# Series in $\mathbb{R}$ Notes

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# Infinite Series of Real Numbers

## Proposition 1. properties of finite sums

Let  $n \in \mathbb{N}$ . Then 1.  $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ . 2.  $\sum_{k=1}^{n} (\lambda a_k) = \lambda \sum_{k=1}^{n} a_k$  for every  $\lambda \in \mathbb{R}$ .

# Definition 2. sequence of partial sums

Let  $(a_n)$  be a sequence of real numbers. Let  $(s_n)$  be a sequence of real numbers defined by  $s_1 = a_1$  and  $s_{n+1} = s_n + a_{n+1}$  for all  $n \in \mathbb{N}$ .

Then  $(s_n)$  is called the sequence of partial sums of  $(a_n)$ .

# Proposition 3. n<sup>th</sup> term of a sequence of partial sums

Let  $(a_n)$  be a sequence of real numbers. Let  $(s_n)$  be a sequence defined by  $s_1 = a_1$  and  $s_{n+1} = s_n + a_{n+1}$  for all  $n \in \mathbb{N}$ .

The  $n^{th}$  term of the sequence  $(s_n)$  is  $s_n = a_1 + a_2 + \ldots + a_n$ .

Therefore, the  $n^{th}$  term of  $(s_n)$  is the sum of the first n terms of  $(a_n)$ . Hence,  $s_n = a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^n a_k$  for all  $n \in \mathbb{N}$ . Observe that  $s_1 = a_1$   $s_2 = a_1 + a_2$   $s_3 = a_1 + a_2 + a_3$   $s_4 = a_1 + a_2 + a_3 + a_4$ ...  $s_n = a_1 + a_2 + ... + a_n$  $s_{n+1} = a_1 + a_2 + ... + a_n + a_{n+1} = s_n + a_{n+1}$ .

# Definition 4. summable sequence

Let  $(a_n)$  be a sequence of real numbers. Let  $(s_n)$  be the sequence of partial sums of  $(a_n)$ . If  $(s_n)$  is convergent, then we say that  $(a_n)$  is **summable**. We denote  $\lim_{n\to\infty} s_n$  by  $\sum_{n=1}^{\infty} a_n$ .

A series is the sum of the terms of a sequence. Therefore, a series is a sum of an infinite number of terms.

#### **Definition 5.** infinite series

Let  $(a_n)$  be a sequence of real numbers. The expression  $\sum_{n=1}^{\infty} a_n$  is called an infinite series. The number  $a_n$  is called the  $n^{th}$  term of the series.

Therefore,  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$ 

# Definition 6. convergent/divergent series

Let  $(a_n)$  be a sequence of real numbers.

Let  $(s_n)$  be the sequence of partial sums of  $(a_n)$ .

If there exists a real number S such that  $\lim_{n\to\infty} s_n = S$ , then the infinite series  $\sum_{n=1}^{\infty} a_n$  is said to be **convergent**.

We call S the sum of the series and write  $\sum_{n=1}^{\infty} a_n = S$ .

If  $(s_n)$  diverges, the infinite series is said to be **divergent**.

Thus, if the sequence of partial sums of a sequence  $(a_n)$  is convergent, then the series  $\sum a_n$  is convergent.

A series is either convergent or divergent.

Let  $\sum_{n=1}^{\infty} a_n$  be a convergent series of real numbers.

Then there exists a real number S such that  $\sum_{n=1}^{\infty} a_n = S$ . Let  $(s_n)$  be the sequence of partial sums of the sequence  $(a_n)$ . Then  $s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^n a_k$  for all  $n \in \mathbb{N}$  and  $\lim_{n \to \infty} s_n = S$ . Therefore,  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots = S = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \sum_{k=1}^n a_k$ .

# Theorem 7. uniqueness of a sum of a convergent series

The sum of a convergent series of real numbers is unique.

Example 8. telescoping series Show that  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$ 

**Definition 9. harmonic series** The series  $\sum_{n=1}^{\infty} \frac{1}{n}$  is called the **harmonic series**.

#### Example 10. harmonic series is divergent

Let  $(a_n)$  be a sequence defined by  $a_n = \frac{1}{n}$ . Let  $(s_n)$  be the sequence of partial sums defined by  $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4}$  $\dots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$  for all  $n \in \mathbb{N}$ . Then  $(s_n)$  is divergent, so the sequence of terms  $(a_n)$  is not summable. Therefore, the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$  diverges.

**Example 11.** The series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent.

In fact, it can be shown that  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ .

# Theorem 12. $n^{th}$ term test for divergence

If the series  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $\lim_{n\to\infty} a_n = 0$ .

Therefore, if  $\lim_{n\to\infty} a_n \neq 0$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.

**Example 13.** Since  $\lim_{n\to\infty} \frac{1}{n} = 0$  and the series  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent, then the converse is false.

#### Definition 14. geometric series

Let  $r \in \mathbb{R}$ . Let  $(r^n)$  be a geometric sequence. The series  $\sum_{n=1}^{\infty} r^n$  is called a **geometric series**.

**Example 15.** Show that  $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ .

**Proposition 16.** sum of n + 1 terms of a geometric series formula Let  $r \in \mathbb{R}, r \neq 1$ . Then

$$\sum_{k=0}^{n} r^{k} = \frac{r^{n+1} - 1}{r - 1} \text{ for all } n \in \mathbb{Z}^{+}.$$

Proposition 17. sum of a convergent geometric series

Let  $r \in \mathbb{R}$ . Then  $\sum_{n=1}^{\infty} r^n$  is convergent iff |r| < 1. If |r| < 1, then  $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ .

Therefore,  $\sum_{n=1}^{\infty} r^n$  is divergent iff  $|r| \ge 1$ .

#### Theorem 18. algebraic summation rules for convergent series

If  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are convergent series of real numbers, then 1. Scalar Multiple Rule  $\sum_{n=1}^{\infty} (\lambda a_n) = \lambda \sum_{n=1}^{\infty} a_n$  for every  $\lambda \in \mathbb{R}$ . 2. Sum Rule  $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ . 3. Difference Rule  $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$ . If  $\sum_{n=1}^{\infty} a_n = A$  and  $\sum_{n=1}^{\infty} b_n = B$ , then 1. Scalar Multiple Rule  $\sum_{n=1}^{\infty} \lambda a_n = \lambda A$  for every  $\lambda \in \mathbb{R}$ . Therefore a scalar multiple of a convergent series is convergent. 2. Sum Rule  $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$ . Therefore the sum of two convergent series is convergent. 3. Difference Rule  $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$ .

Therefore the difference of two convergent series is convergent.

#### Theorem 19. inequality rule for convergent series

If the series  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are convergent and  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ , then  $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$ .

# Theorem 20. tail of a series determines convergence of a series

Let M be any positive integer.

The series  $\sum_{n=1}^{\infty} a_n$  is convergent iff the series  $\sum_{n=1}^{\infty} a_{M+n}$  is convergent.

Therefore, the convergence of a series is not affected by changing a finite number of its initial terms.

# **Convergence Tests for Series of Real Numbers**

### Proposition 21. Cauchy convergence criterion for series

The infinite series of real numbers  $\sum a_n$  is convergent iff for every  $\epsilon > 0$ there exists  $N \in \mathbb{N}$  such that if n > m > N, then  $|\sum_{k=m+1}^{n} a_k| < \epsilon$ .

# Theorem 22. Boundedness convergence criterion for series of nonnegative terms

If  $(a_n)$  is a sequence of nonnegative terms, then the series  $\sum a_n$  is convergent iff the sequence of partial sums of  $(a_n)$  is bounded.

Let  $(a_n)$  be a sequence of nonnegative terms. Let  $(s_n)$  be the sequence of partial sums of  $(a_n)$ . Then the series  $\sum a_n$  is convergent iff  $(s_n)$  is bounded. Therefore the series  $\sum a_n$  is divergent iff  $(s_n)$  is unbounded.

# Theorem 23. direct comparison test

Let  $(a_n)$  and  $(b_n)$  be sequences such that  $0 \leq a_n \leq b_n$  for all  $n \in \mathbb{N}$ . If  $\sum b_n$  is convergent, then  $\sum a_n$  is convergent.

Let  $(a_n)$  and  $(b_n)$  be sequences such that  $0 \le a_n \le b_n$  for all  $n \in \mathbb{N}$ . If the series  $\sum b_n$  is convergent, then the series  $\sum a_n$  is convergent. Therefore, if the series  $\sum a_n$  is divergent, then the series  $\sum b_n$  is divergent.

Let  $(a_n)$  and  $(b_n)$  be sequences such that  $0 \le a_n \le b_n$  for all  $n \in \mathbb{N}$ .

Then  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ .

Suppose the series  $\sum b_n$  is convergent.

Then, by DCT, the series  $\sum a_n$  is convergent.

Since  $\sum a_n$  is convergent and  $\sum b_n$  is convergent and  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ , then by the inequality rule for convergent series,  $\sum a_n \leq \sum b_n$ .

#### Example 24. applying the direct comparison test

a. The series  $\sum \frac{1}{\sqrt{n}}$  and  $\sum \frac{n+1}{n^2+1}$  are divergent. b. The series  $\sum \frac{1}{n^2+1}$  and  $\sum \frac{1}{n^3}$  are convergent.

#### Theorem 25. limit comparison test

Let  $(a_n)$  and  $(b_n)$  be sequences of real numbers such that  $a_n > 0$  and  $b_n > 0$ for all  $n \in \mathbb{N}$ .

If there exists a positive real number L such that  $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ , then the series  $\sum a_n$  is convergent iff the series  $\sum b_n$  is convergent.

Therefore, the series  $\sum a_n$  is divergent iff the series  $\sum b_n$  is divergent.

#### Example 26. applying the limit comparison test

The series  $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$  is convergent.

### **Lemma 27.** Let $(a_n)$ be a sequence in $\mathbb{R}$ .

If there exists a real number L such that  $\lim_{n\to\infty} a_{2n} = L$  and  $\lim_{n\to\infty} a_{2n-1} = L$ L, then  $\lim_{n\to\infty} a_n = L$ .

#### Theorem 28. alternating series test

Let  $(a_n)$  be a sequence of positive terms in  $\mathbb{R}$ .

If  $(a_n)$  is monotonic decreasing and  $\lim_{n\to\infty} a_n = 0$ , then the series  $\sum (-1)^n a_n$ is convergent.

#### Example 29. alternating harmonic series is convergent

The series  $\sum \frac{(-1)^n}{n}$  is convergent.

**Example 30.** The series  $\sum \frac{(-1)^n}{n^2}$  converges.

# Definition 31. absolute and conditional convergence

A series  $\sum a_n$  is said to be **absolutely convergent** iff the series  $\sum |a_n|$  is convergent.

A series  $\sum a_n$  is said to be **conditionally convergent** iff  $\sum a_n$  is convergent and  $\sum |a_n|$  is divergent.

Let  $\sum a_n$  be a convergent series.

The series  $\sum |a_n|$  is either convergent or divergent.

If  $\sum |a_n|$  is convergent, then  $\sum a_n$  is absolutely convergent. If  $\sum |a_n|$  is divergent, then  $\sum a_n$  is conditionally convergent.

Therefore, a convergent series is either absolutely convergent or conditionally convergent (but not both)

**Example 32.** The series  $\sum \frac{(-1)^n}{n^2}$  is absolutely convergent since the series  $\sum \left|\frac{(-1)^n}{n^2}\right| = \sum \frac{1}{n^2}$  is convergent.

## Theorem 33. absolute convergence implies convergence

Let  $(a_n)$  be a sequence in  $\mathbb{R}$ .

If the series  $\sum |a_n|$  is convergent, then the series  $\sum a_n$  is convergent.

Therefore, if a series  $\sum a_n$  is absolutely convergent, then the series  $\sum a_n$  is convergent.

**Example 34.** The series  $\sum \frac{\sin(n)}{n^2}$  is absolutely convergent, so  $\sum \frac{\sin(n)}{n^2}$  is convergent.

# Example 35. convergence does not imply absolute convergence

The alternating harmonic series  $\sum \frac{(-1)^n}{n}$  is convergent, but the harmonic series  $\sum |\frac{(-1)^n}{n}| = \sum \frac{1}{n}$  is divergent, so the alternating harmonic series  $\sum \frac{(-1)^n}{n}$  is not absolutely convergent.

In fact, the alternating harmonic series  $\sum \frac{(-1)^n}{n}$  is conditionally convergent.

# Theorem 36. ratio test

Let  $(a_n)$  be a sequence of nonzero real numbers. a. If  $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent. b. If  $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent. c. If  $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$ , then the ratio test is inconclusive.