence, $\frac{1}{n^2-1} > 0$.
Since $n > 0$, then $n^2 > 0$, so $\frac{1}{n^2} > 0$.
Thus, $\frac{1}{n^2-1} > 0$ and $\frac{1}{n^2} > 0$ for all $n > 1$.
Since $\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1 > 0$ and the series $\sum \frac{1}{n^2}$ is convergent, then by LCT, the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ is convergent. $\square$

Example 7. alternating harmonic series is convergent
The series $\sum \frac{(-1)^n}{n}$ is convergent.

Solution.
Let (a_n) be the sequence of real numbers defined by $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$.
Since $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$ and the sequence $(\frac{1}{n})$ is decreasing and $\lim_{n\to\infty} \frac{1}{n} = 0$, then by AST, the series $\sum \frac{(-1)^n}{n}$ is convergent. $\square$

Example 8. The series $\sum \frac{(-1)^n}{n^2}$ converges.

Solution.
Since $(\frac{1}{n^2})$ is a decreasing sequence of positive terms and $\lim_{n\to\infty} \frac{1}{n^2} = 0$, then by AST, the series $\sum \frac{(-1)^n}{n^2}$ converges.
In fact, the series converges to a negative real number. $\square$

Example 9. The series $\sum \frac{\sin(n)}{n^2}$ is absolutely convergent, so $\sum \frac{\sin(n)}{n^2}$ is convergent.

Solution. Let $n \in \mathbb{N}$ be given.
Then $n > 0$, so $n^2 > 0$.
Since $|\sin x| \leq 1$ for all $x \in \mathbb{R}$ and $n \in \mathbb{R}$, then $0 \leq |\sin n| \leq 1$.
Thus, $0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$, so $0 \leq |\frac{\sin n}{n^2}| \leq \frac{1}{n^2}$.
Since the series $\sum \frac{1}{n^2}$ is convergent, then by DCT, the series $\sum |\frac{\sin n}{n^2}|$ is convergent.
Therefore, the series $\sum \frac{\sin(n)}{n^2}$ is absolutely convergent, so $\sum \frac{\sin(n)}{n^2}$ is convergent. $\square$