# Functions Theory

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May 10, 2025

## Functions

**Proposition 1.** ***A function value is unique.***
*Let $f$ be a function.*
*Let $a, b \in domf$.*
*If $a = b$, then $f(a) = f(b)$.*

*Proof.* Suppose $a = b$.
Since $a \in domf$ and $f$ is a relation, then $(a, f(a)) \in f$.
Since $b \in domf$ and $f$ is a relation, then $(b, f(b)) \in f$.
Since $b = a$ and $(b, f(b)) \in f$, then $(a, f(b)) \in f$.
Since $f$ is a function and $(a, f(a)) \in f$ and $(a, f(b)) \in f$, then $f(a) = f(b)$. $\square$

**Theorem 2.** ***equality of functions***
*Let $f$ and $g$ be functions.*
*Let $domf$ be the domain of $f$.*
*Let $domg$ be the domain of $g$.*
*Then $f = g$ iff*
*1. $domf = domg$.*
*2. $f(x) = g(x)$ for all $x \in domf \cap domg$.*

*Proof.* We prove if $domf = domg$ and $f(x) = g(x)$ for all $x \in domf \cap domg$, then $f = g$.
Suppose $domf = domg$ and $f(x) = g(x)$ for all $x \in domf \cap domg$.

We first prove $f \subset g$.
Let $x \in domf$.
Then $(x, f(x)) \in f$.
Since $x \in domf$ and $domf = domg$, then $x \in domg$, so $(x, g(x)) \in g$.
Since $x \in domf$ and $x \in domg$, then $x \in domf \cap domg$, so $f(x) = g(x)$.
Hence, $(x, f(x)) \in g$.
Thus, if $(x, f(x)) \in f$, then $(x, f(x)) \in g$, so $f \subset g$.

We prove $g \subset f$.
Let $y \in domg$.
Then $(y, g(y)) \in g$.
Since $y \in domg$ and $domg = domf$, then $y \in domf$, so $(y, f(y)) \in f$.
Since $y \in domf$ and $y \in domg$, then $y \in domf \cap domg$, so $f(y) = g(y)$.
Hence, $(y, g(y)) \in f$.
Thus, if $(y, g(y)) \in g$, then $(y, g(y)) \in f$, so $g \subset f$.

Since $f \subset g$ and $g \subset f$, then $f = g$, as desired. ☐

*Proof.* Conversely, we prove if $f = g$, then $domf = domg$ and $f(x) = g(x)$ for all $x \in domf \cap domg$.
Suppose $f = g$.

We first prove $domf = domg$.
Let $x \in domf$.
Then $(x, f(x)) \in f$.
Since $f = g$, then $(x, f(x)) \in g$, so $x \in domg$.
Thus, $domf \subset domg$.

Let $y \in domg$.
Then $(y, g(y)) \in g$.
Since $g = f$, then $(y, g(y)) \in f$, so $y \in domf$.
Thus, $domg \subset domf$.

Since $domf \subset domg$ and $domg \subset domf$, then $domf = domg$, as desired.

We next prove $f(x) = g(x)$ for all $x \in domf \cap domg$.
Let $x \in domf \cap domg$.
Then $x \in domf$ and $x \in domg$, so $(x, f(x)) \in f$ and $(x, g(x)) \in g$.
Since $(x, g(x)) \in g$ and $g = f$, then $(x, g(x)) \in f$.
Since $f$ is a function and $(x, f(x)) \in f$ and $(x, g(x)) \in f$, then $f(x) = g(x)$, as desired. ☐

**Theorem 3.** ***equality of maps***
*The maps $f : A \to B$ and $g : C \to D$ are equal iff*
*1.* $A = C$.
*2.* $B = D$.
*3.* $f(x) = g(x)$ *for all* $x \in A$.

*Proof.* Let $f : A \to B$ and $g : C \to D$ be maps.
Suppose $f : A \to B$ and $g : C \to D$ are equal.
Then $f = g$ and $B = D$, by definition of equal maps.
Since $f = g$, then $domf = domg$, and $f(x) = g(x)$ for all $x \in domf \cap domg$, by theorem 2.
Since $A = domf$ and $domf = domg$ and $domg = C$, then $A = C$.

Observe that $domf \cap domg = domf \cap domf = domf = A$.

Since $domf \cap domg = A$, and $f(x) = g(x)$ for all $x \in domf \cap domg$, then $f(x) = g(x)$ for all $x \in A$.

Therefore, $A = C$, and $B = D$, and $f(x) = g(x)$ for all $x \in A$, as desired. $\square$

*Proof.* Conversely, suppose $A = C$, and $B = D$, and $f(x) = g(x)$ for all $x \in A$.

Since $f : A \to B$ is a map, then $f$ is a function, and $domf = A$.

Since $g : C \to D$ is a map, then $g$ is a function, and $domg = C$.

Since $domf = A$ and $A = C$ and $C = domg$, then $domf = domg$.

Let $x \in domf \cap domg$.

Then $x \in A \cap C$, so $x \in A$ and $x \in C$.

Thus, $x \in A$, so $f(x) = g(x)$.

Hence $f(x) = g(x)$ for all $x \in domf \cap domg$.

Since $f$ and $g$ are functions and $domf = domg$ and $f(x) = g(x)$ for all $x \in domf \cap domg$, then $f = g$, by theorem 2.

Since $f = g$ and $B = D$, then the maps $f : A \to B$ and $g : C \to D$ are equal, as desired. $\square$

**Proposition 4.** ***The restriction of a map is a map.***

*Let $f : A \to B$ be a map.*

*Let $S \subset A$.*

*Let $f|_S$ be the restriction of $f$ to $S$.*

*Then $f|_S : S \to B$ is a map.*

*Proof.* We prove $f|_S$ is a function.

Observe that $f|_S : S \to B$ is a relation from $S$ to $B$.

Let $(a, b) \in f|_S$ and $(a, b') \in f|_S$.

Then $a \in S$ and $f|_S(a) = b$ and $f|_S(a) = b'$.

Since $a \in S$, then $f|_S(a) = f(a)$, so $f(a) = f|_S(a)$.

Since $f(a) = f|_S(a)$ and $f|_S(a) = b$, then $f(a) = b$, so $(a, b) \in f$.

Since $f(a) = f|_S(a)$ and $f|_S(a) = b'$, then $f(a) = b'$, so $(a, b') \in f$.

Since $f$ is a function and $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.

Therefore, $(a, b) \in f|_S$ and $(a, b') \in f|_S$ implies $b = b'$, so $f|_S$ is a function.

We prove $domf|_S = S$.

Since $f|_S : S \to B$ is a relation, then $domf|_S \subset S$.

Let $x \in S$.

Since $S \subset A$, then $x \in A$.

Since $f : A \to B$ is a map, then there exists $y \in B$ such that $f(x) = y$.

Since $x \in S$, then $f|_S(x) = f(x)$.

Since $f|_S(x) = f(x)$ and $f(x) = y$, then $f|_S(x) = y$, so $(x, y) \in f|_S$.

Since $x \in S$ and there exists $y \in B$ such that $(x, y) \in f|_S$, then $x \in domf|_S$.

Hence, $x \in S$ implies $x \in domf|_S$, so $S \subset domf|_S$.

Since $domf|_S \subset S$ and $S \subset domf|_S$, then $domf|_S = S$.

Since $f|_S : S \to B$ is a relation, then $rngf|_S \subset B$.

Since $f|_S$ is a function, and $domf|_S = S$, and $rngf|_S \subset B$, then $f|_S : S \to B$ is a map. $\square$

**Theorem 5.** ***Composition of functions is a function.***

*Let $f$ and $g$ be functions. Then*
*1. $g \circ f$ is a function.*
*2. dom $g \circ f = \{x \in domf : f(x) \in domg\}$.*
*3. $(g \circ f)(x) = g(f(x))$ for all $x \in$ dom $g \circ f$.*

*Proof.* We prove 1.
Since $f$ and $g$ are relations, then $g \circ f = \{(a, b) : (\exists c)((a, c) \in f \wedge (c, b) \in g\}$, so $g \circ f$ is a relation.
Let $(a, b) \in g \circ f$ and $(a, b') \in g \circ f$.
Since $(a, b) \in g \circ f$, then there exists $c$ such that $(a, c) \in f$ and $(c, b) \in g$.
Since $(a, b') \in g \circ f$, then there exists $d$ such that $(a, d) \in f$ and $(d, b') \in g$.
Since $f$ is a function and $(a, c) \in f$ and $(a, d) \in f$, then $c = d$.
Since $(d, b') \in g$, then $(c, b') \in g$.
Since $g$ is a function and $(c, b) \in g$ and $(c, b') \in g$, then $b = b'$.
Therefore, $g \circ f$ is a function. $\square$

*Proof.* We prove 2.
Observe that *dom* g $\circ f = \{a : (\exists b)((a, b) \in g \circ f\}$.
Let $S = \{x \in domf : f(x) \in domg\}$.
We must prove *dom* g $\circ f = S$.

Suppose $x \in$ *dom* g $\circ f$.
Then there exists $y$ such that $(x, y) \in g \circ f$.
Thus, there exists $z$ such that $(x, z) \in f$ and $(z, y) \in g$.
Since $(x, z) \in f$ and $f$ is a function, then $x \in domf$ and $f(x) = z$.
Since $(z, y) \in g$, then $z \in domg$, so $f(x) \in domg$.
Since $x \in domf$ and $f(x) \in domg$, then $x \in S$, so *dom* g $\circ f \subset S$.

Suppose $x \in S$.
Then $x \in domf$ and $f(x) \in domg$.
Let $z = f(x)$.
Since $x \in domf$, then $(x, f(x)) \in f$, so $(x, z) \in f$.
Since $f(x) \in domg$ and $f(x) = z$, then $z \in domg$, so there exists $y$ such that $(z, y) \in g$.
Since $(x, z) \in f$ and $(z, y) \in g$, then $(x, y) \in g \circ f$.
Thus, there exists $y$ such that $(x, y) \in g \circ f$, so $x \in$ *dom* g $\circ f$.
Therefore, $S \subset$ *dom* g $\circ f$.

Since $dom$ $\mathrm{g} \circ f \subset S$ and $S \subset dom$ $\mathrm{g} \circ f$, then $dom$ $\mathrm{g} \circ f = S$, as desired. $\square$

*Proof.* We prove 3.
Let $x \in dom$ $\mathrm{g} \circ f$.
Since $g \circ f$ is a function, then $(g \circ f)(x)$ exists.
Let $z = (g \circ f)(x)$.
Then $(x, z) \in g \circ f$, so there exists $y$ such that $(x, y) \in f$ and $(y, z) \in g$.
Since $f$ and $g$ are functions, then $f(x) = y$ and $g(y) = z$.
Thus, $(g \circ f)(x) = z = g(y) = g(f(x))$, as desired. $\square$

**Theorem 6.** ***Function composition is associative.***
*Let $f$, $g$, and $h$ be functions.*
*Then $(f \circ g) \circ h = f \circ (g \circ h)$.*

*Proof.* Since $f$ and $g$ are functions, then $f \circ g$ is a function.
Since $h$ is a function, then $(f \circ g) \circ h$ is a function.
Since $g$ and $h$ are functions, then $g \circ h$ is a function.
Since $f$ is a function, then $f \circ (g \circ h)$ is a function.

We first prove $dom(f \circ g) \circ h = domf \circ (g \circ h)$.

Let $x \in dom(f \circ g) \circ h$.
Then $x \in domh$ and $h(x) \in domf \circ g$.
Since $h(x) \in domf \circ g$, then $h(x) \in domg$ and $g(h(x)) \in domf$.
Since $x \in domh$ and $h(x) \in domg$, then $x \in domg \circ h$.
Since $g(h(x)) \in domf$, then $(g \circ h)(x) \in domf$.
Since $x \in domg \circ h$ and $(g \circ h)(x) \in domf$, then $x \in domf \circ (g \circ h)$.
Thus, $dom(f \circ g) \circ h \subset domf \circ (g \circ h)$.

Let $y \in domf \circ (g \circ h)$.
Then $y \in domg \circ h$ and $(g \circ h)(y) \in domf$.
Since $y \in domg \circ h$, then $y \in domh$ and $h(y) \in domg$.
Since $(g \circ h)(y) \in domf$, then $g(h(y)) \in domf$.
Since $h(y) \in domg$ and $g(h(y)) \in domf$, then $h(y) \in domf \circ g$.
Since $y \in domh$ and $h(y) \in domf \circ g$, then $y \in dom(f \circ g) \circ h$.
Thus, $domf \circ (g \circ h) \subset dom(f \circ g) \circ h$.

Since $dom(f \circ g) \circ h \subset domf \circ (g \circ h)$ and $domf \circ (g \circ h) \subset dom(f \circ g) \circ h$, then $dom(f \circ g) \circ h = domf \circ (g \circ h)$.

Let $x \in dom(f \circ g) \circ h \cap dom f \circ (g \circ h)$.
Then $x \in dom(f \circ g) \circ h \cap dom(f \circ g) \circ h = dom(f \circ g) \circ h$ and

$$\begin{aligned}
[(f \circ g) \circ h](x) &= (f \circ g)(h(x)) \\
&= f[g(h(x))] \\
&= f[(g \circ h)(x)] \\
&= [f \circ (g \circ h)](x).
\end{aligned}$$

Therefore, $[(f \circ g) \circ h](x) = [f \circ (g \circ h)](x)$ for all $x$ in the common domain.

Since $dom(f \circ g) \circ h = dom f \circ (g \circ h)$ and $[(f \circ g) \circ h](x) = [f \circ (g \circ h)](x)$ for all $x$ in the common domain, then $(f \circ g) \circ h = f \circ (g \circ h)$, as desired. □

**Proposition 7.** ***Composition of maps***
*Let $f : A \to B$ and $g : B \to C$ be maps.*
*Then $g \circ f : A \to C$ is a map and $(g \circ f)(x) = g(f(x))$ for all $x \in A$.*

*Proof.* Since $f : A \to B$ and $g : B \to C$ are maps, then $f$ and $g$ are functions, so $g \circ f$ is a function and $dom$ g $\circ f = \{x \in dom f : f(x) \in dom g\}$ and $(g \circ f)(x) = g(f(x))$ for all $x \in dom$ g $\circ f$.
Since $dom f = A$ and $dom g = B$ and $dom$ g$\circ f = \{x \in dom f : f(x) \in dom g\}$, then $dom$ g $\circ f = \{x \in A : f(x) \in B\}$, so $dom$ g $\circ f \subset A$.

Let $x \in A$.
Since $f : A \to B$ is a map, then $f(x) \in B$.
Since $x \in A$ and $f(x) \in B$, then $x \in dom$ g $\circ f$.
Hence, $A \subset dom$ g $\circ f$.

Since $dom$ g $\circ f \subset A$ and $A \subset dom$ g $\circ f$, then $dom$ g $\circ f = A$.

Since $(g \circ f)(x) = g(f(x))$ for all $x \in dom$ g $\circ f$, then $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

We prove $rng$ g $\circ f \subset C$.
Let $y \in rng$ g $\circ f$.
Then there exists $x$ such that $(x, y) \in g \circ f$.
Since $(x, y) \in g \circ f$, then $x \in dom$ g $\circ f$, so $x \in A$.
Since $g \circ f$ is a function and $(x, y) \in g \circ f$, then $(g \circ f)(x) = y$, so $y = (g \circ f)(x) = g(f(x))$.
Since $f : A \to B$ is a map and $x \in A$, then $f(x) \in B$.
Since $g : B \to C$ is a map, then $g(f(x)) \in C$.
Thus, $y \in C$, so $rng$ g $\circ f \subset C$.
Since $g \circ f$ is a function and $dom$ g $\circ f = A$ and $rng$ g $\circ f \subset C$, then $g \circ f : A \to C$ is a map. □

**Proposition 8.** *Let $f : A \to B$ be a map.*

*Let $I_A$ be the identity map on $A$ and $I_B$ be the identity map on $B$.*

*Then $f \circ I_A = I_B \circ f = f$.*

*Proof.* We prove $f \circ I_A = f$.

Since $I_A : A \to A$ is a map and $f : A \to B$ is a map, then $f \circ I_A : A \to B$ is a map and $(f \circ I_A)(x) = f(I_A(x))$ for all $x \in A$.

Since the domain of $f \circ I_A$ and $f$ is $A$, then $f \circ I_A$ and $f$ have the same domain.

Since the codomain of $f \circ I_A$ and $f$ is $B$, then $f \circ I_A$ and $f$ have the same codomain.

Let $x \in A$.

Then $(f \circ I_A)(x) = f(I_A(x)) = f(x)$, so $(f \circ I_A)(x) = f(x)$ for all $x \in A$.

Therefore, $f \circ I_A = f$. $\square$

*Proof.* We prove $I_B \circ f = f$.

Since $f : A \to B$ is a map and $I_B : B \to B$ is a map, then $I_B \circ f : A \to B$ is a map and $(I_B \circ f)(x) = I_B(f(x))$ for all $x \in A$.

Since the domain of $I_B \circ f$ and $f$ is $A$, then $I_B \circ f$ and $f$ have the same domain.

Since the codomain of $I_B \circ f$ and $f$ is $B$, then $I_B \circ f$ and $f$ have the same codomain.

Let $x \in A$.

Then $(I_B \circ f)(x) = I_B(f(x)) = f(x)$, so $(I_B \circ f)(x) = f(x)$ for all $x \in A$.

Therefore, $I_B \circ f = f$.

Since $f \circ I_A = f$ and $I_B \circ f = f$, then $f \circ I_A = f = I_B \circ f$, as desired. $\square$

**Theorem 9.** ***Left cancellation property of injective maps***

*Let $f : X \to Y$ be a map.*

*Then $f$ is injective iff for every set $W$ and every map $g : W \to X$ and $h : W \to X$ such that $f \circ g = f \circ h$ we have $g = h$.*

*Proof.* We prove if $f$ is injective, then for every set $W$ and every map $g : W \to X$ and $h : W \to X$ such that $f \circ g = f \circ h$ we have $g = h$.

Suppose $f$ is injective.

Let $W$ be a set and let $g : W \to X$ and $h : W \to X$ be maps such that $f \circ g = f \circ h$.

We must prove $g = h$.

Since $g : W \to X$ is a map and $h : W \to X$ is a map, then $domg = W = domh$.

Let $x \in W$.

Since $f \circ g = f \circ h$, then $(f \circ g)(x) = (f \circ h)(x)$, so $f(g(x)) = f(h(x))$.

Since $f$ is injective, then $g(x) = h(x)$.

Thus, $g(x) = h(x)$ for all $x \in W$, so $g = h$, as desired. $\square$

*Proof.* Conversely, we prove if for every set $W$ and every map $g : W \to X$ and $h : W \to X$ such that $f \circ g = f \circ h$ implies $g = h$, then $f$ is injective.

We prove by contrapositive.

Suppose $f$ is not injective.

We must prove there exists a set $W$ and there exist maps $g : W \to X$ and $h : W \to X$ such that $f \circ g = f \circ h$ and $g \neq h$.

Since $f$ is not injective, then there exist $a, b \in X$ such that $a \neq b$ and $f(a) = f(b)$.

Let $W = \{a, b\}$.

Let $g = \{(a, a), (b, a)\}$.

Then $g$ is a function and $domg = \{a, b\} = W$ and $rngg = \{a\} \subset X$.

Thus, $g : W \to X$ is a map and $g(a) = a = g(b)$.

Let $h = \{(a, b), (b, b)\}$.

Then $h$ is a function and $domh = \{a, b\} = W$ and $rngh = \{b\} \subset X$.

Thus, $h : W \to X$ is a map and $h(a) = b = h(b)$.

Since $(a, a) \in h$ iff $a = b$ and $a \neq b$, then $(a, a) \notin h$.

Since $(a, a) \in g$, but $(a, a) \notin h$, then $g \neq h$.

Since $g : W \to X$ is a map and $f : X \to Y$ is a map, then $f \circ g : W \to Y$ is a map and $(f \circ g)(x) = f(g(x))$ for all $x \in W$.

Since $h : W \to X$ is a map and $f : X \to Y$ is a map, then $f \circ h : W \to Y$ is a map and $(f \circ h)(x) = f(h(x))$ for all $x \in W$.

Observe that $dom(f \circ g) = W = dom(f \circ h)$.

Observe that $(f \circ g)(a) = f(g(a)) = f(a) = f(b) = f(h(a)) = (f \circ h)(a)$.

Observe that $(f \circ g)(b) = f(g(b)) = f(a) = f(b) = f(h(b)) = (f \circ h)(b)$.

Since $dom(f \circ g) = W = dom(f \circ h)$ and $(f \circ g)(a) = (f \circ h)(a)$ and $(f \circ g)(b) = (f \circ h)(b)$, then $f \circ g = f \circ h$. ∎

**Proposition 10.** *A map $f : A \to B$ is surjective iff $(\forall b \in B)(\exists a \in A)(f(a) = b)$.*

*Proof.* Let $f : A \to B$ be a map.

We first prove if $f$ is surjective, then $(\forall b \in B)(\exists a \in A)(f(a) = b)$.

Suppose $f$ is surjective.

Let $b \in B$.

Since $f$ is surjective, then $rngf = B$.

Since $b \in B$, then $b \in rngf$, so there exists $a \in A$ such that $f(a) = b$.

Conversely, we prove if $(\forall b \in B)(\exists a \in A)(f(a) = b)$, then $f$ is surjective.

Suppose $(\forall b \in B)(\exists a \in A)(f(a) = b)$.

Since $f : A \to B$ is a map, then $rngf \subset B$.

We prove $B \subset rngf$.

Suppose $b \in B$.

Then there exists $a \in A$ such that $f(a) = b$.

Hence, $b \in rngf$, so $B \subset rngf$.

Since $rngf \subset B$ and $B \subset rngf$, then $rngf = B$, so $f$ is surjective. ∎

**Theorem 11.** ***Right cancellation property of surjective maps***

*Let $X$ be a nonempty set.*

*Let $f : X \to Y$ be a map.*

*Then $f$ is surjective iff for every set $Z$ and every map $g : Y \to Z$ and $h : Y \to Z$ such that $g \circ f = h \circ f$ we have $g = h$.*

*Proof.* We prove if $f$ is surjective, then for every set $Z$ and every map $g : Y \to Z$ and $h : Y \to Z$ such that $g \circ f = h \circ f$ we have $g = h$.

Suppose $f$ is surjective.

Let $Z$ be a set and let $g : Y \to Z$ and $h : Y \to Z$ be maps such that $g \circ f = h \circ f$.

We must prove $g = h$.

Since $g : Y \to Z$ is a map and $h : Y \to Z$ is a map, then $g$ and $h$ are functions and $domg = Y = domh$ and the codomain of $g$ is $Z$ which is the codomain of $h$.

Since $X \neq \emptyset$ and $f : X \to Y$ is a map, then there exists $x \in X$, so $f(x) \in Y$.

Hence, $Y \neq \emptyset$.

Let $y \in Y$.

Since $f$ is surjective, then there exists $x \in X$ such that $f(x) = y$.

Since $g \circ f = h \circ f$ and $x \in X$, then $(g \circ f)(x) = (h \circ f)(x)$.

Observe that

$$\begin{aligned} g(y) &= g(f(x)) \\ &= (g \circ f)(x) \\ &= (h \circ f)(x) \\ &= h(f(x)) \\ &= h(y). \end{aligned}$$

Therefore, $g(y) = h(y)$ for all $y \in Y$, so $g = h$, as desired. □

*Proof.* Conversely, we prove if for every set $Z$ and every map $g : Y \to Z$ and $h : Y \to Z$ such that $g \circ f = h \circ f$ implies $g = h$, then $f$ is surjective.

We prove by contrapositive.

Suppose $f$ is not surjective.

We must prove there exists a set $Z$ and there exist maps $g : Y \to Z$ and $h : Y \to Z$ such that $g \circ f = h \circ f$ and $g \neq h$.

Since $f$ is not surjective, then there exists $y_0 \in Y$ such that for all $x \in X$, $f(x) \neq y_0$.

Since $X \neq \emptyset$, then there exists $x_0 \in X$.

Since $f : X \to Y$ is a map, then $f(x_0) \in Y$.

Let $Z = Y$.

Let $g : Y \to Z$ be the identity map on $Y$ defined by $g(y) = y$.

Let $h : Y \to Z$ be a map defined by $h(y) = y$ if $y \neq y_0$ and $h(y_0) = f(x_0)$.

We prove $g \neq h$.

Since $x_0 \in X$, then $f(x_0) \neq y_0$.

Since $g(y_0) = y_0 \neq f(x_0) = h(y_0)$, then $g(y_0) \neq h(y_0)$, so $g \neq h$.

We prove $g \circ f = h \circ f$.

Since $f : X \to Y$ is a map and $g : Y \to Z$ is a map, then $g \circ f : X \to Z$ is a map and $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

Since $f : X \to Y$ is a map and $h : Y \to Z$ is a map, then $h \circ f : X \to Z$ is a map and $(h \circ f)(x) = h(f(x))$ for all $x \in X$.

Observe that $dom(g \circ f) = X = dom(h \circ f)$.

Let $x \in X$.

Since $f : X \to Y$ is a map, then $f(x) \in Y$.

Since $x \in X$, then $f(x) \neq y_0$.

Observe that

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= f(x) \\ &= h(f(x)) \\ &= (h \circ f)(x). \end{aligned}$$

Hence, $(g \circ f)(x) = (h \circ f)(x)$ for all $x \in X$.

Therefore, $g \circ f = h \circ f$, as desired. □

**Proposition 12.** ***identity map is bijective.***

*Let $S$ be a set.*

*The identity map $I_S : S \to S$ on $S$ is a bijection.*

*Proof.* Let $I_S : S \to S$ be the map defined by $I_S(x) = x$ for all $x \in S$.

We prove $I_S$ is injective.

Let $a, b \in S$ such that $I_S(a) = I_S(b)$.

Then $a = b$.

Therefore, $I_S$ is injective.

We prove $I_S$ is surjective.

Let $b \in S$ be arbitrary.

Let $a = b$.

Then $a \in S$ and $I(a) = a = b$.

Thus, there exists $a \in S$ such that $I_S(a) = b$.

Therefore, $I_S$ is surjective.

Since $I_S$ is injective and surjective, then $I_S$ is bijective, as desired. □

**Theorem 13.** *Let $f : A \to B$ and $g : B \to C$ be maps.*

*1. If $f$ and $g$ are injective, then $g \circ f$ is injective.*

*A composition of injections is an injection.*

*2. If $f$ and $g$ are surjective, then $g \circ f$ is surjective.*

*A composition of surjections is a surjection.*

*3. If $g \circ f$ is injective, then $f$ is injective.*

*4. If $g \circ f$ is surjective, then $g$ is surjective.*

*Proof.* We prove 1.

Suppose $f$ and $g$ are injective.

Since $f : A \to B$ is a map and $g : B \to C$ is a map, then $g \circ f : A \to C$ is a map.

Let $a, b \in A$ such that $(g \circ f)(a) = (g \circ f)(b)$.

Then $g(f(a)) = g(f(b))$.

Since $g$ is injective, then $f(a) = f(b)$.

Since $f$ is injective, then $a = b$.

Therefore, $g \circ f$ is injective. $\square$

*Proof.* We prove 2.

Suppose $f$ and $g$ are surjective.

Since $f : A \to B$ is a map and $g : B \to C$ is a map, then $g \circ f : A \to C$ is a map.

Let $c \in C$ be arbitrary.

Since $g$ is surjective, then there exists $b \in B$ such that $g(b) = c$.

Since $f$ is surjective, then there exists $a \in A$ such that $f(a) = b$.

Observe that $(g \circ f)(a) = g(f(a)) = g(b) = c$.

Therefore, there exists $a \in A$ such that $(g \circ f)(a) = c$, so $g \circ f$ is surjective. $\square$

*Proof.* We prove 3.

Suppose $g \circ f$ is injective.

Since $f : A \to B$ is a map and $g : B \to C$ is a map, then $g \circ f : A \to C$ is a map.

Let $a, b \in A$ such that $f(a) = f(b)$.

Then $(g \circ f)(a) = g(f(a)) = g(f(b)) = (g \circ f)(b)$, so $(g \circ f)(a) = (g \circ f)(b)$.

Since $g \circ f$ is injective, then $a = b$.

Therefore, $f$ is injective. $\square$

*Proof.* We prove 4.

Suppose $g \circ f$ is surjective.

Since $f : A \to B$ is a map and $g : B \to C$ is a map, then $g \circ f : A \to C$ is a map.

Let $c \in C$ be arbitrary.

Since $g \circ f$ is surjective, then there exists $a \in A$ such that $(g \circ f)(a) = c$.

Since $a \in A$ and $f : A \to B$ is a map, then $f(a) \in B$.

Observe that $g(f(a)) = (g \circ f)(a) = c$.

Thus, there exists $f(a) \in B$ such that $g(f(a)) = c$.

Therefore, $g$ is surjective. $\square$

**Corollary 14.** *Let $f : A \to B$ and $g : B \to C$ be maps.*

*1. If $f$ and $g$ are bijective, then $g \circ f$ is bijective.*

*A composition of bijections is a bijection.*

*2. If $g \circ f$ is bijective, then $f$ is injective and $g$ is surjective.*

*Proof.* We prove 1.

Suppose $f$ and $g$ are bijective.

Since $f : A \to B$ is a map and $g : B \to C$ is a map, then $g \circ f : A \to C$ is a map.

Since $f$ is bijective, then $f$ is injective and surjective.

Since $g$ is bijective, then $g$ is injective and surjective.

Since $f$ and $g$ are injective, then $g \circ f$ is injective.

Since $f$ and $g$ are surjective, then $g \circ f$ is surjective.

Since $g \circ f$ is injective and surjective, then $g \circ f$ is bijective. □

*Proof.* We prove 2.

Suppose $g \circ f$ is bijective.

Then $g \circ f$ is injective and surjective.

Since $g \circ f$ is injective, then $f$ is injective.

Since $g \circ f$ is surjective, then $g$ is surjective. □

**Theorem 15.** ***existence of inverse function***

*Let $f$ be a function.*

*Then the inverse relation $f^{-1}$ is a function iff $f$ is injective.*

*Proof.* We prove if $f^{-1}$ is a function, then $f$ is injective.

Suppose $f^{-1}$ is a function.

Let $a_1, a_2 \in dom f$ such that $f(a_1) = f(a_2)$.

Since $f$ is a relation, then $(a_1, f(a_1)) \in f$ and $(a_2, f(a_2)) \in f$.

Since $f^{-1}$ is an inverse of $f$, then $(f(a_1), a_1) \in f^{-1}$ and $(f(a_2), a_2) \in f^{-1}$.

Since $f^{-1}$ is a function and $f(a_1) = f(a_2)$, then $a_1 = a_2$.

Therefore, $f$ is injective.

Conversely, we prove if $f$ is injective, then $f^{-1}$ is a function.

Suppose $f$ is injective.

Let $(a, b_1) \in f^{-1}$ and $(a, b_2) \in f^{-1}$.

Since $f^{-1}$ is an inverse of $f$, then $(b_1, a) \in f$ and $(b_2, a) \in f$, so $f(b_1) = a$ and $f(b_2) = a$.

Thus, $f(b_1) = a = f(b_2)$.

Since $f$ is injective, then $b_1 = b_2$.

Therefore, $f^{-1}$ is a function. □

**Theorem 16.** ***The inverse of an invertible map is unique.***

*Let $f : A \to B$ be an invertible map.*

*Then the inverse map is unique.*

*Proof.* Since $f : A \to B$ is an invertible map, then there exists a map that is an inverse of $f$.

Let $g : B \to A$ and $h : B \to A$ be inverse maps of $f$.

To prove the inverse map is unique, we must prove $g = h$.

Observe that the domain of $g$ equals $B$ which equals the domain of $h$ and the codomain of $g$ equals $A$ which equals the codomain of $h$.

Let $x \in B$ be arbitrary.
Since $g : B \to A$ is a map, then $g(x) \in A$.
Since $f$ is a relation, then $(g(x), x) \in f$.
Since $h$ and $f$ are inverses, then $(x, g(x)) \in h$, so $h(x) = g(x)$.
Therefore, $g = h$, as desired. $\square$

**Theorem 17.** *Let $f : A \to B$ and $g : B \to A$ be maps.*
*Then $g$ is an inverse of $f$ iff*
*1. $g \circ f = I_A$*
*2. $f \circ g = I_B$.*

*Proof.* We prove if $g$ is an inverse of $f$, then $g \circ f = I_A$ and $f \circ g = I_B$.

Since $f : A \to B$ and $g : B \to A$ are maps, then $g \circ f : A \to A$ and $f \circ g : B \to B$ are maps and $(g \circ f)(a) = g(f(a))$ for all $a \in A$ and $(f \circ g)(b) = f(g(b))$ for all $b \in B$.

Suppose $g$ is an inverse of $f$.

We prove $g \circ f = I_A$.
Let $I_A$ be the identity map on $A$.
Then $dom(g \circ f) = A = domI_A$.
Let $a \in A$.
Since $f$ is a function, then $(a, f(a)) \in f$.
Since $g$ is an inverse of $f$, then $(f(a), a) \in g$, so $g(f(a)) = a$.
Observe that

$$\begin{aligned}(g \circ f)(a) &= g(f(a)) \\ &= a \\ &= I_A(a).\end{aligned}$$

Hence, $(g \circ f)(a) = I(a)$ for every $a \in A$.
Therefore, $g \circ f = I_A$.

We prove $f \circ g = I_B$.
Let $I_B$ be the identity map on $B$.
Then $dom(f \circ g) = B = domI_B$.
Let $b \in B$.
Since $g$ is a function, then $(b, g(b)) \in g$.
Since $f$ is an inverse of $g$, then $(g(b), b) \in f$, so $f(g(b)) = b$.
Observe that

$$\begin{aligned}(f \circ g)(b) &= f(g(b)) \\ &= b \\ &= I(b).\end{aligned}$$

Hence, $(f \circ g)(b) = I(b)$ for every $b \in B$.
Therefore, $f \circ g = I_B$. $\square$

*Proof.* Conversely, we prove if $g \circ f = I_A$ and $f \circ g = I_B$, then $g$ is an inverse of $f$.

Suppose $g \circ f = I_A$ and $f \circ g = I_B$.

Let $(a,b) \in f$.
Then $a \in A$ and $b \in B$ and $f(a) = b$.
Since $a \in A$, then $a = I_A(a) = (g \circ f)(a) = g(f(a)) = g(b)$, so $(b,a) \in g$.
Hence, if $(a,b) \in f$, then $(b,a) \in g$.

Let $(b,a) \in g$.
Then $b \in B$ and $a \in A$ and $g(b) = a$.
Since $b \in B$, then $b = I_B(b) = (f \circ g)(b) = f(g(b)) = f(a)$, so $(a,b) \in f$.
Hence, if $(b,a) \in g$, then $(a,b) \in f$.
Since $(b,a) \in g$ implies $(a,b) \in f$ and $(a,b) \in f$ implies $(b,a) \in g$, then $(b,a) \in g$ iff $(a,b) \in f$.
Therefore, $g$ is an inverse of $f$. □

**Corollary 18.** *Let $f : A \to B$ be an invertible map. Then*
1. $f^{-1} \circ f = I_A$
2. $f \circ f^{-1} = I_B$.

*Proof.* Since $f : A \to B$ is an invertible map, then the inverse map $f^{-1} : B \to A$ exists, so $f^{-1}$ is an inverse of $f$.
Therefore, $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$. □

**Theorem 19.** ***An invertible map is bijective.***
*Let $f : A \to B$ be a map.*
*Then $f$ is invertible iff $f$ is bijective.*

*Proof.* We prove if $f$ is bijective, then $f$ is invertible.
Suppose $f$ is bijective.
Then $f$ is injective and surjective.
Since $f$ is injective, then the inverse relation $f^{-1}$ is a function.
Since $f^{-1}$ is a relation, then $dom f^{-1} = rng f$ and $rng f^{-1} = dom f$.
Since $f$ is surjective, then $rng f = B$.
Thus, $dom f^{-1} = rng f = B$ and $rng f^{-1} = dom f = A \subset A$.
Since $f^{-1}$ is a function and $dom f^{-1} = B$ and $rng f^{-1} \subset A$, then $f^{-1} : B \to A$ is a map.
Since $f^{-1}$ is the inverse of $f$, then $f$ is invertible. □

*Proof.* Conversely, we prove if $f$ is invertible, then $f$ is bijective.
Suppose $f$ is invertible.
Then the inverse map $f^{-1} : B \to A$ exists.
Hence, the inverse relation $f^{-1}$ is a function, so $f$ is injective.

Let $b \in B$.
Since $f^{-1} : B \to A$ is a map, then $f^{-1}(b) \in A$.
Let $a = f^{-1}(b)$.
Then $a \in A$.
Since $f^{-1}$ is the inverse of $f$ and $f^{-1}(b) = a$, then $f(a) = b$.
Therefore, there exists $a \in A$ such that $f(a) = b$, so $f$ is surjective.
Since $f$ is injective and surjective, then $f$ is bijective. □

**Lemma 20.** *Let $f : A \to B$ be a map.*
*If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.*

*Proof.* Suppose the map $f : A \to B$ is a bijection.
Then $f$ is bijective, so $f$ is invertible.
Hence, the map $f : A \to B$ is invertible, so the inverse map $f^{-1} : B \to A$ exists.

We prove $f^{-1}$ is injective.
Let $b_1, b_2 \in B$ such that $f^{-1}(b_1) = f^{-1}(b_2)$.
Let $a = f^{-1}(b_1) = f^{-1}(b_2)$.
Then $f^{-1}(b_1) = a$ and $f^{-1}(b_2) = a$, so $(b_1, a) \in f^{-1}$ and $(b_2, a) \in f^{-1}$.
Since $f^{-1}$ is the inverse of $f$, then $(a, b_1) \in f$ and $(a, b_2) \in f$.
Since $f$ is a function and $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$.
Therefore, $f^{-1}$ is injective.

We prove $f^{-1}$ is surjective.
Let $a \in A$.
Since $f : A \to B$ is a map, then $f(a) \in B$.
Let $b = f(a)$.
Then $b \in B$.
Since $f^{-1}$ is the inverse of $f$ and $f(a) = b$, then $f^{-1}(b) = a$.
Thus, there exists $b \in B$ such that $f^{-1}(b) = a$, so $f^{-1}$ is surjective.

Since $f^{-1}$ is injective and surjective, then $f^{-1}$ is bijective.
Since $f^{-1} : B \to A$ is a map and $f^{-1}$ is bijective, then $f^{-1} : B \to A$ is a bijection. □

**Theorem 21.** *Let $f : A \to B$ be a bijection. Then*
*1. $(f^{-1})^{-1} : A \to B$ is a bijection.*
*2. $(f^{-1})^{-1} = f$.*

*Proof.* Since $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection, so $(f^{-1})^{-1} : A \to B$ is a bijection.
Observe that $(f^{-1})^{-1} : A \to B$ and $f : A \to B$ have the same domain $A$ and same codomain $B$.
Let $a \in A$ be arbitrary.
Since $f$ is a function, then there is a unique $b \in B$ such that $f(a) = b$.
Since $f^{-1}$ is the inverse of $f$, then $f^{-1}(b) = a$.

Since $(f^{-1})^{-1}$ is the inverse of $f^{-1}$, then $(f^{-1})^{-1}(a) = b$.
Thus, $(f^{-1})^{-1}(a) = b = f(a)$.
Hence, $(f^{-1})^{-1}(a) = f(a)$ for all $a \in A$.
Therefore, $(f^{-1})^{-1} = f$. $\square$

**Theorem 22.** *Let $f : A \to B$ and $g : B \to C$ be bijections. Then*
1. *$(g \circ f)^{-1} : C \to A$ is a bijection.*
2. *$f^{-1} \circ g^{-1} : C \to A$ is a bijection.*
3. *$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

*Proof.* Since $f : A \to B$ and $g : B \to C$ are bijections, then the composition $g \circ f : A \to C$ is a bijection, so $(g \circ f)^{-1} : C \to A$ is a bijection.
Since $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.
Since $g : B \to C$ is a bijection, then $g^{-1} : C \to B$ is a bijection.
Thus, the composition $f^{-1} \circ g^{-1} : C \to A$ is a bijection.
Observe that $(g \circ f)^{-1} : C \to A$ and $f^{-1} \circ g^{-1} : C \to A$ have the same domain $C$ and same codomain $A$.

Let $c \in C$ be arbitrary.
Since $(g \circ f)^{-1}$ is a function, then there exists a unique $a \in A$ such that $(g \circ f)^{-1}(c) = a$.
Since $(g \circ f)^{-1}$ is the inverse of $g \circ f$, then $(g \circ f)(a) = c$.
Since $f$ is a function and $a \in A$, then there exists a unique $b \in B$ such that $f(a) = b$.
Thus, $c = (g \circ f)(a) = g(f(a)) = g(b)$.
Since $g^{-1}$ is the inverse of $g$ and $g(b) = c$, then $g^{-1}(c) = b$.
Since $f^{-1}$ is the inverse of $f$ and $f(a) = b$, then $f^{-1}(b) = a$.
Observe that

$$\begin{aligned} (g \circ f)^{-1}(c) &= a \\ &= f^{-1}(b) \\ &= f^{-1}(g^{-1}(c)) \\ &= (f^{-1} \circ g^{-1})(c). \end{aligned}$$

Thus, $(g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c)$ for all $c \in C$.
Therefore, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. $\square$

## Image and inverse image of functions

**Proposition 23.** *Let $f : A \to B$ be a map.*
1. *Then $f$ is injective iff every $b \in B$ has at most one pre-image.*
2. *Then $f$ is surjective iff every $b \in B$ has at least one pre-image.*
3. *Then $f$ is bijective iff every $b \in B$ has exactly one pre-image.*

*Proof.* We prove 1.

We prove if $f$ is injective, then every $b \in B$ has at most one pre-image.

Suppose $f$ is injective.

Let $b \in B$.

Either there exists $a \in A$ such that $f(a) = b$ or there does not exist $a \in A$ such that $f(a) = b$.

We consider each case separately.

**Case 1:** Suppose there does not exist $a \in A$ such that $f(a) = b$.

Then $b$ has no pre-image.

**Case 2:** Suppose there exists $a \in A$ such that $f(a) = b$.

Then $a$ is a pre-image of $b$, so $b$ has at least one pre-image.

Suppose $a_1, a_2 \in A$ are pre-images of $b$.

Then $f(a_1) = b$ and $f(a_2) = b$, so $f(a_1) = f(a_2)$.

Since $f$ is injective, then $a_1 = a_2$, so there is at most one pre-image of $b$.

In either case, at most one pre-image of $b$ exists. $\square$

*Proof.* Conversely, we prove if every $b \in B$ has at most one pre-image, then $f$ is injective.

Suppose every $b \in B$ has at most one pre-image.

To prove $f$ is injective, let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$.

Let $b = f(a_1) = f(a_2)$.

Since $f : A \to B$ is a map, then $b \in B$.

Hence, $b$ has at most one pre-image, so there is at most one $a \in A$ such that $f(a) = b$.

Therefore, $a_1 = a_2$. $\square$

*Proof.* We prove 2.

We prove if $f$ is surjective, then every $b \in B$ has at least one pre-image.

Suppose $f$ is surjective.

Let $b \in B$ be arbitrary.

Since $f$ is surjective, then there exists $a \in A$ such that $f(a) = b$.

Hence, $a$ is a pre-image of $b$, so $b$ has at least one pre-image. $\square$

*Proof.* We prove 3.

We prove if $f$ is bijective, then every $b \in B$ has exactly one pre-image.

Suppose $f$ is bijective.

Then $f$ is injective and surjective.

Let $b \in B$.

Since $f$ is surjective, then $b$ has at least one pre-image.

Since $f$ is injective, then $b$ has at most one pre-image.

Since $b$ has at least one pre-image and $b$ has at most one pre-image, then $b$ has exactly one pre-image. $\square$

**Proposition 24.** *Let $f : A \to B$ be a map. Then*

*1. $f(\emptyset) = \emptyset$.*

*The image of the empty set is the empty set.*

*2. $f^{-1}(\emptyset) = \emptyset$.*

*The inverse image of the empty set is the empty set.*
*3.* $f(A) = rngf$.
*The image of the domain of* $f$ *is the range of* $f$.
*4.* $f^{-1}(B) = A$.
*The inverse image of the codomain of* $f$ *is the domain of* $f$.

*Proof.* We prove 1.
We prove $f(\emptyset) = \emptyset$ by contradiction.
Suppose $f(\emptyset) \neq \emptyset$.
Then there exists $b \in f(\emptyset)$, so there exists $x \in \emptyset$ such that $f(x) = b$.
Since $\emptyset$ is empty, then $x \notin \emptyset$.
Thus, we have $x \in \emptyset$ and $x \notin \emptyset$, a contradiction.
Therefore, $f(\emptyset) = \emptyset$. ☐

*Proof.* We prove 2.
We prove $f^{-1}(\emptyset) = \emptyset$ by contradiction.
Suppose $f^{-1}(\emptyset) \neq \emptyset$.
Then there exists $x \in f^{-1}(\emptyset)$, so $x \in A$ and $f(x) \in \emptyset$.
Since $\emptyset$ is empty, then $f(x) \notin \emptyset$.
Thus, we have $f(x) \in \emptyset$ and $f(x) \notin \emptyset$, a contradiction.
Therefore, $f^{-1}(\emptyset) = \emptyset$. ☐

*Proof.* We prove 3.
We prove $f(A) = rngf$.
Since $b \in f(A)$ iff there exists $a \in A$ such that $f(a) = b$ iff $b \in rngf$, then $b \in f(A)$ iff $b \in rngf$.
Therefore, $f(A) = rngf$. ☐

*Proof.* We prove 4.
We prove $f^{-1}(B) = A$.
Since $f^{-1}(B) = \{x \in A : f(x) \in B\}$, then $f^{-1}(B) \subset A$.

Let $x \in A$.
Since $f : A \to B$ is a map, then $f(x) \in B$.
Since $x \in A$ and $f(x) \in B$, then $x \in f^{-1}(B)$.
Thus, $A \subset f^{-1}(B)$.
Since $f^{-1}(B) \subset A$ and $A \subset f^{-1}(B)$, then $f^{-1}(B) = A$. ☐

**Proposition 25.** *Let* $f : X \to Y$ *be a map.*
*1. For every subset* $A$ *and* $B$ *of* $X$*, if* $A \subset B$*, then* $f(A) \subset f(B)$.
*2.* $f(A \cup B) = f(A) \cup f(B)$ *for every subset* $A$ *and* $B$ *of* $X$.
*The image of a union equals the union of the images.*
*3.* $f(A \cap B) \subset f(A) \cap f(B)$ *for every subset* $A$ *and* $B$ *of* $X$.
*The image of an intersection is a subset of the intersection of the images.*
*4.* $f(A \cap B) = f(A) \cap f(B)$ *for every subset* $A$ *and* $B$ *of* $X$ *iff* $f$ *is injective.*

*Proof.* We prove 1.
Let $A$ and $B$ be subsets of $X$ such that $A \subset B$.
We must prove $f(A) \subset f(B)$.
Let $y \in f(A)$.
Then there exists $x \in A$ such that $f(x) = y$.
Since $x \in A$ and $A \subset B$, then $x \in B$.
Thus, there exists $x \in B$ such that $f(x) = y$, so $y \in f(B)$.
Therefore, $f(A) \subset f(B)$. $\square$

*Proof.* We prove 2.
Let $A$ and $B$ be subsets of $X$.
We must prove $f(A \cup B) = f(A) \cup f(B)$.
Observe that

$$\begin{aligned} y \in f(A \cup B) &\Leftrightarrow \text{there exists } x \in A \cup B \text{ such that } y = f(x) \\ &\Leftrightarrow \text{either there exists } x \in A \text{ or there exists } x \in B \text{ and } y = f(x) \\ &\Leftrightarrow \text{either there exists } x \in A \text{ and } y = f(x) \text{ or there exists } x \in B \text{ and } y = f(x) \\ &\Leftrightarrow \text{either } y \in f(A) \text{ or } y \in f(B) \\ &\Leftrightarrow y \in f(A) \cup f(B). \end{aligned}$$

Therefore, $y \in f(A \cup B)$ iff $y \in f(A) \cup f(B)$, so $f(A \cup B) = f(A) \cup f(B)$. $\square$

*Proof.* We prove 2.
Let $A$ and $B$ be subsets of $X$.
We first prove $f(A \cup B) \subset f(A) \cup f(B)$.
Let $y \in f(A \cup B)$.
Then there exists $x \in A \cup B$ such that $f(x) = y$.
Since $x \in A \cup B$, then either $x \in A$ or $x \in B$.
**Case 1:** Suppose $x \in A$.
Since $x \in A$ and $y = f(x)$, then $y \in f(A)$.
**Case 2:** Suppose $x \in B$.
Since $x \in B$ and $y = f(x)$, then $y \in f(B)$.
Thus, either $y \in f(A)$ or $y \in f(B)$, so $y \in f(A) \cup f(B)$.
Therefore, $f(A \cup B) \subset f(A) \cup f(B)$.

We next prove $f(A) \cup f(B) \subset f(A \cup B)$.
Let $y \in f(A) \cup f(B)$.
Then either $y \in f(A)$ or $y \in f(B)$.
**Case 1:** Suppose $y \in f(A)$.
Then there exists $a \in A$ such that $f(a) = y$.
Since $a \in A$, then either $a \in A$ or $a \in B$, so $a \in A \cup B$.
Since $a \in A \cup B$ and $f(a) = y$, then $y \in f(A \cup B)$.
**Case 2:** Suppose $y \in f(B)$.
Then there exists $b \in B$ such that $f(b) = y$.
Since $b \in B$, then either $b \in A$ or $b \in B$, so $b \in A \cup B$.
Since $b \in A \cup B$ and $f(b) = y$, then $y \in f(A \cup B)$.

Hence, in either case, $y \in f(A \cup B)$.
Therefore, $f(A) \cup f(B) \subset f(A \cup B)$.

Since $f(A \cup B) \subset f(A) \cup f(B)$ and $f(A) \cup f(B) \subset f(A \cup B)$, then $f(A \cup B) = f(A) \cup f(B)$. $\square$

*Proof.* We prove 3.
Let $A$ and $B$ be subsets of $X$.
We prove $f(A \cap B) \subset f(A) \cap f(B)$.
Let $y \in f(A \cap B)$.
Then there exists $x \in A \cap B$ such that $f(x) = y$.
Since $x \in A \cap B$, then $x \in A$ and $x \in B$.
Since $x \in A$ and $f(x) = y$, then $y \in f(A)$.
Since $x \in B$ and $f(x) = y$, then $y \in f(B)$.
Thus, $y \in f(A)$ and $y \in f(B)$, so $y \in f(A) \cap f(B)$.
Therefore, $f(A \cap B) \subset f(A) \cap f(B)$. $\square$

*Proof.* We prove 4.
We prove $f(A \cap B) = f(A) \cap f(B)$ for every subset $A$ and $B$ of $X$ iff $f$ is injective.

We first prove if $f$ is injective, then $f(A \cap B) = f(A) \cap f(B)$ for every subset $A$ and $B$ of $X$.
Suppose $f$ is injective.
Let $A$ and $B$ be subsets of $X$.
We prove $f(A) \cap f(B) \subset f(A \cap B)$.

Let $y \in f(A) \cap f(B)$.
Then $y \in f(A)$ and $y \in f(B)$.
Since $y \in f(A)$, then $y = f(a)$ for some $a \in A$.
Since $y \in f(B)$, then $y = f(b)$ for some $b \in B$.
Hence, $f(a) = y = f(b)$.
Since $f$ is injective and $f(a) = f(b)$, then $a = b$.
Since $a = b$ and $b \in B$, then $a \in B$.
Since $a \in A$ and $a \in B$, then $a \in A \cap B$.
Since $a \in A \cap B$ and $f(a) = y$, then $y \in f(A \cap B)$.
Therefore, $f(A) \cap f(B) \subset f(A \cap B)$.
Since $f(A \cap B) \subset f(A) \cap f(B)$ and $f(A) \cap f(B) \subset f(A \cap B)$, then $f(A \cap B) = f(A) \cap f(B)$.

Conversely, we prove if $f(A \cap B) = f(A) \cap f(B)$ for every subset $A$ and $B$ of $X$, then $f$ is injective.
We prove by contrapositive.
Suppose $f$ is not injective.
Then there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

We must prove there exist subsets $A$ and $B$ of $X$ such that $f(A \cap B) \neq f(A) \cap f(B)$.

Let $A = \{x_1\}$ and $B = \{x_2\}$.

Since $x_1 \in X$ and $A = \{x_1\}$, then $A \subset X$.

Since $x_2 \in X$ and $B = \{x_1\}$, then $B \subset X$.

We prove $f(A \cap B) \neq f(A) \cap f(B)$.

If $A \cap B \neq \emptyset$, then there exists $x$ such that $x \in A \cap B$, so $x \in A$ and $x \in B$.

Hence, $x \in \{x_1\}$ and $x \in \{x_2\}$, so $x = x_1$ and $x = x_2$.

Thus, $x_1 = x = x_2$.

Therefore, if $A \cap B \neq \emptyset$, then $x_1 = x_2$, so if $x_1 \neq x_2$, then $A \cap B = \emptyset$.

Since $x_1 \neq x_2$, then we conclude $A \cap B = \emptyset$.

Since $x_1 \in A$, then $f(x_1) \in f(A)$.

Since $x_2 \in B$, then $f(x_2) \in f(B)$.

Since $f(x_1) = f(x_2)$ and $f(x_2) \in f(B)$, then $f(x_1) \in f(B)$.

Thus, $f(x_1) \in f(A)$ and $f(x_1) \in f(B)$, so $f(x_1) \in f(A) \cap f(B)$.

Hence, $f(A) \cap f(B) \neq \emptyset$.

Therefore, $f(A \cap B) = f(\emptyset) = \emptyset \neq f(A) \cap f(B)$, as desired. □

**Proposition 26.** *Let $f : X \to Y$ be a map.*

*1. For every subset $C$ and $D$ of $Y$, if $C \subset D$, then $f^{-1}(C) \subset f^{-1}(D)$.*

*2. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ for every subset $C$ and $D$ of $Y$.*

*The inverse image of a union equals the union of the inverse images.*

*3. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$ for every subset $C$ and $D$ of $Y$.*

*The inverse image of an intersection equals the intersection of the inverse images.*

*Proof.* We prove 1.

Let $C$ and $D$ be subsets of $Y$ such that $C \subset D$.

Let $x \in f^{-1}(C)$.

Then $x \in X$ and $f(x) \in C$.

Since $f(x) \in C$ and $C \subset D$, then $f(x) \in D$.

Hence, $x \in X$ and $f(x) \in D$, so $x \in f^{-1}(D)$.

Therefore, $f^{-1}(C) \subset f^{-1}(D)$. □

*Proof.* We prove 2.

Let $C$ and $D$ be subsets of $Y$.

We must prove $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

Observe that

$$\begin{aligned}
x \in f^{-1}(C \cup D) &\Leftrightarrow x \in X \text{ and } f(x) \in C \cup D \\
&\Leftrightarrow x \in X \text{ and either } f(x) \in C \text{ or } f(x) \in D \\
&\Leftrightarrow \text{either } x \in X \text{ and } f(x) \in C \text{ or } x \in X \text{ and } f(x) \in D \\
&\Leftrightarrow \text{either } x \in f^{-1}(C) \text{ or } x \in f^{-1}(D) \\
&\Leftrightarrow x \in f^{-1}(C) \cup f^{-1}(D).
\end{aligned}$$

Therefore, $x \in f^{-1}(C \cup D)$ iff $x \in f^{-1}(C) \cup f^{-1}(D)$, so $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$. ☐

*Proof.* We prove 3.

Let $C$ and $D$ be subsets of $Y$.

We must prove $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Observe that

$$\begin{aligned} x \in f^{-1}(C \cap D) &\Leftrightarrow x \in X \text{ and } f(x) \in C \cap D \\ &\Leftrightarrow x \in X \text{ and } f(x) \in C \text{ and } f(x) \in D \\ &\Leftrightarrow x \in X \text{ and } f(x) \in C \text{ and } x \in X \text{ and } f(x) \in D \\ &\Leftrightarrow x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \\ &\Leftrightarrow x \in f^{-1}(C) \cap f^{-1}(D). \end{aligned}$$

Therefore, $x \in f^{-1}(C \cap D)$ iff $x \in f^{-1}(C) \cap f^{-1}(D)$, so $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. ☐

**Proposition 27.** ***inverse image of the image of a subset of the domain of a map***

*Let $f : A \to B$ be a map. Then*

*1. $S \subset f^{-1}(f(S))$ for every subset $S$ of $A$.*

*2. $f^{-1}(f(S)) = S$ for every subset $S$ of $A$ iff $f$ is injective.*

*Proof.* We prove 1.

We prove $S \subset f^{-1}(f(S))$ for every subset $S$ of $A$.

Let $S \subset A$.

Suppose $x \in S$.

Then $f(x) \in f(S)$.

Since $x \in S$ and $S \subset A$, then $x \in A$.

Since $x \in A$ and $f(x) \in f(S)$, then $x \in f^{-1}(f(S))$.

Therefore, $S \subset f^{-1}(f(S))$. ☐

*Proof.* We prove 2.

We prove $f^{-1}(f(S)) = S$ for every subset $S$ of $A$ iff $f$ is injective.

We first prove if $f^{-1}(f(S)) = S$ for every subset $S$ of $A$, then $f$ is injective.

Suppose $f^{-1}(f(S)) = S$ for every subset $S$ of $A$.

To prove $f$ is injective, let $a, b \in A$ such that $f(a) = f(b)$.

We must prove $a = b$.

Let $S = \{a\}$.

Since $a \in A$, then $S \subset A$.

Hence, $f^{-1}(f(S)) = S$.

Since $a \in S$, then $f(a) \in f(S)$.

Since $f(b) = f(a)$, then $f(b) \in f(S)$.

Since $b \in A$ and $f(b) \in f(S)$, then $b \in f^{-1}(f(S))$.

Thus, $b \in S$, so $b \in \{a\}$.

Therefore, $b = a$, as desired.

Conversely, we prove if $f$ is injective, then $f^{-1}(f(S)) = S$ for every subset $S$ of $A$.

Suppose $f$ is injective.
Let $S \subset A$.
We must prove $f^{-1}(f(S)) = S$.
Let $x \in f^{-1}(f(S))$.
Then $x \in A$ and $f(x) \in f(S)$.
Since $f(x) \in f(S)$, then there exists $s \in S$ such that $f(s) = f(x)$.
Since $f$ is injective, then $s = x$.
Since $s \in S$, then $x \in S$.
Therefore, $f^{-1}(f(S)) \subset S$.
Since $f^{-1}(f(S)) \subset S$ and $S \subset f^{-1}(f(S))$, then $f^{-1}(f(S)) = S$. $\square$

**Proposition 28.** ***image of the inverse image of a subset of the codomain of a map***

*Let $f : A \to B$ be a map. Then*
*1. $f(f^{-1}(T)) \subset T$ for every subset $T$ of $B$.*
*2. $f(f^{-1}(T)) = T$ for every subset $T$ of $B$ iff $f$ is surjective.*

*Proof.* We prove 1.
Let $T \subset B$.
We prove $f(f^{-1}(T)) \subset T$.

Let $y \in f(f^{-1}(T))$.
Then there exists $x \in f^{-1}(T)$ such that $f(x) = y$.
Since $x \in f^{-1}(T)$, then $x \in A$ and $f(x) \in T$.
Thus, $y \in T$.
Therefore, $f(f^{-1}(T)) \subset T$. $\square$

*Proof.* We prove 2.
We must prove $f(f^{-1}(T)) = T$ for every subset $T$ of $B$ iff $f$ is surjective.

We first prove if $f$ is surjective, then $f(f^{-1}(T)) = T$ for every subset $T$ of $B$.
Suppose $f$ is surjective.
Let $T \subset B$.
Let $y \in T$.
Since $f$ is surjective, then there exists $x \in A$ such that $f(x) = y$.
Since $x \in A$ and $f(x) \in T$, then $x \in f^{-1}(T)$.
Since $y = f(x)$ and $x \in f^{-1}(T)$, then $y \in f(f^{-1}(T))$.
Therefore, $T \subset f(f^{-1}(T))$.
Since $f(f^{-1}(T)) \subset T$ and $T \subset f(f^{-1}(T))$, then $f(f^{-1}(T)) = T$.

Conversely, we prove if $f(f^{-1}(T)) = T$ for every subset $T$ of $B$, then $f$ is surjective.

Suppose $f(f^{-1}(T)) = T$ for every subset $T$ of $B$.
Since $B \subset B$, then $f(f^{-1}(B)) = B$.
Observe that

$$\begin{aligned} B &= f(f^{-1}(B)) \\ &= f(A) \\ &= rngf. \end{aligned}$$

Therefore, $rngf = B$, so $f$ is surjective, as desired. $\square$