Series in $\mathbb{R}$ Exercises

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Infinite Series of Real Numbers

Exercise 1. Obtain a formula for $\sum_{k=1}^{n}(-1)^k$.

Solution. After various attempts, we have $\sum_{k=1}^{n}(-1)^k = \frac{-1}{2} + \frac{1}{2}(-1)^n$ for all $n \in \mathbb{N}$.

We shall prove $\sum_{k=1}^{n}(-1)^k = \frac{-1}{2} + \frac{1}{2}(-1)^n$ for all $n \in \mathbb{N}$ by induction on n. $\square$

Exercise 2. Show that $\sum_{n=1}^{\infty} \frac{1}{n+2} - \frac{1}{n+3} = \frac{1}{3}$.

Solution. Let (a_n) be a sequence of real numbers defined by $a_n = \frac{1}{n+2} - \frac{1}{n+3}$ for all $n \in \mathbb{N}$.

Let (s_n) be the sequence of partial sums of (a_n).
We first prove $s_n = \frac{1}{3} - \frac{1}{n+3}$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : s_n = \frac{1}{3} - \frac{1}{n+3}\}$.
Since $s_1 = a_1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{3} - \frac{1}{1+3}$, then $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $s_k = \frac{1}{3} - \frac{1}{k+3}$.
Observe that

$$\begin{aligned}
s_{k+1} &= s_k + a_{k+1} \\
&= (\frac{1}{3} - \frac{1}{k+3}) + (\frac{1}{[(k+1)+2]} - \frac{1}{[(k+1)+3]}) \\
&= \frac{1}{3} - \frac{1}{k+3} + \frac{1}{k+3} - \frac{1}{k+4} \\
&= \frac{1}{3} - \frac{1}{k+4} \\
&= \frac{1}{4} - \frac{1}{[(k+1)+3]}.
\end{aligned}$$

Thus, $k+1 \in S$.

Hence, by PMI, $S = \mathbb{N}$, so $s_n = \frac{1}{3} - \frac{1}{n+3}$ for all $n \in \mathbb{N}$.
Let $n \in \mathbb{N}$ be given.
Then $n \geq 1 > 0$, so $n > 0$.

Since $n+3>n>0$, then $0<\frac{1}{n+3}<\frac{1}{n}$.

Hence, $0<\frac{1}{n+3}<\frac{1}{n}$ for all $n\in\mathbb{N}$.

Since $0\leq\frac{1}{n+3}\leq\frac{1}{n}$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}0=0$ and $\lim_{n\to\infty}\frac{1}{n}=0$, then by the squeeze rule, $\lim_{n\to\infty}\frac{1}{n+3}=0$.

Observe that

$$\begin{aligned}\frac{1}{3} &= \frac{1}{3}-0\\ &= \lim_{n\to\infty}\frac{1}{3}-\lim_{n\to\infty}\frac{1}{n+3}\\ &= \lim_{n\to\infty}(\frac{1}{3}-\frac{1}{n+3})\\ &= \lim_{n\to\infty}s_n.\end{aligned}$$

Therefore, $\frac{1}{3}=\lim_{n\to\infty}s_n=\sum_{n=1}^{\infty}\frac{1}{n+2}-\frac{1}{n+3}$. □

Exercise 3. Show that $\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}=\frac{1}{4}$.

Solution. Let (a_n) be a sequence of real numbers defined by $a_n=\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}$ for all $n\in\mathbb{N}$.

Let (s_n) be the sequence of partial sums of (a_n).

We first prove $s_n=\frac{1}{4}-\frac{1}{(n+2)^2}$ for all $n\in\mathbb{N}$ by induction on n.

Let $S=\{n\in\mathbb{N}:s_n=\frac{1}{4}-\frac{1}{(n+2)^2}\}$.

Since $s_1=a_1=\frac{1}{2^2}-\frac{1}{3^2}=\frac{1}{4}-\frac{1}{(1+2)^2}$, then $1\in S$.

Suppose $k\in S$.

Then $k\in\mathbb{N}$ and $s_k=\frac{1}{4}-\frac{1}{(k+2)^2}$.

Observe that

$$\begin{aligned}s_{k+1} &= s_k+a_{k+1}\\ &= [\frac{1}{4}-\frac{1}{(k+2)^2}]+(\frac{1}{[(k+1)+1]^2}-\frac{1}{[(k+1)+2]^2})\\ &= \frac{1}{4}-\frac{1}{(k+2)^2}+\frac{1}{(k+2)^2}-\frac{1}{(k+3)^2}\\ &= \frac{1}{4}-\frac{1}{(k+3)^2}\\ &= \frac{1}{4}-\frac{1}{[(k+1)+2]^2}.\end{aligned}$$

Thus, $k+1\in S$.

Hence, by PMI, $S=\mathbb{N}$, so $s_n=\frac{1}{4}-\frac{1}{(n+2)^2}$ for all $n\in\mathbb{N}$.

Let $n\in\mathbb{N}$ be given.

Then $n\geq 1>0$, so $n>0$.

Since $n+2>n>0$, then $0<\frac{1}{n+2}<\frac{1}{n}$.

Hence, $0<\frac{1}{n+2}<\frac{1}{n}$ for all $n\in\mathbb{N}$.

Since $0 \leq \frac{1}{n+2} \leq \frac{1}{n}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, then by the squeeze rule, $\lim_{n\to\infty} \frac{1}{n+2} = 0$.

Observe that

$$\begin{aligned}
\frac{1}{4} &= \frac{1}{4} - 0 \cdot 0 \\
&= \lim_{n\to\infty} \frac{1}{4} - \lim_{n\to\infty} \frac{1}{n+2} \cdot \lim_{n\to\infty} \frac{1}{n+2} \\
&= \lim_{n\to\infty} \frac{1}{4} - \lim_{n\to\infty} \frac{1}{(n+2)^2} \\
&= \lim_{n\to\infty} [\frac{1}{4} - \frac{1}{(n+2)^2}] \\
&= \lim_{n\to\infty} s_n.
\end{aligned}$$

Therefore, $\frac{1}{4} = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2}$. □

Exercise 4. Show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}} = \frac{1}{\sqrt{2}}$.

Solution. Let (a_n) be a sequence of real numbers defined by $a_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}$ for all $n \in \mathbb{N}$.

Let (s_n) be the sequence of partial sums of (a_n).

We first prove $s_n = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+2}}$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : s_n = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+2}}\}$.

Since $s_1 = a_1 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{1+2}}$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $s_k = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{k+2}}$.

Observe that

$$\begin{aligned}
s_{k+1} &= s_k + a_{k+1} \\
&= (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{k+2}}) + (\frac{1}{\sqrt{(k+1)+1}} - \frac{1}{\sqrt{(k+1)+2}}) \\
&= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{k+2}} + \frac{1}{\sqrt{k+2}} - \frac{1}{\sqrt{k+3}} \\
&= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{k+3}} \\
&= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{(k+1)+2}}.
\end{aligned}$$

Thus, $k+1 \in S$.

Hence, by PMI, $S = \mathbb{N}$, so $s_n = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+2}}$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1 > 0$, so $n > 0$.

Since $n+2 > n > 0$, then $\sqrt{n+2} > \sqrt{n} > 0$, so $0 < \frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n}}$.

Hence, $0 < \frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$.

Since $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, then by a previous exercise, $\lim_{n\to\infty} \sqrt{\frac{1}{n}} = 0$, so $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Since $0 \leq \frac{1}{\sqrt{n+2}} \leq \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, then by the squeeze rule, $\lim_{n\to\infty} \frac{1}{\sqrt{n+2}} = 0$.

Observe that

$$\begin{aligned} \frac{1}{\sqrt{2}} &= \frac{1}{\sqrt{2}} - 0 \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{2}} - \lim_{n\to\infty} \frac{1}{\sqrt{n+2}} \\ &= \lim_{n\to\infty} (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+2}}) \\ &= \lim_{n\to\infty} s_n. \end{aligned}$$

Therefore, $\frac{1}{\sqrt{2}} = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}$. ∎

Exercise 5. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent.

Solution. Let (a_n) be a sequence of real numbers defined by $a_n = \frac{1}{n^3}$ for all $n \in \mathbb{N}$.

Let (s_n) be the sequence of partial sums of (a_n).

We first prove (s_n) is strictly increasing.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1$, so $n + 1 \geq 2 > 0$.

Thus, $n + 1 > 0$, so $(n + 1)^3 > 0$.

Hence, $\frac{1}{(n+1)^3} > 0$.

Observe that $s_{n+1} - s_n = a_{n+1} = \frac{1}{(n+1)^3} > 0$, so $s_{n+1} - s_n > 0$.

Thus, $s_{n+1} > s_n$, so $s_n < s_{n+1}$.

Therefore, (s_n) is strictly increasing, so (s_n) is monotonic and $s_1 = a_1 = 1$ is a lower bound of (s_n).

We next prove $s_n \leq 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : s_n \leq 2 - \frac{1}{n}\}$.

Since $s_1 = a_1 = 1 = 2 - \frac{1}{1}$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $s_k \leq 2 - \frac{1}{k}$.

Since $k \in \mathbb{N}$, then $k \geq 1 > 0$, so $k > 0$ and $k^2 > 0$.

Since $k^2 > 0$ and $k > 0$, then $k^2 + k + 1 > 0$, so $(k + 1)^2 = k^2 + 2k + 1 > k$.

Thus, $(k + 1)^2 > k$, so $(k + 1)^2 > k > 0$.

Hence, $(k + 1)^3 > k(k + 1) > 0$, so $\frac{1}{(k+1)^3} < \frac{1}{k(k+1)}$.

Observe that

$$\begin{aligned}
\frac{1}{(k+1)^3} < \frac{1}{k(k+1)} \quad &\Rightarrow \quad s_k + \frac{1}{(k+1)^3} < s_k + \frac{1}{k(k+1)} \\
&\Rightarrow \quad s_k + a_{k+1} < s_k + \frac{1}{k(k+1)} \\
&\Rightarrow \quad s_k + a_{k+1} < s_k + \frac{1}{k(k+1)} \leq 2 - \frac{1}{k} + \frac{1}{k(k+1)} \\
&\Rightarrow \quad s_k + a_{k+1} < 2 - \frac{1}{k} + \frac{1}{k(k+1)} \\
&\Rightarrow \quad s_{k+1} < 2 - \frac{1}{k}(1 - \frac{1}{k+1}) \\
&\Rightarrow \quad s_{k+1} < 2 - \frac{1}{k}(\frac{k}{k+1}) \\
&\Rightarrow \quad s_{k+1} < 2 - \frac{1}{k+1} \\
&\Rightarrow \quad k+1 \in S.
\end{aligned}$$

Thus, $k \in S$ implies $k+1 \in S$.

Hence, by PMI, $S = \mathbb{N}$, so $s_n \leq 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

We next prove $2 - \frac{1}{n} < 2$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1 > 0$, so $n > 0$.

Hence, $\frac{1}{n} > 0$, so $\frac{1}{n} > 2 - 2$.

Therefore, $2 > 2 - \frac{1}{n}$, so $2 - \frac{1}{n} < 2$ for all $n \in \mathbb{N}$.

Since $s_n \leq 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$ and $2 - \frac{1}{n} < 2$ for all $n \in \mathbb{N}$, then $s_n \leq 2 - \frac{1}{n} < 2$ for all $n \in \mathbb{N}$, so $s_n < 2$ for all $n \in \mathbb{N}$.

Since 1 is a lower bound of (s_n), then $1 \leq s_n$ for all $n \in \mathbb{N}$.

Thus, $1 \leq s_n < 2$ for all $n \in \mathbb{N}$, so (s_n) is bounded.

Since (s_n) is bounded and monotonic, then by MCT, (s_n) is convergent.

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent. □

Exercise 6. Show that the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent.

Solution. Since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, then by the n^{th} term test for divergence, the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent. □

Exercise 7. Find a sequence whose n^{th} partial sum is $\frac{n-1}{n+1}$.

Solution. Let (a_n) be a sequence of real numbers.

Let (s_n) be the sequence of partial sums of (a_n).

Then $s_n = \frac{n-1}{n+1}$ for all $n \in \mathbb{N}$.

Thus, $0 = s_1 = a_1$.

Since $\frac{1}{3} = s_2 = a_1 + a_2 = 0 + a_2$, then $a_2 = \frac{1}{3}$.

Since $\frac{1}{2} = s_3 = a_1 + a_2 + a_3 = 0 + \frac{1}{3} + a_3$, then $a_3 = \frac{1}{6}$.

Since $\frac{3}{5} = s_4 = a_1 + a_2 + a_3 + a_4 = 0 + \frac{1}{3} + \frac{1}{6} + a_4$, then $a_4 = \frac{1}{10}$.

We can continue this process and see that the sequence has terms $a_1 = 0, a_2 = \frac{1}{3}, a_3 = \frac{1}{6}, a_4 = \frac{1}{10}, a_5 = \frac{1}{15}, a_6 = \frac{1}{21}, a_7 = \frac{1}{28}, ...$

The sequence is defined recursively by $a_1 = 0$ and $a_2 = \frac{1}{3}$ and $a_{n+1} = \frac{a_n}{1+(n+1)a_n}$ for $n > 2$.

Equivalently, the sequence is $a_1 = 0$ and $a_n = \frac{2}{n(n+1)}$ for $n > 1$. □

Proof. Let the sequence (a_n) be given by $a_1 = 0$ and $a_n = \frac{2}{n(n+1)}$ for $n > 1$.

We prove the sequence of partial sums of (a_n) is $\frac{n-1}{n+1}$.

Let (s_n) be the sequence of partial sums of (a_n).

Then for all $n \in \mathbb{N}$, we have $s_n = a_1 + (a_2 + a_3 + ... + a_n) = 0 + (a_2 + a_3 + ... + a_n) = a_2 + a_3 + ... + a_n = \sum_{k=2}^{n} a_k$.

To prove $s_n = \frac{n-1}{n+1}$ for all $n \in \mathbb{N}$, we must prove $\sum_{k=2}^{n} a_k = \frac{n-1}{n+1}$ for all $n \in \mathbb{N}$.

We prove by induction on n.

For $n = 1$, observe that $\frac{1-1}{1+1} = 0 = \sum_{k=2}^{1} a_k$.

For $n = 2$, observe that $\frac{2-1}{2+1} = \frac{1}{3} = \frac{2}{2 \cdot 3} = a_2 = \sum_{k=2}^{2} a_k$.

Let $m \in \mathbb{N}$ such that $\sum_{k=2}^{m} a_k = \frac{m-1}{m+1}$.

We must prove $\sum_{k=2}^{m+1} a_k = \frac{m}{m+2}$.

Since $m \in \mathbb{N}$, then $m \geq 1$, so $m + 1 \geq 2 > 1$.

Hence, $m + 1 > 1$, so $a_{m+1} = \frac{2}{(m+1)(m+2)}$.

Observe that

$$\begin{aligned}
\sum_{k=2}^{m+1} a_k &= \sum_{k=2}^{m} a_k + a_{m+1} \\
&= \frac{m-1}{m+1} + \frac{2}{(m+1)(m+2)} \\
&= \frac{(m-1)(m+2)+2}{(m+1)(m+2)} \\
&= \frac{m^2+m}{(m+1)(m+2)} \\
&= \frac{m(m+1)}{(m+1)(m+2)} \\
&= \frac{m}{m+2}.
\end{aligned}$$

Thus, by PMI, $\sum_{k=2}^{n} a_k = \frac{n-1}{n+1}$ for all $n \in \mathbb{N}$, so $s_n = \frac{n-1}{n+1}$ for all $n \in \mathbb{N}$, as desired. □

Exercise 8. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is convergent.

Solution. Let $n \in \mathbb{N}$ be given.

Then $n > 0$, so $n^3 > 0$.

Since $0 < n^3 < n^3 + 1$, then $0 < \frac{1}{n^3+1} < \frac{1}{n^3}$.

Thus, $0 < \frac{1}{n^3+1} < \frac{1}{n^3}$ for all $n \in \mathbb{N}$.

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, then by DCT, the series $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converges. $\square$

Exercise 9. Determine if the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$ is convergent.

Solution. Let $n \in \mathbb{N}$ be given.

Then $n \geq 1$.

Since $1 \leq n$, then $n^2 + 1 \leq n^2 + n = n(n+1)$, so $0 < n^2 + 1 \leq n(n+1)$.

Thus, $0 < \frac{1}{n} \leq \frac{n+1}{n^2+1}$ for all $n \in \mathbb{N}$.

Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then by DCT, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$ diverges. $\square$

Exercise 10. Determine if the series $\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$ is convergent.

Solution. Since $\frac{n+1}{n^3+1} > 0$ for all $n \in \mathbb{N}$ and $\frac{1}{n^2} > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{\frac{n+1}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3+1} = 1 > 0$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, then by LCT, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$ is convergent. $\square$

Exercise 11. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ is convergent.

Solution. Observe that $\frac{1}{\sqrt{n^2+1}} > 0$ for all $n \in \mathbb{N}$ and $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$.

Observe that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2+1}}} &= \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n} \\
&= \lim_{n\to\infty} \frac{\sqrt{n^2(1+\frac{1}{n^2})}}{n} \\
&= \lim_{n\to\infty} \sqrt{1+\frac{1}{n^2}} \\
&= 1 \\
&> 0.
\end{aligned}$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, then by LCT, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ is divergent too. $\square$

Exercise 12. Determine if the series $\sum_{n=1}^{\infty} (\frac{5}{3})^{-n}$ is convergent.

Solution. Observe that

$$\begin{aligned}
\sum_{n=1}^{\infty} (\frac{5}{3})^{-n} &= \sum_{n=1}^{\infty} \frac{1}{(\frac{5}{3})^n} \\
&= \sum_{n=1}^{\infty} \frac{1}{\frac{5^n}{3^n}} \\
&= \sum_{n=1}^{\infty} \frac{3^n}{5^n} \\
&= \sum_{n=1}^{\infty} (\frac{3}{5})^n.
\end{aligned}$$

Thus, the series is a geometric series with common ratio $\frac{3}{5}$.

Since $|\frac{3}{5}| = \frac{3}{5} < 1$, then the geometric series converges to $\frac{\frac{3}{5}}{1-\frac{3}{5}} = \frac{3}{2}$.

Therefore, $\sum_{n=1}^{\infty}(\frac{5}{3})^{-n} = \frac{3}{2}$. ∎

Exercise 13. Let $\sum_{n=1}^{\infty} a_n$ be a series of nonnegative terms.
If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

Proof. Let (a_n) be a sequence in $\mathbb{R}$ of nonnegative terms.
Then $a_n \geq 0$ for all $n \in \mathbb{N}$.
Let (s_n) be the sequence of partial sums of (a_n).
Suppose the series $\sum_{n=1}^{\infty} a_n$ is convergent.
Then the sequence (s_n) is convergent, so (s_n) is bounded.
Hence, there exists $M \in \mathbb{R}$ such that $|s_n| \leq M$ for all $n \in \mathbb{N}$.
We first prove $s_n \geq 0$ for all $n \in \mathbb{N}$ by induction on n.
Let $S = \{n \in \mathbb{N} : s_n \geq 0\}$.
Since $s_1 = a_1 \geq 0$, then $s_1 \geq 0$, so $1 \in S$.
Suppose $k \in S$.
Then $k \in \mathbb{N}$ and $s_k \geq 0$.
Since $k+1 \in \mathbb{N}$, then $a_{k+1} \geq 0$.
Since $s_{k+1} = s_k + a_{k+1}$ and $s_k \geq 0$ and $a_{k+1} \geq 0$, then $s_{k+1} \geq 0$.
Hence, $k+1 \in S$.
Therefore, by PMI, $S = \mathbb{N}$, so $s_n \geq 0$ for all $n \in \mathbb{N}$.
Thus, $0 \leq |s_n| = s_n \leq M$ for all $n \in \mathbb{N}$, so $0 \leq s_n \leq M$ for all $n \in \mathbb{N}$.
We next prove $a_n \leq M$ for all $n \in \mathbb{N}$ by induction on n.
Let $T = \{n \in \mathbb{N} : a_n \leq M\}$.
Since $a_1 = s_1 \leq M$, then $a_1 \leq M$, so $1 \in T$.
Suppose $m \in T$.
Then $m \in \mathbb{N}$.
Since $m+1 \in \mathbb{N}$, then $s_{m+1} \leq M$, so $s_m + a_{m+1} \leq M$.
Thus, $a_{m+1} \leq M - s_m$.
Since $s_m \geq 0$, then $-s_m \leq 0$, so $M - s_m \leq M$.
Hence, $a_{m+1} \leq M - s_m \leq M$, so $a_{m+1} \leq M$.
Thus, $m+1 \in T$.
Therefore, by PMI, $T = \mathbb{N}$, so $a_n \leq M$ for all $n \in \mathbb{N}$.
Since $0 \leq a_n$ for all $n \in \mathbb{N}$ and $a_n \leq M$ for all $n \in \mathbb{N}$, then $0 \leq a_n \leq M$ for all $n \in \mathbb{N}$.
Hence, $0 \leq a_n^2 \leq Ma_n$ for all $n \in \mathbb{N}$.
Since $\sum_{n=1}^{\infty} Ma_n = M\sum_{n=1}^{\infty} a_n$ and the series $\sum_{n=1}^{\infty} a_n$ is convergent then by the scalar multiple rule, the series $\sum_{n=1}^{\infty} Ma_n$ is convergent.
Therefore, by DCT, the series $\sum_{n=1}^{\infty} a_n^2$ is convergent. ∎

Exercise 14. Give an example of a divergent series whose sequence of partial sums is bounded.

Solution. Consider the series $\sum_{n=1}^{\infty}(-1)^n$.

Since the sequence given by $a_n = (-1)^n$ for all $n \in \mathbb{N}$ oscillates between -1 and 1, then the sequence (a_n) is divergent.

Hence, (a_n) is not convergent, so in particular, (a_n) does not converge to 0.

Therefore, by the n^{th} term test, the series diverges.

Let (s_n) be the sequence of partial sums of (a_n).

Then the sequence (s_n) has terms $-1, 0, -1, 0, -1, 0...$ so the set of all terms of (s_n) is $\{-1, 0\}$.

Therefore, $-1 \leq s_n \leq 0$ for all $n \in \mathbb{N}$, so the sequence (s_n) is bounded. $\square$

Exercise 15. Let $\sum_{n=1}^{\infty} a_n$ be a series and let $\sum_{n=1}^{\infty} b_n$ be a series in which the terms are the same and in the same order as $\sum_{n=1}^{\infty} a_n$ except that all terms such that $a_n = 0$ are omitted. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} b_n$ is convergent.

Proof. Let (a_n) and (b_n) be sequences of real numbers.

Let (s_n) be the sequence of partial sums of (a_n) and let (t_n) be the sequence of partial sums of (b_n).

Suppose the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Then the sequence (s_n) is convergent.

Either there exists $n \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, if $N \geq n$, then $a_N = 0$ or there does not.

We consider each case separately.

Case 1: Suppose there exists $n \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, if $N \geq n$, then $a_N = 0$.

Then this case implies that the sequence (a_n) has only a finite number of nonzero terms.

So, this is the baseline case in which the series is just a finite sum + 0, so this should be easy to prove.

Case 2: Suppose there does not exist $n \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, if $N \geq n$, then $a_N = 0$.

Then for each $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $N \geq n$ and $a_N \neq 0$.

Let $n \in \mathbb{N}$ be given.

Then there exists $N \in \mathbb{N}$ such that $N \geq n$ and $a_N \neq 0$.

Let $T = \{k \in \mathbb{N} : k \geq n \wedge a_k \neq 0\}$.

Then $T \subset \mathbb{N}$ and $N \in T$, so T is not empty.

Hence, by WOP, T has a least element, so there exists a smallest natural number m such that $m \geq n$ and $a_m \neq 0$.

Thus, there exists a smallest natural number m such that $m \geq n$ and $a_m \neq 0$ for each $n \in \mathbb{N}$.

Since (b_n) consists of the terms of (a_n) in the same order with the zero terms omitted, then (b_n) is a subsequence of (a_n).

Hence, there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{g(n)}$ for all $n \in \mathbb{N}$.

Let $g(n)$ be the smallest natural number such that $g(n) \geq n$ and $a_{g(n)} \neq 0$ for each $n \in \mathbb{N}$.

We prove $t_n = s_{g(n)}$ for all $n \in \mathbb{N}$ by induction on n.

Let $S = \{n \in \mathbb{N} : t_n = s_{g(n)}\}$.

Since $g(1) \geq 1$, then either $g(1) > 1$ or $g(1) = 1$.

If $g(1) = 1$, then $t_1 = b_1 = a_{g(1)} = a_1 = s_1 = s_{g(1)}$, so $t_1 = s_{g(1)}$.

If $g(1) > 1$, then the first $g(1) - 1$ terms of the sequence (a_n) are zero, so $a_i = 0$ for each $i \in \{1, ..., g(1) - 1\}$.

Observe that

$$\begin{aligned} t_1 &= b_1 \\ &= a_{g(1)} \\ &= 0 + a_{g(1)} \\ &= \sum_{i=1}^{g(1)-1} a_i + a_{g(1)} \\ &= \sum_{i=1}^{g(1)} a_i \\ &= s_{g(1)}. \end{aligned}$$

Therefore, in either case, $t_1 = s_{g(1)}$, so $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{N}$ and $t_k = s_{g(k)}$.

Observe that

$$\begin{aligned} t_{k+1} &= t_k + b_{k+1} \\ &= s_{g(k)} + a_{g(k+1)} \\ &= \sum_{i=1}^{g(k)} a_i + a_{g(k+1)} \\ &= \sum_{i=1}^{g(k)} a_i + 0 + a_{g(k+1)} \\ &= \sum_{i=1}^{g(k)} a_i + \sum_{i=g(k)+1}^{g(k+1)-1} a_i + a_{g(k+1)} \\ &= \sum_{i=1}^{g(k+1)} a_i \\ &= s_{g(k+1)}. \end{aligned}$$

Hence, $k + 1 \in S$.

Therefore, by PMI, $S = \mathbb{N}$, so $t_n = s_{g(n)}$ for all $n \in \mathbb{N}$.

Since g is a strictly increasing function and $t_n = s_{g(n)}$ for all $n \in \mathbb{N}$, then (t_n) is a subsequence of (s_n).

Since (s_n) is convergent, then (t_n) is convergent, so $\sum_{n=1}^{\infty} b_n$ is convergent. ☐

Exercise 16. Discuss the convergence behavior of the series $\sum_{n=1}^{\infty} \frac{(-1)^n(3n+2)}{n^3+1}$.

Solution. Since $\frac{3n+2}{n^3+1} > 0$ and $\frac{1}{n^2} > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{\frac{3n+2}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2(3n+2)}{n^3+1} = \lim_{n\to\infty} \frac{3n^3+2n^2}{n^3+1} = 3 > 0$ and the series $\sum \frac{1}{n^2}$ is convergent, then by LCT, the series $\sum \frac{3n+2}{n^3+1}$ is convergent.

Since $\sum |\frac{(-1)^n(3n+2)}{n^3+1}| = \sum \frac{3n+2}{n^3+1}$, then the series $\sum_{n=1}^{\infty} \frac{(-1)^n(3n+2)}{n^3+1}$ is absolutely convergent, so the series $\sum_{n=1}^{\infty} \frac{(-1)^n(3n+2)}{n^3+1}$ is convergent. □

Exercise 17. Discuss the convergence behavior of the series $\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{n^2+1}$.

Solution. Since $\frac{2n+1}{n^2+1} > 0$ and $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{\frac{2n+1}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n(2n+1)}{n^2+1} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+1} = 2 > 0$ and the series $\sum \frac{1}{n}$ is divergent, then by LCT, the series $\sum \frac{2n+1}{n^2+1}$ is divergent.

Let $a_n = \frac{2n+1}{n^2+1}$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Then $n \geq 1$.

Thus,

$$\begin{aligned} n \geq 1 \quad &\Rightarrow \quad n+2 \geq 3 \\ &\Rightarrow \quad n(n+2) \geq 3 > \frac{1}{2} \\ &\Rightarrow \quad n(n+2) > \frac{1}{2} \\ &\Rightarrow \quad 2n(n+2) > 1 \\ &\Rightarrow \quad 2n^2 + 4n > 1 \\ &\Rightarrow \quad 2n^2 + 4n - 1 > 0. \end{aligned}$$

Hence, $2n^2 + 4n - 1 > 0$.

Since $n > 0$, then $n^2+1 > 0$ and $[(n+1)^2+1] > 0$, so $(n^2+1)[(n+1)^2+1] > 0$.

Observe that

$$\begin{aligned} a_n - a_{n+1} &= \frac{2n+1}{n^2+1} - \frac{2(n+1)+1}{(n+1)^2+1} \\ &= \frac{(2n+1)[(n+1)^2+1] - (n^2+1)[2(n+1)+1]}{(n^2+1)[(n+1)^2+1]} \\ &= \frac{(2n+1)(n^2+2n+2) - (n^2+1)(2n+3)}{(n^2+1)[(n+1)^2+1]} \\ &= \frac{(2n^3+5n^2+6n+2) - (2n^3+3n^2+2n+3)}{(n^2+1)[(n+1)^2+1]} \\ &= \frac{2n^2+4n-1}{(n^2+1)[(n+1)^2+1]} \\ &> 0. \end{aligned}$$

Hence, $a_n - a_{n+1} > 0$, so $a_n > a_{n+1}$.

Thus, the sequence (a_n) is strictly decreasing, so (a_n) is monotonic decreasing.

Since $\frac{2n+1}{n^2+1} > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{2n+1}{n^2+1} = 0$, then by AST, the series $\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{n^2+1}$ is convergent.

Since the series $\sum |\frac{(-1)^n(2n+1)}{n^2+1}| = \sum \frac{2n+1}{n^2+1}$ is divergent, then the series $\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{n^2+1}$ is conditionally convergent. $\square$

Exercise 18. Let (a_n) be a sequence of real numbers. Let S be a real number. Then $\lim_{n\to\infty} a_n = S$ iff the series $a_1 + \sum_{n=2}^{\infty}(a_n - a_{n-1})$ converges and has sum S.

Proof. Let (s_n) be the sequence of partial sums of the sequence $(a_n - a_{n-1})$.

Then $s_n = \sum_{k=2}^{n}(a_k - a_{k-1})$ for each $n \geq 2$.

We first prove $s_n = a_n - a_1$ for each natural number $n \geq 2$ by induction on n.

Let $T = \{n \in \mathbb{N} : s_n = a_n - a_1, n \geq 2\}$.

Since $2 = 2$ and $s_2 = \sum_{k=2}^{2}(a_k - a_{k-1}) = a_2 - a_1$, then $2 \in T$.

Suppose $k \in T$.

Then $k \in \mathbb{N}$ and $k \geq 2$ and $s_k = a_k - a_1$.

Thus,

$$\begin{aligned} s_{k+1} &= s_k + (a_{k+1} - a_k) \\ &= (a_k - a_1) + (a_{k+1} - a_k) \\ &= a_{k+1} - a_1. \end{aligned}$$

Since $k \geq 2$ and $s_{k+1} = a_{k+1} - a_1$, then $k + 1 \in T$.

Therefore, by PMI, $s_n = a_n - a_1$ for each natural number $n \geq 2$.

Suppose $\lim_{n\to\infty} a_n = S$.

Then

$$\begin{aligned} S - a_1 &= \lim_{n\to\infty} a_n - \lim_{n\to\infty} a_1 \\ &= \lim_{n\to\infty} (a_n - a_1) \\ &= \lim_{n\to\infty} s_n. \end{aligned}$$

Hence, $\sum_{n=2}^{\infty}(a_n - a_{n-1}) = \lim_{n\to\infty} s_n = S - a_1$.

Therefore, $a_1 + \sum_{n=2}^{\infty}(a_n - a_{n-1}) = S$, as desired.

Conversely, suppose the series $a_1 + \sum_{n=2}^{\infty}(a_n - a_{n-1})$ converges and has sum S.

Then $\sum_{n=2}^{\infty}(a_n - a_{n-1}) = S - a_1$.

Observe that

$$\begin{aligned} S - a_1 &= \lim_{n\to\infty} s_n \\ &= \lim_{n\to\infty} (a_n - a_1) \\ &= \lim_{n\to\infty} a_n - \lim_{n\to\infty} a_1 \\ &= \lim_{n\to\infty} a_n - a_1. \end{aligned}$$

Therefore, $S - a_1 = \lim_{n\to\infty} a_n - a_1$, so $S = \lim_{n\to\infty} a_n$, as desired. ☐

Exercise 19. Let $0 < r < 1$ be a real number.
Then the series $\sum_{n=1}^{\infty} nr^n$ is convergent.

Proof. Let $n \in \mathbb{N}$.
Then $n > 0$.
Since $0 < r < 1$, then $0 < r$ and $r < 1$.
Since $r > 0$ and $n > 0$, then $r^n > 0$, so $nr^n > 0$.
Since $n + 1 > n > 0$, then $\frac{n+1}{n} > 0$ for all $n \in \mathbb{N}$.
Observe that

$$\begin{aligned}
\lim_{n\to\infty} |\frac{(n+1)r^{n+1}}{nr^n}| &= \lim_{n\to\infty} |\frac{n+1}{n} r| \\
&= \lim_{n\to\infty} |\frac{n+1}{n}||r| \\
&= |r| \lim_{n\to\infty} |\frac{n+1}{n}| \\
&= r \lim_{n\to\infty} \frac{n+1}{n} \\
&= r \cdot 1 \\
&= r \\
&< 1.
\end{aligned}$$

Therefore, by the ratio test, the series $\sum_{n=1}^{\infty} nr^n$ is absolutely convergent, so the series $\sum_{n=1}^{\infty} nr^n$ is convergent. ☐