Series in $\mathbb{R}$ Notes

Jason Sass

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Infinite Series of Real Numbers

Proposition 1. ***properties of finite sums***

Let $n \in \mathbb{N}$. Then

1. $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.

2. $\sum_{k=1}^{n}(\lambda a_k) = \lambda \sum_{k=1}^{n} a_k$ *for every* $\lambda \in \mathbb{R}$.

Definition 2. sequence of partial sums

Let (a_n) be a sequence of real numbers.

Let (s_n) be a sequence of real numbers defined by $s_1 = a_1$ and $s_{n+1} = s_n + a_{n+1}$ for all $n \in \mathbb{N}$.

Then (s_n) is called the **sequence of partial sums of** (a_n).

Proposition 3. n^{th} ***term of a sequence of partial sums***

Let (a_n) be a sequence of real numbers.

Let (s_n) be a sequence defined by $s_1 = a_1$ and $s_{n+1} = s_n + a_{n+1}$ for all $n \in \mathbb{N}$.

The n^{th} term of the sequence (s_n) is $s_n = a_1 + a_2 + ... + a_n$.

Therefore, the n^{th} term of (s_n) is the sum of the first n terms of (a_n).

Hence, $s_n = a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n} a_k$ for all $n \in \mathbb{N}$.

Observe that

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$s_4 = a_1 + a_2 + a_3 + a_4$

...

$s_n = a_1 + a_2 + ... + a_n$

$s_{n+1} = a_1 + a_2 + ... + a_n + a_{n+1} = s_n + a_{n+1}$.

Definition 4. summable sequence

Let (a_n) be a sequence of real numbers.

Let (s_n) be the sequence of partial sums of (a_n).

If (s_n) is convergent, then we say that (a_n) is **summable**.

We denote $\lim_{n\to\infty} s_n$ by $\sum_{n=1}^{\infty} a_n$.

A series is the sum of the terms of a sequence.
Therefore, a series is a sum of an infinite number of terms.

Definition 5. infinite series
Let (a_n) be a sequence of real numbers.
The expression $\sum_{n=1}^{\infty} a_n$ is called an **infinite series**.
The number a_n is called the n^{th} **term of the series**.

Therefore, $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$

Definition 6. convergent/divergent series
Let (a_n) be a sequence of real numbers.
Let (s_n) be the sequence of partial sums of (a_n).
If there exists a real number S such that $\lim_{n\to\infty} s_n = S$, then the infinite series $\sum_{n=1}^{\infty} a_n$ is said to be **convergent**.
We call S the **sum of the series** and write $\sum_{n=1}^{\infty} a_n = S$.
If (s_n) diverges, the infinite series is said to be **divergent**.

Thus, if the sequence of partial sums of a sequence (a_n) is convergent, then the series $\sum a_n$ is convergent.
A series is either convergent or divergent.

Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of real numbers.
Then there exists a real number S such that $\sum_{n=1}^{\infty} a_n = S$.
Let (s_n) be the sequence of partial sums of the sequence (a_n).
Then $s_n = a_1+a_2+a_3+...+a_n = \sum_{k=1}^{n} a_k$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = S$.
Therefore, $\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+... = S = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$.

Theorem 7. ***uniqueness of a sum of a convergent series***
The sum of a convergent series of real numbers is unique.

Example 8. telescoping series
Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Definition 9. harmonic series
The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the **harmonic series**.

Example 10. harmonic series is divergent
Let (a_n) be a sequence defined by $a_n = \frac{1}{n}$.
Let (s_n) be the sequence of partial sums defined by $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$ for all $n \in \mathbb{N}$.
Then (s_n) is divergent, so the sequence of terms (a_n) is not summable.
Therefore, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$ diverges.

Example 11. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

In fact, it can be shown that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Theorem 12. n^{th} ***term test for divergence***
If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Therefore, if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 13. Since $\lim_{n\to\infty} \frac{1}{n} = 0$ and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, then the converse is false.

Definition 14. geometric series
Let $r \in \mathbb{R}$.
Let (r^n) be a geometric sequence.
The series $\sum_{n=1}^{\infty} r^n$ is called a **geometric series**.

Example 15. Show that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Proposition 16. ***sum of n terms of a geometric series formula***
Let $r \in \mathbb{R}, r \neq 1$.
Then

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1} \text{ for all } n \in \mathbb{Z}^+$$

Proposition 17. ***sum of a convergent geometric series***
Let $r \in \mathbb{R}$.
Then $\sum_{n=1}^{\infty} r^n$ is convergent iff $|r| < 1$.
If $|r| < 1$, then $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$.

Therefore, $\sum_{n=1}^{\infty} r^n$ is divergent iff $|r| \geq 1$.

Theorem 18. ***algebraic summation rules for convergent series***
If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of real numbers, then
1. Scalar Multiple Rule
$\sum_{n=1}^{\infty} (\lambda a_n) = \lambda \sum_{n=1}^{\infty} a_n$ *for every* $\lambda \in \mathbb{R}$.
2. Sum Rule
$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
3. Difference Rule
$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then
1. Scalar Multiple Rule
$\sum_{n=1}^{\infty} \lambda a_n = \lambda A$ for every $\lambda \in \mathbb{R}$.
Therefore a scalar multiple of a convergent series is convergent.
2. Sum Rule
$\sum_{n=1}^{\infty} (a_n + b_n) = A + B$.
Therefore the sum of two convergent series is convergent.
3. Difference Rule
$\sum_{n=1}^{\infty} (a_n - b_n) = A - B$.
Therefore the difference of two convergent series is convergent.

Theorem 19. ***inequality rule for convergent series***
If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent and $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Theorem 20. ***tail of a series determines convergence of a series***

Let M be any positive integer.

The series $\sum_{n=1}^{\infty} a_n$ is convergent iff the series $\sum_{n=1}^{\infty} a_{M+n}$ is convergent.

Therefore, the convergence of a series is not affected by changing a finite number of its initial terms.

Convergence Tests for Series of Real Numbers

Proposition 21. ***Cauchy convergence criterion for series***

The infinite series of real numbers $\sum a_n$ is convergent iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > m > N$, then $|\sum_{k=m+1}^{n} a_k| < \epsilon$.

Theorem 22. ***Boundedness convergence criterion for series of nonnegative terms***

If (a_n) is a sequence of nonnegative terms, then the series $\sum a_n$ is convergent iff the sequence of partial sums of (a_n) is bounded.

Let (a_n) be a sequence of nonnegative terms.
Let (s_n) be the sequence of partial sums of (a_n).
Then the series $\sum a_n$ is convergent iff (s_n) is bounded.
Therefore the series $\sum a_n$ is divergent iff (s_n) is unbounded.

Theorem 23. ***direct comparison test***

Let (a_n) and (b_n) be sequences such that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.
If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

Let (a_n) and (b_n) be sequences such that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.
If the series $\sum b_n$ is convergent, then the series $\sum a_n$ is convergent.
Therefore, if the series $\sum a_n$ is divergent, then the series $\sum b_n$ is divergent.

Let (a_n) and (b_n) be sequences such that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.
Then $a_n \leq b_n$ for all $n \in \mathbb{N}$.
Suppose the series $\sum b_n$ is convergent.
Then, by DCT, the series $\sum a_n$ is convergent.
Since $\sum a_n$ is convergent and $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n \in \mathbb{N}$, then by the inequality rule for convergent series, $\sum a_n \leq \sum b_n$.

Example 24. applying the direct comparison test

a. The series $\sum \frac{1}{\sqrt{n}}$ and $\sum \frac{n+1}{n^2+1}$ are divergent.

b. The series $\sum \frac{1}{n^2+1}$ and $\sum \frac{1}{n^3}$ are convergent.

Theorem 25. ***limit comparison test***

Let (a_n) and (b_n) be sequences of real numbers such that $a_n > 0$ and $b_n > 0$ for all $n \in \mathbb{N}$.

If there exists a positive real number L such that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, then the series $\sum a_n$ is convergent iff the series $\sum b_n$ is convergent.

Therefore, the series $\sum a_n$ is divergent iff the series $\sum b_n$ is divergent.

Example 26. applying the limit comparison test

The series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ is convergent.

Lemma 27. *Let (a_n) be a sequence in $\mathbb{R}$.*

If there exists a real number L such that $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n-1} = L$, then $\lim_{n\to\infty} a_n = L$.

Theorem 28. ***alternating series test***

Let (a_n) be a sequence of positive terms in $\mathbb{R}$.

If (a_n) is monotonic decreasing and $\lim_{n\to\infty} a_n = 0$, then the series $\sum(-1)^n a_n$ is convergent.

Example 29. alternating harmonic series is convergent

The series $\sum \frac{(-1)^n}{n}$ is convergent.

Example 30. The series $\sum \frac{(-1)^n}{n^2}$ converges.

Definition 31. absolute and conditional convergence

A series $\sum a_n$ is said to be **absolutely convergent** iff the series $\sum |a_n|$ is convergent.

A series $\sum a_n$ is said to be **conditionally convergent** iff $\sum a_n$ is convergent and $\sum |a_n|$ is divergent.

Let $\sum a_n$ be a convergent series.
The series $\sum |a_n|$ is either convergent or divergent.
If $\sum |a_n|$ is convergent, then $\sum a_n$ is absolutely convergent.
If $\sum |a_n|$ is divergent, then $\sum a_n$ is conditionally convergent.
Therefore, a convergent series is either absolutely convergent or conditionally convergent (but not both).

Example 32. The series $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent since the series $\sum |\frac{(-1)^n}{n^2}| = \sum \frac{1}{n^2}$ is convergent.

Theorem 33. ***absolute convergence implies convergence***

Let (a_n) be a sequence in $\mathbb{R}$.

If the series $\sum |a_n|$ is convergent, then the series $\sum a_n$ is convergent.

Therefore, if a series $\sum a_n$ is absolutely convergent, then the series $\sum a_n$ is convergent.

Example 34. The series $\sum \frac{\sin(n)}{n^2}$ is absolutely convergent, so $\sum \frac{\sin(n)}{n^2}$ is convergent.

Example 35. convergence does not imply absolute convergence

The alternating harmonic series $\sum \frac{(-1)^n}{n}$ is convergent, but the the harmonic series $\sum |\frac{(-1)^n}{n}| = \sum \frac{1}{n}$ is divergent, so the alternating harmonic series $\sum \frac{(-1)^n}{n}$ is not absolutely convergent.

In fact, the alternating harmonic series $\sum \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 36. ***ratio test***

Let (a_n) be a sequence of nonzero real numbers.

a. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

b. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

c. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, then the ratio test is inconclusive.