Functions Theory Exercises

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Functions

Exercise 1. Let $f : \mathbb{Q} \to \mathbb{Z}$ be defined by $f(a/b) = a$.
What can we deduce about f?

Solution. We know f is a binary relation since $f \subset \mathbb{Q} \times \mathbb{Z}$.
Is f a function?
We observe that $1/2 = 2/4$, but $f(1/2) = 1$ and $f(2/4) = 2$.
Thus, f is not a function, by definition of function. □

Exercise 2. Let $f : \mathbb{Q} \to \mathbb{Q}$ be a relation defined by $f(\frac{p}{q}) = \frac{p+1}{p-2}$.
Then f is not a function(map).

Proof. Since division by zero is not defined, then $f(\frac{2}{3})$ is undefined, so f is not a function. □

Exercise 3. Let $f : \mathbb{Q} \to \mathbb{Q}$ be a relation defined by $f(\frac{p}{q}) = \frac{3p}{3q}$.
Then f is a function.

Proof. Let $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ such that $\frac{p}{q} = \frac{r}{s}$.
Then $f(\frac{p}{q}) = \frac{3p}{3q} = \frac{p}{q} = \frac{r}{s} = \frac{3r}{3s} = f(\frac{r}{s})$, so $f(\frac{p}{q}) = f(\frac{r}{s})$.
Therefore, f is a function. □

Exercise 4. Let $f : \mathbb{Q} \to \mathbb{Q}$ be a relation defined by $f(\frac{p}{q}) = \frac{p+q}{q^2}$.
Then f is not a function.

Proof. Since $f(\frac{1}{2}) = \frac{1+2}{2^2} = \frac{3}{4}$ and $f(\frac{2}{4}) = \frac{2+4}{4^2} = \frac{3}{8}$, then $f(\frac{1}{2}) \neq f(\frac{2}{4})$, even though $\frac{1}{2} = \frac{2}{4}$.
Therefore, f is not a function. □

Exercise 5. Let $f : \mathbb{Q} \to \mathbb{Q}$ be a relation defined by $f(\frac{p}{q}) = \frac{3p^2}{7q^2} - \frac{p}{q}$.
Then f is a function.

Proof. Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ such that $\frac{a}{b} = \frac{c}{d}$.
Then

$$
\begin{aligned}
f(\frac{a}{b}) &= \frac{3a^2}{7b^2} - \frac{a}{b} \\
&= \frac{3}{7}(\frac{a}{b})^2 - \frac{a}{b} \\
&= \frac{3}{7}(\frac{c}{d})^2 - \frac{c}{d} \\
&= \frac{3c^2}{7d^2} - \frac{c}{d} \\
&= f(\frac{c}{d}).
\end{aligned}
$$

Therefore, f is a function. $\square$

Exercise 6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = e^x$.
Then f is injective, but not surjective.

Proof. Let $a, b \in \mathbb{R}$ such that $f(a) = f(b)$.
Then $e^a = f(a) = f(b) = e^b$, so $a = b$.
Therefore, f is injective.

Since $e^x > 0$ for all $x \in \mathbb{R}$, then $e^x \neq 0$ for all $x \in \mathbb{R}$, so $f(x) \neq 0$ for all $x \in \mathbb{R}$.
Since 0 is a real number and $f(x) \neq 0$ for all $x \in \mathbb{R}$, then f is not surjective.

The range of f is the set $\{f(x) : x \in \mathbb{R}\} = \{e^x : x \in \mathbb{R}\} = (0, \infty)$.
Therefore, the range of f is the interval $(0, \infty)$. $\square$

Exercise 7. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(n) = n^2 + 3$.
Then f is not injective and not surjective.

Proof. Observe that $f(-1) = (-1)^2 + 3 = 4 = 1^2 + 3 = f(1)$, so $f(-1) = f(1)$.
Since $-1 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$ and $-1 \neq 1$ and $f(-1) = f(1)$, then f is not injective. $\square$

Proof. We next prove f is not surjective.
Suppose there is an integer n such that $f(n) = 5$.
Then $5 = f(n) = n^2 + 3$, so $2 = n^2$.
But, there is no integer whose square is 2, so we conclude there is no integer n such that $f(n) = 5$.
Hence, $f(n) \neq 5$ for all $n \in \mathbb{Z}$.
Since $5 \in \mathbb{Z}$ and $f(n) \neq 5$ for all $n \in \mathbb{Z}$, then f is not surjective.

The range of f is the set $\{f(n) : n \in \mathbb{Z}\} = \{n^2+3 : n \in \mathbb{Z}\} = \{3, 4, 7, 12, 19, 28, ...\}$. $\square$

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \sin x$.
Then f is not injective and not surjective.

Proof. Since $f(\pi) = \sin \pi = 0 = \sin 0 = f(0)$, then $f(\pi) = f(0)$.
Since $\pi \in \mathbb{R}$ and $0 \in \mathbb{R}$ and $\pi \neq 0$ and $f(\pi) = f(0)$, then f is not injective.

The range of f is the set $\{f(x) : x \in \mathbb{R}\} = \{\sin x : x \in \mathbb{R}\} = [-1, 1]$.
Therefore, the range of f is the interval $[-1, 1]$ and f is not surjective. $\square$

Exercise 9. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $f(n) = n^2$.
Then f is not injective and not surjective.

Proof. Since $1 \in \mathbb{Z}$ and $-1 \in \mathbb{Z}$ and $1 \neq -1$ and $f(1) = 1^2 = 1 = (-1)^2 = f(-1)$, then f is not injective.

Since $n^2 \geq 0$ for all $n \in \mathbb{Z}$, then $n^2 > -1$ for all $n \in \mathbb{Z}$, so $n^2 \neq -1$ for all $n \in \mathbb{Z}$.
Hence, $f(n) \neq -1$ for all $n \in \mathbb{Z}$.
Since $-1 \in \mathbb{Z}$ and $f(n) \neq -1$ for all $n \in \mathbb{Z}$, then f is not surjective. $\square$

Exercise 10. Let $f : \mathbb{Z} \to \mathbb{Q}$ be a relation defined by $f(n) = \frac{n}{1}$.
Then f is a function and f is injective, but f is not surjective.

Proof. We first prove the relation f is actually a function.

Let $a, b \in \mathbb{Z}$ such that $a = b$.
Then $f(a) = \frac{a}{1} = a = b = \frac{b}{1} = f(b)$, so $f(a) = f(b)$.
Therefore, f is a function.

We next prove f is injective.
Let $a, b \in \mathbb{Z}$ such that $f(a) = f(b)$.
Then $a = \frac{a}{1} = f(a) = f(b) = \frac{b}{1} = b$, so $a = b$.
Therefore, f is injective.

We next prove f is not surjective.
Let $b = \frac{1}{2} \in \mathbb{Q}$.
Let $a \in \mathbb{Z}$ be arbitrary.
Then $f(a) = \frac{a}{1} = a \neq \frac{1}{2} = b$, so $f(a) \neq b$.
Therefore, f is not surjective. $\square$

Exercise 11. Let $g : \mathbb{Q} \to \mathbb{Z}$ be defined by $g(\frac{p}{q}) = p$ for $\frac{p}{q}$ expressed in lowest terms with a positive denominator.
Then the function g is surjective, but g is not injective.

Proof. Let $b \in \mathbb{Z}$ be arbitrary.
Since $\frac{b}{1} \in \mathbb{Q}$ and $g(\frac{b}{1}) = b$, then g is surjective.

Observe that $\frac{1}{2} \in \mathbb{Q}$ and $\frac{1}{3} \in \mathbb{Q}$ and $\frac{1}{2} \neq \frac{1}{3}$, but $g(\frac{1}{2}) = 1 = g(\frac{1}{3})$.
Therefore, g is not injective. $\square$

Exercise 12. Let $f : \mathbb{R} \to \mathbb{R}$ be a map defined by $f(x) = x^3$.
Then f is injective.

Proof. Let $x, y \in \mathbb{R}$ such that $x \neq y$.
To prove f is injective, we must prove $x^3 \neq y^3$.
Since $x \neq y$, then either $x < y$ or $x > y$.
Without loss of generality, assume $x < y$.
Then either $0 < x < y$ or $0 = x < y$ or $x < 0 < y$ or $x < 0 = y$ or $x < y < 0$.
We consider these cases separately.
Case 1: Suppose $0 < x < y$.
Then $0 < x^2 < y^2$, so $0 < x^3 < y^3$.
Hence, $x^3 < y^3$.
Case 2: Suppose $0 = x < y$.
Since $y > 0$, then $y^3 > 0 = x^3$, so $x^3 < y^3$.
Case 3: Suppose $x < 0 < y$.
Then $x < 0$ and $y > 0$, so $x^3 < 0$ and $y^3 > 0$.
Thus, $x^3 < 0 < y^3$, so $x^3 < y^3$.
Case 4: Suppose $x < 0 = y$.
Since $x < 0$, then $x^3 < 0 = y^3$, so $x^3 < y^3$.
Case 5: Suppose $x < y < 0$.
Then $-x > -y > 0$, so $x^2 > y^2 > 0$.
Hence, $-x^3 > -y^3 > 0$, so $-x^3 > -y^3$.
Thus, $x^3 < y^3$.
In all cases, $x^3 < y^3$, so $x^3 \neq y^3$, as desired. $\square$

Exercise 13. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^4 + 5x^2$.
Then f is injective.

Proof. Let $a, b \in [0, \infty)$ such that $f(a) = f(b)$.
Then $a \geq 0$ and $b \geq 0$ and $a^4 + 5a^2 = b^4 + 5b^2$.
Since $a^4 + 5a^2 = b^4 + 5b^2$, then $a^4 - b^4 + 5a^2 - 5b^2 = 0$, so $(a^2 - b^2)(a^2 + b^2) + 5(a^2 - b^2) = 0$.
Hence, $(a^2 - b^2)(a^2 + b^2 + 5) = 0$, so either $a^2 - b^2 = 0$ or $a^2 + b^2 + 5 = 0$.
Since $a \geq 0$ and $b \geq 0$, then $a^2 \geq 0$ and $b^2 \geq 0$, so $a^2 + b^2 \geq 0$.
Thus, $a^2 + b^2 + 5 \geq 5 > 0$, so $a^2 + b^2 + 5 > 0$.
Hence, $a^2 + b^2 + 5 \neq 0$, so $a^2 - b^2 = 0$.
Therefore, $a^2 = b^2$, so $|a| = |b|$.
Since $a \geq 0$ and $b \geq 0$, then $a = b$, as desired. $\square$

Exercise 14. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - x$.
Then f is not injective.

Solution. Since $0 \neq 1$ and $f(0) = 0^3 - 0 = 0 = 1^3 - 1 = f(1)$, then f is not injective. $\square$

Exercise 15. Let $f : A \to B$ and $g : B \to C$ be functions.
If $g \circ f$ is onto, then g is onto.

Proof. Suppose $g \circ f$ is onto.
To prove g is onto, let $c \in C$.
We must prove there exists $b \in B$ such that $g(b) = c$.
Since $g \circ f$ is onto, then there exists $a \in A$ such that $(g \circ f)(a) = c$.
Since f is a function, then there exists $b \in B$ such that $f(a) = b$.
Thus,

$$\begin{aligned} c &= (g \circ f)(a) \\ &= g(f(a)) \\ &= g(b). \end{aligned}$$

Therefore, $g(b) = c$, so g is onto. $\square$

Exercise 16. Let $f : A \to B$ and $g : B \to C$ be functions.
If $g \circ f$ is one to one, then f is one to one.

Proof. Suppose $g \circ f$ is one to one.
To prove f is one to one, let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$.
Then $g(f(a_1)) = g(f(a_2))$, so $(g \circ f)(a_1) = (g \circ f)(a_2)$.
Since $g \circ f$ is one to one, then this implies $a_1 = a_2$.
Therefore, f is one to one. $\square$

Exercise 17. Let $f : A \to B$ and $g : B \to C$ be functions.
If $g \circ f$ is one to one and f is onto, then g is one to one.

Proof. Suppose $g \circ f$ is one to one and f is onto.
To prove g is one to one, let $b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$.
We must prove $b_1 = b_2$.
Since f is onto, then there exist $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$.
Thus,

$$\begin{aligned} (g \circ f)(a_1) &= g(f(a_1)) \\ &= g(b_1) \\ &= g(b_2) \\ &= g(f(a_2)) \\ &= (g \circ f)(a_2). \end{aligned}$$

Since $g \circ f$ is one to one, then this implies $a_1 = a_2$.
Hence,

$$\begin{aligned} b_1 &= f(a_1) \\ &= f(a_2) \\ &= b_2. \end{aligned}$$

Therefore, $b_1 = b_2$, so g is one to one, as desired. $\square$

Exercise 18. Let $f : A \to B$ and $g : B \to C$ be functions.
If $g \circ f$ is onto and g is one to one, then f is onto.

Proof. Suppose $g \circ f$ is onto and g is one to one.
To prove f is onto, let $b \in B$.
We must prove there exists $a \in A$ such that $f(a) = b$.
Since g is a function, then there is a unique $c \in C$ such that $g(b) = c$.
Since $g \circ f$ is onto, then there exists at least one $a \in A$ such that $(g \circ f)(a) = c$.
Thus,

$$\begin{aligned} g(b) &= c \\ &= (g \circ f)(a) \\ &= g(f(a)). \end{aligned}$$

Hence, $g(b) = g(f(a))$.
Since g is one to one, then this implies $b = f(a)$.
Thus, there exists $a \in A$ such that $f(a) = b$.
Therefore, f is onto. $\square$

Exercise 19. Devise a bijection from $S \times T$ to $T \times S$.

Solution. Observe that $S \times T = \{(s,t) : s \in S, t \in T\}$ and $T \times S = \{(t,s) : t \in T, s \in S\}$.
Let $\phi : S \times T \mapsto T \times S$ be a function defined by $\phi(x,y) = (y,x)$.
We can prove that ϕ is both 1-1 and onto.

We prove ϕ is injective.
Let $(a,b), (c,d) \in S \times T$.
Suppose $\phi(a,b) = \phi(c,d)$.
Then $(b,a) = (d,c)$.
Hence $b = d$ and $a = c$ so $(a,b) = (c,d)$.
Therefore $\phi(a,b) = \phi(c,d) \to (a,b) = (c,d)$ so ϕ is injective.

We prove ϕ is surjective.
Let $(a,b) \in T \times S$.
Then $a \in T$ and $b \in S$ so $(b,a) \in S \times T$.
Observe that $\phi(b,a) = (a,b)$.
Hence ϕ is surjective.
Since ϕ is injective and surjective then ϕ is bijective.
Therefore ϕ is a one to one correspondence from $S \times T$ to $T \times S$. $\square$

Exercise 20. Let $A = \{a,b,c\}$.
Let $f : A \to \mathbb{Z}$ be the function defined by $f(a) = 1$ and $f(b) = 5$ and $f(c) = 5$.
1. The image of f is the set $\{1, 5\}$.

2. The pre-image of 5 is the set $\{b, c\}$.
3. The function f is not one to one.

Proof. The image of f is the set $\{f(x) \in \mathbb{Z} : x \in A\} = \{1, 5\}$.

The pre-image of 5 is the set $\{x \in A : f(x) = 5\} = \{b, c\}$.

Since $b \neq c$ and $f(b) = 5 = f(c)$, then f is not one to one. $\square$

Exercise 21. Let $f : S \to T$ be a surjective function.

Let $\{P_a\}_{a \in I}$ be a partition of T.

Then $\{f^{-1}(P_a) : a \in I\}$ is a partition of S.

Proof. Let $T' = \{P_a\}_{a \in I}$ for some index set I.

Then T' is a partition of T.

Thus, T' is a collection of nonempty subsets of T such that $T = \cup(P_a)_{a \in I}$ and if $P_a \neq P_b$ for $a, b \in I$, then $P_a \cap P_b = \emptyset$.

Since T' is a partition of T, then there exists a subset of T in T'.

Hence, there exists $a \in I$ such that $P_a \in T'$.

Since $P_a \in T'$, then $P_a \subset T$ and $P_a \neq \emptyset$.

Since P_a is not empty, then there exists $y \in P_a$.

Since $P_a \subset T$, then $y \in T$.

Since f is surjective, then there exists $x \in S$ such that $f(x) = y$.

Hence, $x \in S$ and $f(x) \in P_a$, so $x \in f^{-1}(P_a)$.

Therefore, $f^{-1}(P_a) \neq \emptyset$.

Let $S' = \{f^{-1}(P_a) : a \in I\}$.

Then S' is a collection of nonempty sets $f^{-1}(P_a)$ for some $a \in I$.

We prove S' is a partition of S.

We first prove $\cup f^{-1}(P_a) = S$.

Let $x \in \cup f^{-1}(P_a)$.

Then there exists $a \in I$ such that $x \in f^{-1}(P_a)$.

Thus, $x \in S$.

Hence, $x \in \cup f^{-1}(P_a)$ implies $x \in S$, so $\cup f^{-1}(P_a) \subset S$.

Let $y \in S$.

Then $f(y) \in T$.

Since T' is a partition of T, then every element of T is contained in some set in T'.

In particular, $f(y)$ is contained in some set in T'.

Hence, there exists $a \in I$ such that $f(y) \in P_a$ and $P_a \in T'$.

Since $y \in S$ and $f(y) \in P_a$, then $y \in f^{-1}(P_a)$.

Thus, there exists $a \in I$ such that $y \in f^{-1}(P_a)$, so $y \in \cup f^{-1}(P_a)$.

Therefore, $y \in S$ implies $y \in \cup f^{-1}(P_a)$, so $S \subset \cup f^{-1}(P_a)$.

Since $\cup f^{-1}(P_a) \subset S$ and $S \subset \cup f^{-1}(P_a)$, then $S = \cup f^{-1}(P_a)$.

Let $a, b \in I$ such that $a \neq b$.

Since T' is a partition of T and $P_a, P_b \in T'$, then $P_a \neq P_b$.

Hence, $P_a \cap P_b = \emptyset$.

Since $P_a \neq P_b$, then $f^{-1}P_a \neq f^{-1}P_b$.

Observe that

$$
\begin{aligned}
f^{-1}(P_a) \cap f^{-1}(P_b) &= f^{-1}(P_a \cap P_b) \\
&= f^{-1}(\emptyset) \\
&= \emptyset.
\end{aligned}
$$

Therefore, if $f^{-1}P_a$ and $f^{-1}P_b$ are distinct, then $f^{-1}P_a$ and $f^{-1}P_b$ are disjoint.
Hence, S' is a partition of S. ☐

Exercise 22. If $f : A \to B$ is a bijective function, then $f^{-1} : B \to A$ is a bijective function.

Proof. Let $f : A \to B$ be a bijective function.
Since f is bijective, then the inverse function $f^{-1} : B \to A$ exists.
Hence, $f \circ f^{-1} = f^{-1} \circ f = id$.

We prove f^{-1} is injective.
Let $a, b \in B$ such that $f^{-1}(a) = f^{-1}(b)$.
Since $f \circ f^{-1} = id$, then $(f \circ f^{-1})(x) = x$ for all $x \in B$.
In particular, $(f \circ f^{-1})(a) = a$.
Thus,

$$
\begin{aligned}
a &= (f \circ f^{-1})(a) \\
&= f(f^{-1}(a)) \\
&= f(f^{-1}(b)) \\
&= (f \circ f^{-1})(b) \\
&= id(b) \\
&= b.
\end{aligned}
$$

Hence, $f^{-1}(a) = f^{-1}(b)$ implies $a = b$, so f^{-1} is injective.

We prove f^{-1} is surjective.
Let $a \in A$.
Since f is a function, then there exists $b \in B$ such that $f(a) = b$.
Observe that

$$
\begin{aligned}
f^{-1}(b) &= f^{-1}(f(a)) \\
&= (f^{-1} \circ f)(a) \\
&= id(a) \\
&= a.
\end{aligned}
$$

Therefore, f^{-1} is surjective.
Hence, f^{-1} is bijective. ☐

Exercise 23. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 + x$.
Then f is injective.

Proof. Let $a, b \in \mathbb{R}$ such that $f(a) = f(b)$.

Then $a^3 + a = b^3 + b$, so $a^3 - b^3 + a - b = 0$.

Hence, $(a - b)(a^2 + ab + b^2) + a - b = 0$, so $(a - b)(a^2 + ab + b^2 + 1) = 0$.

Thus, either $a - b = 0$ or $a^2 + ab + b^2 + 1 = 0$.

Suppose $a^2 + ab + b^2 + 1 = 0$.

Then $a^2 + ab + b^2 = -1$.

Since $a, b \in \mathbb{R}$, then either $a \neq 0$ or $a = 0$ and either $b \neq 0$ or $b = 0$.

Thus, either $a \neq 0$ and $b \neq 0$ or $a \neq 0$ and $b = 0$ or $a = 0$ and $b \neq 0$ or $a = 0$ and $b = 0$.

Since $a \neq 0$ and $b \neq 0$ iff $a > 0$ or $a < 0$ and $b > 0$ or $b < 0$, then either $a > 0$ and $b > 0$ or $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$ or $a < 0$ and $b < 0$.

We consider these cases separately.

Case 1: Suppose $a > 0$ and $b > 0$.

Then $a^2 > 0$ and $b^2 > 0$ and $ab > 0$, so $a^2 + ab + b^2 > 0$.

Hence, $-1 > 0$, a contradiction.

Case 2: Suppose $a > 0$ and $b < 0$.

Since $a > 0$, then $a^2 > 0$, so $a^2 + 1 > 0$.

Hence, $f(a) = a^3 + a = a(a^2 + 1) > 0$.

Since $b < 0$, then $b^2 > 0$, so $b^2 + 1 > 0$.

Thus, $f(b) = b^3 + b = b(b^2 + 1) < 0$, so $f(b) < 0 < f(a)$.

Hence, $f(b) < f(a)$, so $f(b) \neq f(a)$.

Thus, we have $f(a) \neq f(b)$ and $f(a) = f(b)$, a contradiction.

Case 3: Suppose $a < 0$ and $b > 0$.

Since $a < 0$, then $a^2 > 0$, so $a^2 + 1 > 0$.

Hence, $f(a) = a^3 + a = a(a^2 + 1) < 0$.

Since $b > 0$, then $b^2 > 0$, so $b^2 + 1 > 0$.

Thus, $f(b) = b^3 + b = b(b^2 + 1) > 0$, so $f(a) < 0 < f(b)$.

Hence, $f(a) < f(b)$, so $f(a) \neq f(b)$.

Thus, we have $f(a) \neq f(b)$ and $f(a) = f(b)$, a contradiction.

Case 4: Suppose $a < 0$ and $b < 0$.

Then $a^2 > 0$ and $b^2 > 0$ and $ab > 0$, so $a^2 + ab + b^2 > 0$.

Hence, $-1 > 0$, a contradiction.

Case 5: Suppose $a \neq 0$ and $b = 0$.

Then $-1 = a^2 + ab + b^2 = a^2 + a0 + 0^2 = a^2 + 0 + 0 = a^2 > 0$, so $-1 > 0$, a contradiction.

Case 6: Suppose $a = 0$ and $b \neq 0$.

Then $-1 = a^2 + ab + b^2 = 0^2 + 0b + b^2 = 0 + 0 + b^2 = b^2 > 0$, so $-1 > 0$, a contradiction.

Case 7: Suppose $a = 0$ and $b = 0$.

Then $-1 = a^2 + ab + b^2 = 0^2 + 00 + 0^2 = 0 + 0 + 0 = 0$, so $-1 = 0$, a contradiction.

Since a contradiction is reached in all cases, then $a^2 + ab + b^2 + 1 \neq 0$.

Therefore, $a - b = 0$, so $a = b$, as desired. $\square$

Exercise 24. Let $f : A \to B$ and $g : B \to C$ be maps.

If $g \circ f$ is injective and f is surjective, then g is injective.

Proof. Let $g \circ f : A \to C$ be the map such that $g \circ f$ is injective and f is surjective.

To prove g is injective, let $b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$.

Since f is surjective and $b_1 \in B$, then there exists $a_1 \in A$ such that $f(a_1) = b_1$.

Since f is surjective and $b_2 \in B$, then there exists $a_2 \in A$ such that $f(a_2) = b_2$.

Observe that

$$\begin{aligned}(g \circ f)(a_1) &= g(f(a_1)) \\ &= g(b_1) \\ &= g(b_2) \\ &= g(f(a_2)) \\ &= (g \circ f)(a_2).\end{aligned}$$

Since $g \circ f$ is injective and $(g \circ f)(a_1) = (g \circ f)(a_2)$, then $a_1 = a_2$.

Since f is a function, then this implies $f(a_1) = f(a_2)$.

Thus, $b_1 = f(a_1) = f(a_2) = b_2$, so $b_1 = b_2$.

Therefore, g is injective. □

Exercise 25. Let $f : A \to B$ and $g : B \to C$ be maps

If $g \circ f$ is surjective and g is injective, then f is surjective.

Proof. Let $g \circ f : A \to C$ be the map such that $g \circ f$ is surjective and g is injective.

To prove f is surjective, let $b \in B$ be arbitrary.

Since $g \circ f$ is surjective and $g(b) \in C$, then there exists $a \in A$ such that $(g \circ f)(a) = g(b)$.

Thus, $g(b) = (g \circ f)(a) = g(f(a))$.

Since g is injective and $g(f(a)) = g(b)$, then $f(a) = b$.

Therefore, there exists $a \in A$ such that $f(a) = b$, so f is surjective. □

Exercise 26. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by $f(n) = 2n - 1$ for each $n \in \mathbb{N}$.

Analyze f.

Proof. The domain of f is the set $\mathbb{N}$.

The range of f is $f(\mathbb{N}) = \{2n-1 : n \in \mathbb{N}\}$, the set of all odd natural numbers.

We prove f is not surjective.

Suppose there exists $a \in \mathbb{N}$ such that $f(a) = 2$.

Then $a \in \mathbb{N}$ and $2a - 1 = 2$, so $2a = 3$.

This implies 3 is even which contradicts the fact that 3 is odd.

Therefore, there is no $a \in \mathbb{N}$ such that $f(a) = 2$, so $f(a) \neq 2$ for each $a \in \mathbb{N}$.

Since $2 \in \mathbb{N}$ and $f(a) \neq 2$ for each $a \in \mathbb{N}$, then f is not surjective.

We prove f is injective.

Let $a, b \in \mathbb{N}$ such that $f(a) = f(b)$.
Then $2a - 1 = 2b - 1$, so $2a = 2b$.
Hence, $a = b$, so f is injective.

Since f is injective, then the inverse relation f^{-1} is a function and $dom f^{-1} = rng f = \{2n - 1 : n \in \mathbb{N}\}$.

Since f^{-1} is the inverse of f, then $f(x) = y$ iff $f^{-1}(y) = x$.
Hence, $2x - 1 = y$ iff $f^{-1}(y) = x$, so $x = \frac{y+1}{2}$ iff $f^{-1}(y) = x$.
Let $(x, y) \in f$.
Then $f(x) = y$, so $f^{-1}(y) = x$.
Thus, $f^{-1}(y) = x = \frac{y+1}{2}$.
Therefore, $f^{-1}(n) = \frac{n+1}{2}$ for each odd natural number n. $\square$

Exercise 27. Define a function $f : \mathbb{N} \to \mathbb{N}$ that is surjective but not injective.

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

.

We prove f is surjective.

Let $b \in \mathbb{N}$.
Then either b is even or b is odd.
If b is even, then $f(b) = \frac{b}{2}$.
If b is odd, then $f(b) = \frac{b+1}{2}$.
Therefore, f is surjective.

We prove f is not injective.

Observe that $f(1) = \frac{1+1}{2} = 1 = \frac{2}{2} = f(2)$.
Since $1 \in \mathbb{N}$ and $2 \in \mathbb{N}$ and $1 \neq 2$ and $f(1) = f(2)$, then f is not injective. $\square$

Exercise 28. Let f be a function defined on $\mathbb{R}$ by $f(x) = \frac{x+1}{x-1}$.

Analyze f.

That is, compute the domain of f, the range of f and inverse of f and $f \circ f^{-1}$ and $f^{-1} \circ f$.

Solution. Since division by zero is not allowed, then the domain of f is the set $dom f = \mathbb{R} - \{1\}$ and the range of f is the set $rng f = \mathbb{R} - \{1\}$.

We prove f is injective.

Let $a, b \in domf$ such that $f(a) = f(b)$.

Then $a, b \in \mathbb{R}$ and $a \neq 1$ and $b \neq 1$ and $\frac{a+1}{a-1} = \frac{b+1}{b-1}$.

Thus, $(a+1)(b-1) = (a-1)(b+1)$, so $ab - a + b - 1 = ab + a - b - 1$.

Hence, $-a + b = a - b$, so $2b = 2a$.

Therefore, $b = a$, so $a = b$.

Consequently, f is injective, so the inverse relation f^{-1} is a function.

The domain of f^{-1} is the set $domf^{-1} = rngf = \mathbb{R} - \{1\}$ and the range of f^{-1} is the set $rngf^{-1} = domf = \mathbb{R} - \{1\}$.

We determine the formula for f^{-1}.

Since f and f^{-1} are inverses, then $f(x) = y$ iff $f^{-1}(y) = x$.

Let $x \in domf$ and $y = f(x)$.

Then $x \in \mathbb{R}$ and $x \neq 1$ and $y \in \mathbb{R}$ and $y \neq 1$ and $y = f(x) = \frac{x+1}{x-1}$, so $y(x-1) = x+1$.

Hence, $yx - y = x + 1$, so $yx - x = y + 1$.

Thus, $x(y-1) = y+1$, so $x = \frac{y+1}{y-1}$.

Since $f(x) = y$ and $f(x) = y$ iff $f^{-1}(y) = x$, then $f^{-1}(y) = x = \frac{y+1}{y-1}$.

Therefore, $f^{-1}(x) = \frac{x+1}{x-1}$.

Let $x \in dom(f \circ f^{-1})$.

Then $(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(\frac{x+1}{x-1}) = x$.

Therefore, $f \circ f^{-1}$ is the identity function $I(x) = x$ on $\mathbb{R} - \{1\}$.

Similarly, $f^{-1} \circ f$ is the identity function $I(x) = x$ on $\mathbb{R} - \{1\}$. ☐

Proposition 29. *When is function composition commutative?*

Solution. We need to determine necessary and sufficient conditions to guarantee $g \circ f = f \circ g$.

Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions.

Suppose $g \circ f = f \circ g$.

Then $g \circ f : A \mapsto C$ and $f \circ g$ are functions.

By definition of equal functions we have domain($g \circ f$) = domain ($f \circ g$) and codomain($g \circ f$) = codomain ($f \circ g$) and $(g \circ f)(x) = (f \circ g)(x)$ for each $x \in A$.

Since domain($g \circ f$) = domain ($f \circ g$) then domain of $f \circ g$ is A.

Since codomain($g \circ f$) = codomain ($f \circ g$) then codomain of $f \circ g$ is C.

By definition of function composition for $f \circ g$, the domain of f is the codomain of g.

Thus, $A = C$.

Hence, the codomain of g is A and the codomain of f is A.

By definition of function composition for $g \circ f$, the domain of g is the codomain of f.

Thus, $B = A$.

Since $A = B = C$ then function f is $f : A \mapsto A$ and function g is $g : A \mapsto A$.

Hence, function $g \circ f$ is $g \circ f : A \mapsto A$ and function $f \circ g$ is $f \circ g : A \mapsto A$.

Assume f and g are inverse functions.

Then $(g \circ f)(x) = (f \circ g)(x) = I(x) = x$ for all $x \in A$ by definition of inverse function.

Thus, $g \circ f = f \circ g$.

Moreover, every invertible function is bijective, so f and g are bijections.

Since $f : A \mapsto A$ and $g : A \mapsto A$ are bijective functions, then f and g are permutation maps on A, by definition of permutation map. Hence, if f and g are permutation maps such that $g = f^{-1}$ then $g \circ f = f \circ g$. $\square$

Exercise 30. Let $f : \mathbb{Z} \mapsto \mathbb{Q}$ be defined by $f(n) = n/1$.

What can we deduce about f? Is f a function? If so, is f one to one or onto?

Solution. We know f is a binary relation from $\mathbb{Z}$ to $\mathbb{Q}$ since $f \subseteq \mathbb{Z} \times \mathbb{Q}$.

For each $n \in \mathbb{Z}$, $f(n)$ exists, so each element in the domain has at least one image.

Let $a \in \mathbb{Z}$ such that $f(a) = b_1 \wedge f(a) = b_2$ with $b_1, b_2 \in \mathbb{Q}$.

Then $a/1 = b_1$ and $a/1 = b_2$, so $b_1 = b_2$.

Hence, each element in the domain has at most one image.

Since each element in the domain has at least one image and each element in the domain has at most one image, then each element in the domain has exactly one image.

Thus, f is a function, by definition of function.

Let $a_1, a_2 \in \mathbb{Z}$ such that $f(a_1) = f(a_2)$.

Then $a_1/1 = a_2/1$, so $a_1 = a_2$.

Hence $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

Therefore, f is one to one(injective).

There is no integer k such that $f(k) = 1/2$, so f is not onto $\mathbb{Q}$.

Thus, f is not surjective. $\square$

Exercise 31. Let f be the function defined on $\mathbb{R}$ by $f(x) = \frac{x}{x+2}$.

Analyze f.

Solution. Since $x \in \mathbb{R}$, then $f(x) \in \mathbb{R}$ iff $\frac{x}{x+2} \in \mathbb{R}$ iff $x + 2 \neq 0$ iff $x \neq -2$.

Since f is a function, then $f(x) \in \mathbb{R}$ iff $x \neq -2$.

Thus, if $x \neq -2$, then $f(x) \in \mathbb{R}$, so for every $x \in \mathbb{R}$ with $x \neq -2$, $f(x) \in \mathbb{R}$.

Therefore, the domain of f is the set $dom f = \mathbb{R} - \{-2\}$.

We prove the range of f is the set $\mathbb{R} - \{1\}$.

Let $y \in rng f$.

Then $y = f(x) = \frac{x}{x+2}$ and $x \in \mathbb{R} - \{-2\}$, so $x \in \mathbb{R}$ and $x \neq -2$.

Since $x \neq -2$, then $x + 2 \neq 0$, so $\frac{x}{x+2} = y \in \mathbb{R}$.

Suppose $y = 1$.

Then $1 = \frac{x}{x+2}$.

Since $x + 2 \neq 0$, then $x + 2 = x$, so $2 = 0$, a contradiction.

Therefore, $y \neq 1$.

Since $y \in \mathbb{R}$ and $y \neq 1$, then $y \in \mathbb{R} - \{1\}$.

Hence, if $y \in rng f$, then $y \in \mathbb{R} - \{1\}$, so $rng f \subset \mathbb{R} - \{1\}$.

Suppose $t \in \mathbb{R} - \{1\}$.
Then $t \in \mathbb{R}$ and $t \neq 1$.
Since $t \neq 1$, then $t - 1 \neq 0$, so $\frac{-2t}{t-1} \in \mathbb{R}$.
Let $x = \frac{-2t}{t-1}$.
Then $x \in \mathbb{R}$.
Since $t - 1 \neq 0$, then $x(t-1) = -2t$, so $xt - x = -2t$.
Hence, $xt + 2t = x$, so $t(x+2) = x$.
Suppose $x + 2 = 0$.
Then $x = -2$ and $x = t \cdot 0 = 0$, so $x = -2$ and $x = 0$, a contradiction.
Thus, $x + 2 \neq 0$.
Since $t(x+2) = x$ and $x + 2 \neq 0$, then $t = \frac{x}{x+2}$.
Since $x + 2 \neq 0$, then $x \neq -2$.
Since $x \in \mathbb{R}$ and $x \neq -2$, then $x \in \mathbb{R} - \{-2\}$.
Therefore, there exists $x \in \mathbb{R} - \{-2\}$ such that $t = \frac{x}{x+2}$, so $t \in rngf$.
Hence, if $t \in \mathbb{R} - \{1\}$, then $t \in rngf$, so $\mathbb{R} - \{1\} \subset rngf$.
Since $rngf \subset \mathbb{R} - \{1\}$ and $\mathbb{R} - \{1\} \subset rngf$, then $rngf = \mathbb{R} - \{1\}$.
Therefore, $rngf = \mathbb{R} - \{1\}$.

We prove f is injective.
Let $a, b \in \mathbb{R} - \{-2\}$ such that $f(a) = f(b)$.
Then $a \in \mathbb{R}$ and $a \neq -2$ and $b \in \mathbb{R}$ and $b \neq -2$ and $\frac{a}{a+2} = \frac{b}{b+2}$.
Since $a \neq -2$, then $a + 2 \neq 0$, so $a = \frac{(a+2)b}{b+2}$.
Since $b \neq -2$, then $b + 2 \neq 0$, so $a(b+2) = (a+2)b$.
Hence, $ab + 2a = ab + 2b$, so $2a = 2b$.
Therefore, $a = b$, so f is injective.

Since f is injective, then the inverse relation f^{-1} is a function and $domf^{-1} = rngf = \mathbb{R} - \{1\}$.
Since f^{-1} is the inverse of f, then $f(x) = y$ iff $f^{-1}(y) = x$.
Hence, $\frac{x}{x+2} = y$ iff $f^{-1}(y) = x$.
Let $x \in domf$.
Then $x \in \mathbb{R}$ and $x \neq -2$ and $f(x) \in rngf$.
Let $y = f(x)$.
Then $y \in rngf$, so $y = \frac{x}{x+2}$ and $y \neq 1$.
Since $x \neq -2$, then $x + 2 \neq 0$.
Since $y = \frac{x}{x+2}$, then $(x+2)y = x$, so $xy + 2y = x$.
Hence, $2y = x - xy = x(1-y)$.
Since $y \neq 1$, then $1 \neq y$, so $1 - y \neq 0$.
Thus, $\frac{2y}{1-y} = x$.
Since $\frac{x}{x+2} = y$, then $f^{-1}(y) = x = \frac{2y}{1-y}$.
Thus, $f^{-1}(y) = \frac{2y}{1-y}$ and $y \neq 1$.
Therefore, $f^{-1}(x) = \frac{2x}{1-x}$ for each $x \neq 1$. $\square$

Exercise 32. Let π be a permutation defined on the set $\{1, 2, 3\}$ by

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

.

What is the inverse permutation π^{-1}?

Solution. The inverse permutation π^{-1} is given by

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

.

$\square$

Exercise 33. If $f : A \to B$ is an injective map, then $f^{-1} : rngf \to A$ is an injective map.

Proof. Suppose $f : A \to B$ is an injective map.

Then f is an injective function.

Hence, the inverse relation f^{-1} is a function.

To prove $f^{-1} : rngf \to A$ is a map, we must prove $domf^{-1} = rngf$ and $rngf^{-1} \subset A$.

We first prove $domf^{-1} = rngf$.

Either $f = \emptyset$ or $f \neq \emptyset$.

Case 1: Suppose $f = \emptyset$.

Then f is the empty relation, so $domf = \emptyset = rngf$.

Thus, f^{-1} is empty, so $f^{-1} = \emptyset$.

Hence, $domf^{-1} = \emptyset = rngf$.

Case 2: Suppose $f \neq \emptyset$.

Then f is a nonempty relation, so $domf^{-1} = rngf$.

Therefore, in either case, $domf^{-1} = rngf$.

We next prove $rngf^{-1} \subset A$.

Suppose $b \in rngf^{-1}$.

Then there exists $a \in domf^{-1}$ such that $(a, b) \in f^{-1}$.

Since f^{-1} is the inverse of f, then $(b, a) \in f$, so $b \in domf$.

Since $domf = A$, then $b \in A$.

Hence, $rngf^{-1} \subset A$.

Since f^{-1} is a function and $domf^{-1} = rngf$ and $rngf^{-1} \subset A$, then $f^{-1} : rngf \to A$ is a map.

To prove f^{-1} is injective, let $b_1, b_2 \in dom f^{-1}$ such that $f^{-1}(b_1) = f^{-1}(b_2)$.

Since $b_1, b_2 \in dom f^{-1}$ and $dom f^{-1} = rng f$, then $b_1, b_2 \in rng f$, so there exist $a_1, a_2 \in dom f$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$.

Since f^{-1} is the inverse of f, then $f^{-1}(b_1) = a_1$ and $f^{-1}(b_2) = a_2$.

Hence, $a_1 = f^{-1}(b_1) = f^{-1}(b_2) = a_2$, so $a_1 = a_2$.

Since f is a function and $a_1 = a_2$ and $f(a_1) = b_1$ and $f(a_2) = b_2$, then $b_1 = b_2$.

Therefore, f^{-1} is injective. $\square$

Exercise 34. Let A be a set and $X \subset A$.

Then the inclusion map $i : X \to A$ is injective.

Proof. Let $a, b \in X$ such that $i(a) = i(b)$.

Since $i(x) = x$ for all $x \in X$ and $a, b \in X$, then $i(a) = a$ and $i(b) = b$.

Therefore, $a = b$, so i is injective. $\square$

Exercise 35. Restriction of an injective map is injective.

Let $f : A \to B$ be a map and $S \subset A$.

If f is injective, then the restriction $f|_S : S \to B$ is injective.

Proof. Suppose f is injective.

Let $a, b \in S$ such that $f|_S(a) = f|_S(b)$.

Since $a, b \in S$, then $f|_S(a) = f(a)$ and $f|_S(b) = f(b)$.

Thus, $f(a) = f|_S(a) = f|_S(b) = f(b)$.

Since f is injective and $f(a) = f(b)$, then $a = b$, so $f|_S$ is injective. $\square$

Exercise 36. Provide an example of two functions f and g such that $f \circ g = g \circ f$.

Solution. Let $f = \{(1,3),(4,7),(9,8)\}$ and $g = \{(1,9),(3,8),(4,1),(7,3)\}$.

Then f and g are functions and $f \circ g = \{(1,8),(4,3)\} = g \circ f$. $\square$

Exercise 37. Composition of linear functions is a linear function

Let $M, B, N, C \in \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a map defined by $f(x) = Mx + B$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a map defined by $g(x) = Nx + C$.

I. Then $g \circ f$ is a linear map with slope MN and $f \circ g$ is a linear map with slope MN.

II. If $B(N-1) = C(M-1)$, then $f \circ g = g \circ f$.

III. Provide an example of two linear functions f and g such that $f \circ g = g \circ f$.

Proof. We prove I.

Since $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are maps, then $g \circ f : \mathbb{R} \to \mathbb{R}$ is a map and $(g \circ f)(x) = g(f(x))$ for all $x \in \mathbb{R}$ and $f \circ g : \mathbb{R} \to \mathbb{R}$ is a map and $(f \circ g)(x) = f(g(x))$ for all $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$.

Then $(g \circ f)(x) = g(f(x)) = g(Mx + B) = N(Mx + B) + C = NMx + NB + C = MNx + (NB + C)$.

Hence, $(g \circ f)(x) = MNx + (NB + C)$, so $g \circ f$ is a linear map with slope MN.

Let $x \in \mathbb{R}$.

Then $(f \circ g)(x) = f(g(x)) = f(Nx+C) = M(Nx+C)+B = MNx+MC+B = MNx + (MC + B)$.

Hence, $(f \circ g)(x) = MNx + (MC + B)$, so $f \circ g$ is a linear map with slope MN. $\square$

Proof. We prove II.

Suppose $B(N - 1) = C(M - 1)$.

Then $BN - B = CM - C$, so $BN + C = CM + B$.

Thus, $NB + C = MC + B$.

The domain of $g \circ f$ is $\mathbb{R}$ which is the same as the domain of $f \circ g$ and the codomain of $g \circ f$ is $\mathbb{R}$ which is the codomain of $f \circ g$.

Let $x \in \mathbb{R}$.

Then

$$\begin{aligned}(f \circ g)(x) &= MNx + (MC + B)\\ &= MNx + (NB + C)\\ &= (g \circ f)(x).\end{aligned}$$

Thus, $(f \circ g)(x) = (g \circ f)(x)$ for all $x \in \mathbb{R}$, so $f \circ g = g \circ f$. $\square$

Solution. III.

Let $f : \mathbb{R} \to \mathbb{R}$ by given by $f(x) = 3x + 5$.

Let $g : \mathbb{R} \to \mathbb{R}$ by given by $g(x) = 7x + 15$.

Let $x \in \mathbb{R}$.

Then $(g \circ f)(x) = g(f(x)) = g(3x + 5) = 7(3x + 5) + 15 = 21x + 35 + 15 = 21x+50$ and $(f \circ g)(x) = f(g(x)) = f(7x+15) = 3(7x+15)+5 = 21x+45+5 = 21x + 50$.

Thus, $(g \circ f)(x) = 21x + 50 = (f \circ g)(x)$. $\square$

Exercise 38. Let f and g be functions such that $f \neq \emptyset$ and $g \neq \emptyset$.

If $rngf \cap domg = \emptyset$, then $g \circ f = \emptyset$.

Proof. Suppose $rngf \cap domg = \emptyset$.

Since f and g are functions, then $g \circ f$ is a function.

We prove $g \circ f = \emptyset$ by contradiction.

Suppose $g \circ f \neq \emptyset$.

Then there exists an ordered pair $(a, b) \in g \circ f$.

Since $f \neq \emptyset$ and $g \neq \emptyset$, then there exists c such that $(a, c) \in f$ and $(c, b) \in g$.

Since $(a, c) \in f$, then $c \in rngf$.

Since $(c, b) \in g$, then $c \in domg$.

Thus, $c \in rngf$ and $c \in domg$, so $c \in rngf \cap domg$.

Hence, $rngf \cap domg \neq \emptyset$.

Thus, we have $rngf \cap domg = \emptyset$ and $rngf \cap domg \neq \emptyset$, a contradiction.

Therefore, $g \circ f = \emptyset$. $\square$

Exercise 39. Let f and g be functions such that $rngf \subset domg$.
Let $E \subset domf$.
Then $(g \circ f)(E) = g(f(E))$.

Proof. We prove $g(f(E)) \subset (g \circ f)(E)$.
Let $b \in g(f(E))$.
Then $b = g(a)$ for some $a \in f(E)$.
Since $a \in f(E)$, then $a = f(x)$ for some $x \in E$.
Thus, $b = g(a) = g(f(x)) = (g \circ f)(x)$ for some $x \in E$, so $b \in (g \circ f)(E)$.
Hence, $g(f(E)) \subset (g \circ f)(E)$.

We prove $(g \circ f)(E) \subset g(f(E))$.
Let $x \in (g \circ f)(E)$.
Then $x = (g \circ f)(y)$ for some $y \in E$.
Since $y \in E$ and $E \subset domf$, then $y \in domf$, so $f(y) \in f(E)$.
Since $E \subset domf$, then $f(E) \subset rngf$, so $f(y) \in rngf$.
Since $rngf \subset domg$, then $f(y) \in domg$.
Thus, $g(f(y)) \in g(f(E))$, so $(g \circ f)(y) \in g(f(E))$.
Hence, $x \in g(f(E))$, so $(g \circ f)(E) \subset g(f(E))$.
Since $g(f(E)) \subset (g \circ f)(E)$ and $(g \circ f)(E) \subset g(f(E))$, then $(g \circ f)(E) = g(f(E))$. □

Exercise 40. Let $f : A \to B$ be a map.
Let I_A be the identity map on A and I_B be the identity map on B.
1. If f is injective, then $f^{-1} \circ f = I_A$.
2. Let $X \subset A$.
Let i be the inclusion map of X into A.
Let $f|_X$ be the restriction of f to X.
Then $f \circ i_X = f|_X$.

Proof. We prove 1.
Suppose f is injective.
Then the inverse relation f^{-1} is a function.
Since f is a function, then $f^{-1} \circ f$ is a function and $domf^{-1} \circ f = \{x \in domf : f(x) \in domf^{-1}\}$ and $(f^{-1} \circ f)(x) = f^{-1}(f(x))$ for all $x \in domf^{-1} \circ f$.
We prove $domf^{-1} \circ f = A$.
Since $domf = A$, then $domf^{-1} \circ f = \{x \in A : f(x) \in domf^{-1}\}$, so $domf^{-1} \circ f \subset A$.
Let $a \in A$.
Then $a \in domf$, so $f(a) \in rngf$.
Since $rngf = domf^{-1}$, then $f(a) \in domf^{-1}$.
Since $a \in domf$ and $f(a) \in domf^{-1}$, then $a \in domf^{-1} \circ f$, so $A \subset domf^{-1} \circ f$.
Thus, $domf^{-1} \circ f \subset A$ and $A \subset domf^{-1} \circ f$, so $domf^{-1} \circ f = A$.
Hence, $(f^{-1} \circ f)(x) = f^{-1}(f(x))$ for all $x \in A$.
Let $x \in A$.
Let $b = (f^{-1} \circ f)(x) = f^{-1}(f(x))$.

Since f^{-1} is the inverse of f, then $f^{-1}(f(x)) = b$ iff $f(b) = f(x)$.
Thus, $f(b) = f(x)$.
Since f is injective, then $b = x$.
Hence, $I_A(x) = x = b = (f^{-1} \circ f)(x)$, so $I_A(x) = (f^{-1} \circ f)(x)$ for all $x \in A$.
Therefore, $I_A = f^{-1} \circ f$. □

Proof. We prove 2.
Since i is the inclusion map of X into A, then $i : X \to A$ is a map defined by $i(x) = x$ for all $x \in X$.
Since $f|_X$ is the restriction of f to X, then $f|_X : X \to B$ is a map defined by $f|_X(x) = f(x)$ for all $x \in X$.
Since $i : X \to A$ is a map and $f : A \to B$ is a map, then $f \circ i : X \to B$ is a map and $(f \circ i)(x) = f(i(x))$ for all $x \in X$.
We prove the maps $f \circ i$ and $f|_X$ are equal.
Observe that $dom f \circ i = X = dom f|_X$ and the codomain of $f \circ i$ is B which is the codomain of $f|_X$.
Let $x \in X$.
Then $(f \circ i)(x) = f(i(x)) = f(x) = f|_X(x)$, so $(f \circ i)(x) = f|_X(x)$ for all $x \in X$.
Therefore, $f \circ i = f|_X$. □

Exercise 41. i. Give an example of maps $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is injective, but g is not injective.
ii. Give an example of maps $f : A \to B$ and $g : B \to C$ such that f is injective, but $g \circ f$ is not injective.

Solution. Let $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 8, 9\}$ and $C = \{5, 6, 7, 10\}$.
i. Here is an example such that $g \circ f$ is injective, but g is not injective.
Let $f : A \to B$ be a map given by $f = \{(1, 3), (2, 5), (3, 4), (7, 8)\}$.
Let $g : B \to C$ be a map given by $g = \{(3, 6), (4, 7), (5, 10), (8, 5), (9, 6)\}$.
Since $g(3) = 6$ and $g(9) = 6$, then g is not injective.
Observe that $g \circ f = \{(1, 6), (2, 10), (3, 7), (7, 5)\}$ is injective.
ii. Here is an example such that f is injective, but $g \circ f$ is not injective.
Let $f : A \to B$ be a map given by $f = \{(1, 3), (2, 5), (3, 4), (7, 8)\}$.
Then f is injective.
Let $g : B \to C$ be a map given by $g = \{(3, 6), (4, 7), (5, 10), (8, 6), (9, 2)\}$.
Then $g \circ f = \{(1, 6), (2, 10), (3, 7), (7, 6)\}$ is not injective since $(g \circ f)(1) = 6$ and $(g \circ f)(7) = 6$. □

Exercise 42. Let $f : A \to B$ and $g : C \to D$ be maps.
If $rng f \subset C$, then $dom(g \circ f) = A$ and $rng(g \circ f) \subset D$.

Proof. Suppose $rng f \subset C$.
Since $f : A \to B$ and $g : C \to D$ are maps, then f and g are functions, so $g \circ f$ is a function and $dom g \circ f = \{x \in dom f : f(x) \in dom g\}$ and $(g \circ f)(x) = g(f(x))$ for all $x \in dom g \circ f$.
Since $dom f = A$ and $dom g = C$, then $dom g \circ f = \{x \in A : f(x) \in C\}$, so $dom g \circ f \subset A$.

Let $a \in A$.

Then $a \in domf$, so $f(a) \in rngf$.

Since $rngf \subset C$, then $f(a) \in C$.

Since $a \in A$ and $f(a) \in C$, then $a \in domg \circ f$, so $A \subset domg \circ f$.

Since $domg \circ f \subset A$ and $A \subset domg \circ f$, then $dom(g \circ f) = A$, so $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

Let $y \in rng(g \circ f)$.

Then there exists $x \in domg \circ f$ such that $(g \circ f)(x) = y$.

Thus, there exists $x \in A$ such that $g(f(x)) = y$.

Since $x \in A$, then $f(x) \in rngf$.

Since $rngf \subset C$, then $f(x) \in C$.

Since $g : C \to D$ is a map, then $g(f(x)) \in D$, so $y \in D$.

Therefore, $rng(g \circ f) \subset D$, as desired. $\square$

Exercise 43. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be maps.

Let $f \vee g$ be a map from $\mathbb{R}$ into $\mathbb{R}$ defined by $(f \vee g)(x) = \max\{f(x), g(x)\}$ for all $x \in \mathbb{R}$.

Let $f \wedge g$ be a map from $\mathbb{R}$ into $\mathbb{R}$ defined by $(f \wedge g)(x) = \min\{f(x), g(x)\}$ for all $x \in \mathbb{R}$.

Provide an example to show that f and g can be one to one, yet $f \vee g$ is not one to one.

Provide an example to show that f and g can be one to one, yet $f \wedge g$ is not one to one.

Solution. Let $f(x) = e^x$ and $g(x) = e^{-x}$ for all $x \in \mathbb{R}$.

Then f and g are one to one functions.

Since $e > 1$, then $1 > \frac{1}{e}$, so $e > 1 > \frac{1}{e}$.

Thus, $e > \frac{1}{e}$, so $\frac{1}{e} < e$.

Since $f(-1) = e^{-1} = \frac{1}{e} < e = e^1 = e^{-(-1)} = g(-1)$, then $f(-1) < g(-1)$, so $(f \vee g)(-1) = \max\{f(-1), g(-1)\} = g(-1) = e$.

Since $f(1) = e > \frac{1}{e} = e^{-1} = g(1)$, then $f(1) > g(1)$, so $(f \vee g)(1) = \max\{f(1), g(1)\} = f(1) = e$.

Thus, $(f \vee g)(-1) = e = (f \vee g)(1)$, so $f \vee g$ is not one to one.

Since $f(-1) = e^{-1} = \frac{1}{e} < e = e^1 = e^{-(-1)} = g(-1)$, then $f(-1) < g(-1)$, so $(f \wedge g)(-1) = \min\{f(-1), g(-1)\} = f(-1) = \frac{1}{e}$.

Since $f(1) = e > \frac{1}{e} = e^{-1} = g(1)$, then $f(1) > g(1)$, so $(f \wedge g)(1) = \min\{f(1), g(1)\} = g(1) = \frac{1}{e}$.

Thus, $(f \wedge g)(-1) = \frac{1}{e} = (f \wedge g)(1)$, so $f \wedge g$ is not one to one. $\square$

Exercise 44. Let $f : A \to B$ and $g : C \to D$ be maps.

a. If $A \cap C = \emptyset$, then $f \cup g : A \cup C \to B \cup D$ is a map.

b. If $A \cap C = B \cap D = \emptyset$ and f and g are injective, then $f \cup g$ is injective.

c. If $A \cap C = \emptyset$, then $(f \cup g)|_A = f$ and $(f \cup g)|_C = g$.

Proof. We prove a.

Suppose $A \cap C = \emptyset$.

To prove $f \cup g : A \cup C \to B \cup D$ is a map, we must prove $f \cup g$ is a function and $dom(f \cup g) = A \cup C$ and $rng(f \cup g) \subset B \cup D$.

Since $f : A \to B$ and $g : C \to D$ are maps, then f and g are functions, so f and g are relations.
Thus, $f \cup g$ is a relation.
To prove $f \cup g$ is a function, let $(a, b) \in f \cup g$ and $(a, b') \in f \cup g$.
Then either $(a, b) \in f$ or $(a, b) \in g$, and either $(a, b') \in f$ or $(a, b') \in g$.
Hence, either $(a, b) \in f$ and $(a, b') \in f$ or $(a, b) \in f$ and $(a, b') \in g$ or $(a, b) \in g$ and $(a, b') \in f$ or $(a, b) \in g$ and $(a, b') \in g$.
Suppose $(a, b) \in f$ and $(a, b') \in g$ or $(a, b) \in g$ and $(a, b') \in f$.
Then $a \in domf$ and $a \in domg$, so $a \in A$ and $a \in C$.
Hence, $a \in A \cap C$, so $A \cap C \neq \emptyset$.
This contradicts the assumption $A \cap C = \emptyset$.
Thus, it cannot be the case that $(a, b) \in f$ and $(a, b') \in g$ or $(a, b) \in g$ and $(a, b') \in f$.
Hence, either $(a, b) \in f$ and $(a, b') \in f$ or $(a, b) \in g$ and $(a, b') \in g$.
We consider these cases separately.
Case 1: Suppose $(a, b) \in f$ and $(a, b') \in f$.
Since f is a function, then $b = b'$.
Case 2: Suppose $(a, b) \in g$ and $(a, b') \in g$.
Since g is a function, then $b = b'$.
Thus, in all cases, $b = b'$, so $f \cup g$ is a function, as desired.

We prove $dom(f \cup g) = A \cup C$.
Let $x \in dom(f \cup g)$.
Then there exists y such that $(x, y) \in f \cup g$, so either $(x, y) \in f$ or $(x, y) \in g$.
Hence, either $x \in domf$ or $x \in domg$, so $x \in domf \cup domg$.
Thus, $x \in A \cup C$, so $dom(f \cup g) \subset A \cup C$.
Let $y \in A \cup C$.
Then either $y \in A$ or $y \in C$, so either $y \in domf$ or $y \in domg$.
We consider these cases separately.
Case 1: Suppose $y \in domf$.
Then there exists z_1 such that $(y, z_1) \in f$.
Since $f \subset f \cup g$, then $(y, z_1) \in f \cup g$.
Thus, there exists z_1 such that $(y, z_1) \in f \cup g$.
Case 2: Suppose $y \in domg$.
Then there exists z_2 such that $(y, z_2) \in g$.
Since $g \subset f \cup g$, then $(y, z_2) \in f \cup g$.
Thus, there exists z_2 such that $(y, z_2) \in f \cup g$.
Hence, in all cases, there exists z such that $(y, z) \in f \cup g$, so $y \in dom(f \cup g)$.
Hence, $A \cup C \subset dom(f \cup g)$.
Since $dom(f \cup g) \subset A \cup C$ and $A \cup C \subset dom(f \cup g)$, then $dom(f \cup g) = A \cup C$, as desired.

We prove $rng(f \cup g) \subset B \cup D$.
Let $b \in rng(f \cup g)$.
Then there exists a such that $(a, b) \in f \cup g$, so either $(a, b) \in f$ or $(a, b) \in g$.

We consider these cases separately.
Case 1: Suppose $(a, b) \in f$.
Since $f : A \to B$ is a map, then $b \in B$.
Case 2: Suppose $(a, b) \in g$.
Since $g : C \to D$ is a map, then $b \in D$.
Thus, either $b \in B$ or $b \in D$, so $b \in B \cup D$.
Therefore, $rng(f \cup g) \subset B \cup D$, as desired. $\square$

Proof. We prove b.
Suppose $A \cap C = B \cap D = \emptyset$ and f and g are injective.
We must prove $f \cup g$ is injective.
Since $A \cap C = \emptyset$, then $f \cup g : A \cup C \to B \cup D$ is a map.
To prove $f \cup g$ is injective, let $a, b \in A \cup C$ such that $(f \cup g)(a) = (f \cup g)(b)$.
We must prove $a = b$.
Let $y = (f \cup g)(a)$.
Then $(a, y) \in f \cup g$, so either $(a, y) \in f$ or $(a, y) \in g$.
Since $a \in A \cup C$, then either $a \in A$ or $a \in C$.
We consider these cases separately.
Case 1: Suppose $a \in A$.
Since $A \cap C = \emptyset$, then $a \notin C$, so $a \notin domg$.
Hence, $(a, y) \notin g$.
Since either $(a, y) \in f$ or $(a, y) \in g$, then this implies $(a, y) \in f$.
Thus, $f(a) = y = (f \cup g)(a) = (f \cup g)(b)$, so $y = (f \cup g)(b)$.
Hence, $(b, y) \in f \cup g$, so either $(b, y) \in f$ or $(b, y) \in g$.
Since $b \in A \cup C$, then either $b \in A$ or $b \in C$.
Case 1a: Suppose $b \in C$.
Since $A \cap C = \emptyset$, then $b \notin A$, so $b \notin domf$.
Thus, $(b, y) \notin f$.
Since either $(b, y) \in f$ or $(b, y) \in g$, then this implies $(b, y) \in g$.
Since $(a, y) \in f$ and $f : A \to B$ is a map, then $y \in B$.
Since $(b, y) \in g$ and $g : C \to D$ is a map, then $y \in D$.
Thus, $y \in B$ and $y \in D$, so $y \in B \cap D$.
Hence, $B \cap D \neq \emptyset$.
But, this contradicts the hypothesis $B \cap D = \emptyset$.
Therefore, it is not possible that $a \in A$ and $b \in C$.
Case 1b: Suppose $b \in A$.
Since $A \cap C = \emptyset$, then $b \notin C$, so $b \notin domg$.
Thus, $(b, y) \notin g$.
Since either $(b, y) \in f$ or $(b, y) \in g$, then this implies $(b, y) \in f$.
Hence, $f(b) = y = f(a)$.
Since $f(a) = f(b)$ and f is injective, then $a = b$.
Case 2: Suppose $a \in C$.
Since $A \cap C = \emptyset$, then $a \notin A$, so $a \notin domf$.
Hence, $(a, y) \notin f$.
Since either $(a, y) \in f$ or $(a, y) \in g$, then this implies $(a, y) \in g$.
Thus, $g(a) = y = (f \cup g)(a) = (f \cup g)(b)$, so $y = (f \cup g)(b)$.

Hence, $(b,y) \in f \cup g$, so either $(b,y) \in f$ or $(b,y) \in g$.
Since $b \in A \cup C$, then either $b \in A$ or $b \in C$.
Case 2a: Suppose $b \in A$.
Since $A \cap C = \emptyset$, then $b \notin C$, so $b \notin domg$.
Thus, $(b,y) \notin g$.
Since either $(b,y) \in f$ or $(b,y) \in g$, then this implies $(b,y) \in f$.
Since $(a,y) \in g$ and $g : C \to D$ is a map, then $y \in D$.
Since $(b,y) \in f$ and $f : A \to B$ is a map, then $y \in B$.
Thus, $y \in B$ and $y \in D$, so $y \in B \cap D$.
Hence, $B \cap D \neq \emptyset$.
But, this contradicts the hypothesis $B \cap D = \emptyset$.
Therefore, it is not possible that $a \in C$ and $b \in A$.
Case 2b: Suppose $b \in C$.
Since $A \cap C = \emptyset$, then $b \notin A$, so $b \notin domf$.
Thus, $(b,y) \notin f$.
Since either $(b,y) \in f$ or $(b,y) \in g$, then this implies $(b,y) \in g$.
Hence, $g(b) = y = g(a)$.
Since $g(a) = g(b)$ and g is injective, then $a = b$.
Thus, we conclude $a = b$, so $f \cup g$ is injective, as desired. □

Proof. We prove c.
Suppose $A \cap C = \emptyset$.
Then $f \cup g : A \cup C \to B \cup D$ is a map.
Since $f \cup g : A \cup C \to B \cup D$ is a map and $A \subset A \cup C$, then the restriction $(f \cup g)|_A : A \to B \cup D$ is a map, so $(f \cup g)|_A$ is a function.

We prove $(f \cup g)|_A = f$.
Since the domain of $(f \cup g)|_A$ is A and the domain of f is A, then the functions $(f \cup g)|_A$ and f have the same domain.
Let $x \in A$.
Let $y = (f \cup g)|_A(x)$.
Since $x \in A$, then $(f \cup g)|_A(x) = (f \cup g)(x)$, so $y = (f \cup g)(x)$.
Hence, $(x,y) \in f \cup g$, so either $(x,y) \in f$ or $(x,y) \in g$.
Since $x \in A$ and $A \cap C = \emptyset$, then $x \notin C$, so $x \notin domg$.
Thus, $(x,y) \notin g$, so $(x,y) \in f$.
Since f is a function, then $f(x) = y$.
Hence, $(f \cup g)|_A(x) = (f \cup g)(x) = y = f(x)$, so $(f \cup g)|_A = f$.

Since $f \cup g : A \cup C \to B \cup D$ is a map and $C \subset A \cup C$, then the restriction $(f \cup g)|_C : C \to B \cup D$ is a map, so $(f \cup g)|_C$ is a function.

We prove $(f \cup g)|_C = g$.
Since the domain of $(f \cup g)|_C$ is C and the domain of g is C, then the functions $(f \cup g)|_C$ and g have the same domain.
Let $x \in C$.

Let $y = (f \cup g)|_C(x)$.
Since $x \in C$, then $(f \cup g)|_C(x) = (f \cup g)(x)$, so $y = (f \cup g)(x)$.
Hence, $(x, y) \in f \cup g$, so either $(x, y) \in f$ or $(x, y) \in g$.
Since $x \in C$ and $A \cap C = \emptyset$, then $x \notin A$, so $x \notin dom f$.
Thus, $(x, y) \notin f$, so $(x, y) \in g$.
Since g is a function, then $g(x) = y$.
Hence, $(f \cup g)|_C(x) = (f \cup g)(x) = y = g(x)$, so $(f \cup g)|_C = g$. $\square$

Exercise 45. A map $g : C \to B$ is an extension of a map $f : A \to B$ iff $f \subset g$.
a. If a map $g : C \to B$ is an extension of a map $f : A \to B$, then $A \subset C$.
b. If $f : A \to B$ and $g : C \to D$ are maps such that $A \cap C = \emptyset$, then $f \cup g : A \cup C \to B \cup D$ is an extension of both f and g.
c. If a map $g : C \to B$ is an extension of a map $f : A \to B$, then $g|_A = f$.
d. Find an extension of the map $f : \mathbb{R} - \{-5\} \to \mathbb{R}$ given by $f(x) = \frac{x^2-25}{x+5}$ whose domain is $\mathbb{R}$ and that is continuous on $\mathbb{R}$.
e. Let $f : \mathbb{R} \to \mathbb{C}$ be a map defined by $f(x) = e^x$.
Let $z = x + yi$ for $x, y \in \mathbb{R}$.
Let $g : \mathbb{C} \to \mathbb{C}$ be defined by $g(z) = e^x(\cos y + i \sin y)$ for all $z \in C$.
Then g is an extension of f.

Proof. We prove a.
Suppose a map $g : C \to B$ is an extension of a map $f : A \to B$.
Then $f \subset g$.
Let $a \in A$.
Since $A = dom f$, then $a \in dom f$, so $(a, f(a)) \in f$.
Since $f \subset g$, then $(a, f(a)) \in g$, so $a \in dom g$.
Since $dom g = C$, then $a \in C$, so $A \subset C$, as desired. $\square$

Proof. We prove b.
Suppose $f : A \to B$ and $g : C \to D$ are maps such that $A \cap C = \emptyset$.
Then, by a previous exercise, $f \cup g : A \cup C \to B \cup D$ is a map.
Since $f : A \to B$ is a map, then f is a function, so f is a relation.
Hence, f is a set.
Since $g : C \to D$ is a map, then g is a function, so g is a relation.
Hence, g is a set.
Since $f \subset f \cup g$, then $f \cup g$ is an extension of f.
Since $g \subset f \cup g$, then $f \cup g$ is an extension of g. $\square$

Proof. We prove c.
Suppose a map $g : C \to B$ is an extension of a map $f : A \to B$.
Then $f \subset g$ and $A \subset C$.
Since $f : A \to B$ is a map, then f is a function.
Since $g : C \to B$ is a map, then g is a function.
Since $g : C \to B$ is a map and $A \subset C$, then the restriction $g|_A : A \to B$ is a map.
Observe that dom $\mathrm{g}|_A = A = dom f$.
Let $a \in A$ be arbitrary.

Then $(g|_A)(a) = g(a)$.
Since $a \in A$ and f is a function, then $(a, f(a)) \in f$.
Since $f \subset g$, then $(a, f(a)) \in g$.
Since g is a function, then $g(a) = f(a)$.
Thus, $(g|_A)(a) = g(a) = f(a)$, so $(g|_A)(a) = f(a)$ for all $a \in A$.
Therefore, $g|_A = f$, as desired. □

Proof. We solve d.
Let $g : \mathbb{R} \to \mathbb{R}$ be a map defined by $g(x) = x - 5$.
Then g is a function and $domg = \mathbb{R}$.
Since g is a polynomial function, then g is continuous, so g is continuous on $\mathbb{R}$.
We prove g is an extension of f.
Let $x \in domf$ be arbitrary.
Then $(x, f(x)) \in f$.
Since $x \in domf$ and $domf = \mathbb{R} - \{-5\}$, then $x \in \mathbb{R} - \{-5\}$, so $x \in \mathbb{R}$ and $x \neq -5$.
Since $x + 5 = 0$ iff $x = -5$ and $x \neq -5$, then $x + 5 \neq 0$.
Thus, $g(x) = x - 5 = (x - 5) \cdot \frac{x+5}{x+5} = \frac{x^2-25}{x+5} = f(x)$, so $g(x) = f(x)$.
Since g is a function, then $(x, g(x)) \in g$, so $(x, f(x)) \in g$.
Hence, $f \subset g$.
Therefore, g is an extension of f. □

Proof. We prove e.
Let $r \in domf$.
Then $r \in \mathbb{R}$ and $f(r) = e^r$.
Since f is a function, then $(r, f(r)) \in f$.
Since $r \in \mathbb{R}$, then $r = r + 0 = r + i \cdot 0$, so $g(r) = e^r(\cos 0 + i \sin 0) = e^r(1 + 0) = e^r \cdot 1 = e^r = f(r)$.
Thus, $g(r) = f(r)$, so $r \in domg$.
Since g is a function, then $(r, g(r)) \in g$, so $(r, f(r)) \in g$.
Thus, $(r, f(r)) \in f$ implies $(r, f(r)) \in g$, so $f \subset g$.
Therefore, g is an extension of f. □

Exercise 46. a. Give an example of maps $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is surjective, but f is not surjective.
b. Give an example of maps $f : A \to B$ and $g : B \to C$ such that g is surjective, but $g \circ f$ is not surjective.
c. Give an example of maps $f : A \to B$ and $g : B \to C$ such that f is surjective, but $g \circ f$ is not surjective.

Solution. Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8, 9\}$ and $C = \{10, 11, 12, 13\}$.
a. Let $f = \{(1, 5), (2, 6), (3, 7), (4, 8)\}$ and $g = \{(5, 10), (6, 11), (7, 12), (8, 13), (9, 10)\}$.
Then f and g are functions.
Since f is a function and $domf = A$ and $rngf = \{5, 6, 7, 8\} \subset B$, then $f : A \to B$ is a map.

Since g is a function and $domg = B$ and $rngg = C \subset C$, then $g : B \to C$ is a map.

Since $f : A \to B$ and $g : B \to C$ are maps, then $g \circ f : A \to C$ is a map and $g \circ f = \{(1, 10), (2, 11), (3, 12), (4, 13)\}$.

Since $rng(g \circ f) = C$, then $g \circ f$ is surjective.

Since $9 \in B$, but $9 \notin rngf$, then $rngf \neq B$, so f is not surjective.

We know that if $g \circ f$ is surjective, then g is surjective.

Since $g \circ f$ is surjective, then g is surjective.

Since $rngg = C$, then g is surjective, as predicted by theory.

b. Let $f = \{(1, 5), (2, 6), (3, 7), (4, 8)\}$ and $g = \{(5, 10), (6, 11), (7, 12), (8, 12), (9, 13)\}$.

Then f and g are functions.

Since f is a function and $domf = A$ and $rngf = \{5, 6, 7, 8\} \subset B$, then $f : A \to B$ is a map.

Since g is a function and $domg = B$ and $rngg = C \subset C$, then $g : B \to C$ is a map.

Since $f : A \to B$ and $g : B \to C$ are maps, then $g \circ f : A \to C$ is a map and $g \circ f = \{(1, 10), (2, 11), (3, 12), (4, 12)\}$.

Since $rngg = C$, then g is surjective.

Observe that $rng(g \circ f) = \{10, 11, 12\}$.

Since $13 \in C$, but $13 \notin rng(g \circ f)$, then $rng(g \circ f) \neq C$, so $g \circ f$ is not surjective.

We know that if f and g are surjective, then $g \circ f$ is surjective, so if $g \circ f$ is not surjective, then either f is not surjective or g is not surjective.

Since $g \circ f$ is not surjective, then either f is not surjective or g is not surjective.

Since g is surjective, then this implies f is not surjective.

Since $9 \in B$, but $9 \notin rngf$, then $rngg \neq B$, so f is not surjective, as predicted by theory.

Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$ and $C = \{10, 11, 12, 13\}$.

c. Let $f = \{(1, 5), (2, 6), (3, 7), (4, 8)\}$ and $g = \{(5, 10), (6, 11), (7, 12), (8, 12)\}$.

Then f and g are functions.

Since f is a function and $domf = A$ and $rngf = B \subset B$, then $f : A \to B$ is a map.

Since g is a function and $domg = B$ and $rngg = \{10, 11, 12\} \subset C$, then $g : B \to C$ is a map.

Since $f : A \to B$ and $g : B \to C$ are maps, then $g \circ f : A \to C$ is a map and $g \circ f = \{(1, 10), (2, 11), (3, 12), (4, 12)\}$.

Since $rngf = B$, then f is surjective.

Observe that $rng(g \circ f) = \{10, 11, 12\}$.

Since $13 \in C$, but $13 \notin rng(g \circ f)$, then $rng(g \circ f) \neq C$, so $g \circ f$ is not surjective.

We know that if f and g are surjective, then $g \circ f$ is surjective, so if $g \circ f$ is not surjective, then either f is not surjective or g is not surjective.

Since $g \circ f$ is not surjective, then either f is not surjective or g is not surjective.

Since f is surjective, then this implies g is not surjective.

Since $13 \in C$, but $13 \notin rngg$, then $rngg \neq C$, so g is not surjective, as predicted by theory. $\square$

Exercise 47. Let $f : A \to B$ and $g : B \to C$ be maps such that f is surjective.

Then $g \circ f : A \to C$ is a surjection iff g is surjective.

Proof. Since $f : A \to B$ and $g : B \to C$ are maps, then $g \circ f : A \to C$ is a map.

We must prove $g \circ f$ is surjective iff g is surjective.

Suppose $g \circ f$ is surjective.

Then g is surjective.

Conversely, suppose g is surjective.

Since f is surjective and g is surjective, then the composition $g \circ f$ is surjective. $\square$

Exercise 48. Let $f : A \to B$ and $g : B \to C$ be maps such that g is injective.

Then $g \circ f : A \to C$ is an injection iff f is injective.

Proof. Since $f : A \to B$ and $g : B \to C$ are maps, then $g \circ f : A \to C$ is a map.

We must prove $g \circ f$ is injective iff f is injective.

Suppose $g \circ f$ is injective.

Then f is injective.

Conversely, suppose f is injective.

Since f is injective and g is injective, then the composition $g \circ f$ is injective. $\square$

Exercise 49. Give an example of maps $f : A \to B$ and $g : B \to C$ such that:

1. $g \circ f : A \to C$ is a bijection, but f is not onto and g is not one to one.
2. f is bijective, but $g \circ f$ is not bijective.
3. g is bijective, but $g \circ f$ is not bijective.

Solution. 1. We give an example such that $g \circ f$ is bijective and f is not onto and g is not one to one.

Let $A = \{1, 2, 3, 4\}$ and $B = \{10, 20, 30, 40, 50\}$ and $C = \{3, 5, 7, 9\}$.

Let $f = \{(1, 10), (2, 20), (3, 30), (4, 40)\}$.

Let $g = \{(10, 3), (20, 5), (30, 7), (40, 9), (50, 9)\}$.

Since f is a function and $domf = A$ and $rngf = \{10, 20, 30, 40\} \subset B$, then $f : A \to B$ is a map.

Since $50 \in B$, but $50 \notin rngf$, then $rngf \neq B$, so f is not onto B.

Since g is a function and $domg = B$ and $rngg = C \subset C$, then $g : B \to C$ is a map.

Since $g(40) = 9 = g(50)$, but $40 \neq 50$, then g is not one to one.

Since $f : A \to B$ is a map and $g : B \to C$ is a map, then the composition $g \circ f : A \to C$ is a map and $g \circ f = \{(1,3),(2,5),(3,7),(4,9)\}$.

Clearly, $g \circ f$ is one to one and onto, so $g \circ f$ is bijective. $\square$

Solution. 2. We give an example such that f is bijective and $g \circ f$ is not bijective:

Let $A = \{1,2,3,4\}$ and $B = \{10,20,30,40\}$ and $C = \{3,5,7,9\}$.

Let $f = \{(1,10),(2,20),(3,30),(4,40)\}$.

Let $g = \{(10,3),(20,5),(30,7),(40,7)\}$.

Since f is a function and $domf = A$ and $rngf = B \subset B$, then $f : A \to B$ is a map.

Clearly, f is one to one.

Since $rngf = B$, then f is onto, so f is bijective.

Since g is a function and $domg = B$ and $rngg = \{3,5,7\} \subset C$, then $g : B \to C$ is a map.

Since $f : A \to B$ is a map and $g : B \to C$ is a map, then the composition $g \circ f : A \to C$ is a map and $g \circ f = \{(1,3),(2,5),(3,7),(4,7)\}$.

Since $(g \circ f)(3) = 7 = (g \circ f)(4)$, but $3 \neq 4$, then $g \circ f$ is not one to one, so $g \circ f$ is not bijective. $\square$

Solution. 3. We give an example such that g is bijective and $g \circ f$ is not bijective:

Let $A = \{1,2,3,4\}$ and $B = \{10,20,30,40\}$ and $C = \{3,5,7,9\}$.

Let $f = \{(1,10),(2,20),(3,30),(4,30)\}$.

Let $g = \{(10,3),(20,5),(30,7),(40,9)\}$.

Since f is a function and $domf = A$ and $rngf = \{10,20,30\} \subset B$, then $f : A \to B$ is a map.

Since g is a function and $domg = B$ and $rngg = C \subset C$, then $g : B \to C$ is a map.

Clearly, g is one to one.

Since $rngg = C$, then g is onto.

Hence, g is bijective.

Since $f : A \to B$ is a map and $g : B \to C$ is a map, then the composition $g \circ f : A \to C$ is a map and $g \circ f = \{(1,3),(2,5),(3,7),(4,7)\}$.

Since $(g \circ f)(3) = 7 = (g \circ f)(4)$, but $3 \neq 4$, then $g \circ f$ is not one to one, so $g \circ f$ is not bijective. $\square$

Exercise 50. Let $f : X \to Y$ and $g : Y \to Z$ be maps.

If $g \circ f$ is injective, then f is injective.

Prove this statement using the left cancellation property of injective maps.

Proof. Suppose $g \circ f$ is injective.

To prove f is injective using the left cancellation property of injective maps, let W be a set and let $h : W \to X$ and $k : W \to X$ be maps such that $f \circ h = f \circ k$.

We must prove $h = k$.

Since $g \circ f$ is injective, then by the left cancellation property of injective maps, if $h : W \to X$ and $k : W \to X$ are maps such that $(g \circ f) \circ h = (g \circ f) \circ k$, then $h = k$.

Since f, g, h are functions, then $(g \circ f) \circ h = g \circ (f \circ h) = g \circ (f \circ k) = (g \circ f) \circ k$.

Since $h : W \to X$ and $k : W \to X$ are maps and $(g \circ f) \circ h = (g \circ f) \circ k$, then we conclude $h = k$, as desired. □

Exercise 51. Let $f : X \to Y$ and $g : Y \to Z$ be maps.

If $g \circ f$ is surjective, then g is surjective.

Prove this statement using the right cancellation property of surjective maps.

Proof. Suppose $g \circ f$ is surjective.

To prove g is surjective using the right cancellation property of surjective maps, let W be a set and let $h : Z \to W$ and $k : Z \to W$ be maps such that $h \circ g = k \circ g$.

We must prove $h = k$.

Since $g \circ f$ is surjective, then by the right cancellation property of surjective maps, if $h : Z \to W$ and $k : Z \to W$ are maps such that $h \circ (g \circ f) = k \circ (g \circ f)$, then $h = k$.

Since f, g, h are functions, then $h \circ (g \circ f) = (h \circ g) \circ f = (k \circ g) \circ f = k \circ (g \circ f)$.

Since $h : Z \to W$ and $k : Z \to W$ are maps and $h \circ (g \circ f) = k \circ (g \circ f)$, then we conclude $h = k$, as desired. □

Exercise 52. Let $f : X \to Y$ and $g : Y \to Z$ be maps.

If f and g are injective, then $g \circ f$ is injective.

Prove this statement using the left cancellation property of injective maps.

Proof. Suppose f and g are injective.

Since $f : X \to Y$ and $g : Y \to Z$ are maps, then $g \circ f : X \to Z$ is a map.

To prove $g \circ f$ is injective using the left cancellation property of injective maps, let W be a set and let $h : W \to X$ and $k : W \to X$ be maps such that $(g \circ f) \circ h = (g \circ f) \circ k$.

We must prove $h = k$.

Since g is injective, then by the left cancellation property of injective maps, if $f \circ h : W \to Y$ and $f \circ k : W \to Y$ are maps such that $g \circ (f \circ h) = g \circ (f \circ k)$, then $f \circ h = f \circ k$.

Since $h : W \to X$ and $f : X \to Y$ are maps, then $f \circ h : W \to Y$ is a map.

Since $k : W \to X$ and $f : X \to Y$ are maps, then $f \circ k : W \to Y$ is a map.

Since f, g, h are functions, then $g \circ (f \circ h) = (g \circ f) \circ h = (g \circ f) \circ k = g \circ (f \circ k)$.

Since $f \circ h : W \to Y$ is a map and $f \circ k : W \to Y$ is a map and $g \circ (f \circ h) = g \circ (f \circ k)$, then we conclude $f \circ h = f \circ k$.

Since f is injective, then by the left cancellation property of injective maps, if $h : W \to X$ and $k : W \to X$ are maps such that $f \circ h = f \circ k$, then $h = k$.

Since $h : W \to X$ and $k : W \to X$ are maps and $f \circ h = f \circ k$, then we conclude $h = k$, as desired. $\square$

Exercise 53. Give an example of a map $f : X \to Y$ and subsets A, B of X such that $A \subset B$, but $f(A) = f(B)$.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be the map defined by $f(x) = \sin x$.
Let $A = [0, \frac{\pi}{2}]$ and $B = [0, \pi]$.
Then $A \subset B$.
Observe that

$$\begin{aligned} f(A) &= \{f(x) : x \in A\} \\ &= \{\sin x : x \in [0, \frac{\pi}{2}]\} \\ &= \{\sin x : 0 \leq x \leq \frac{\pi}{2}\} \\ &= [0, 1] \\ &= \{\sin x : 0 \leq x \leq \pi\} \\ &= \{f(x) : x \in [0, \pi]\} \\ &= \{f(x) : x \in B\} \\ &= f(B). \end{aligned}$$

$\square$

Exercise 54. Let $f : X \to Y$ be a map.
Let A and B be subsets of X.
If $f(A) = f(B)$ and f is one to one, then $A = B$.

Proof. Suppose $f(A) = f(B)$ and f is one to one.

We first prove $A \subset B$.
Let $a \in A$.
Then $f(a) \in f(A)$.
Since $f(A) = f(B)$, then $f(a) \in f(B)$.
Hence, there exists $b \in B$ such that $f(a) = f(b)$.
Since f is one to one, then $a = b$.
Since $b \in B$, then $a \in B$.
Thus, $A \subset B$.

We next prove $B \subset A$.
Let $b \in B$.
Then $f(b) \in f(B)$.
Since $f(B) = f(A)$, then $f(b) \in f(A)$.
Hence, there exists $a \in A$ such that $f(b) = f(a)$.
Since f is one to one, then $b = a$.
Since $a \in A$, then $b \in A$.
Thus, $B \subset A$.

Since $A \subset B$ and $B \subset A$, then $A = B$, as desired. $\square$

Exercise 55. Let $f : A \to B$ be a map.
Let $Y \subset B$.
Then $f(f^{-1}(Y)) = Y$ iff $Y \subset rngf$.

Proof. We must prove $f(f^{-1}(Y)) = Y$ iff $Y \subset rngf$.

We first prove if $f(f^{-1}(Y)) = Y$, then $Y \subset rngf$.
Suppose $f(f^{-1}(Y)) = Y$.
To prove $Y \subset rngf$, let $b \in Y$.
Since $Y = f(f^{-1}(Y))$, then $y \in f(f^{-1}(Y))$.
Hence, $y = f(x)$ for some $x \in f^{-1}(Y)$.
Since $x \in f^{-1}(Y)$, then $x \in A$ and $f(x) \in Y$.
Thus, there exists $x \in A$ such that $f(x) = y$, so $y \in rngf$, as desired.

Conversely, we prove if $Y \subset rngf$, then $f(f^{-1}(Y)) = Y$.
Suppose $Y \subset rngf$.
We first prove $Y \subset f(f^{-1}(Y))$.
Let $y \in Y$.
Since $Y \subset rngf$, then $y \in rngf$, so there exists $x \in A$ such that $f(x) = y$.
Thus, there exists $x \in A$ such that $f(x) \in Y$, so $x \in f^{-1}(Y)$.
Hence, there exists $x \in f^{-1}(Y)$ such that $y = f(x)$, so $y \in f(f^{-1}(Y))$.
Therefore, $Y \subset f(f^{-1}(Y))$.
Since $f(f^{-1}(Y)) \subset Y$ and $Y \subset f(f^{-1}(Y))$, then $f(f^{-1}(Y)) = Y$, as desired.
$\square$

Exercise 56. Let $f : A \to B$ be a map.
Let $X \subset A$.
Then $f^{-1}(f(X)) = X$ iff the restriction of f to the subset $f^{-1}(f(X))$ of A is one to one.

Proof. We must prove $f^{-1}(f(X)) = X$ iff the restriction of f to the subset $f^{-1}(f(X))$ of A is one to one.

We first prove if $f^{-1}(f(X)) = X$, then the restriction of f to the subset $f^{-1}(f(X))$ of A is one to one.
Suppose $f^{-1}(f(X)) = X$.
Then f is one to one.
Let $f|_X : X \to B$ be the restriction of f to X defined by $f|_X(x) = f(x)$ for all $x \in X$.
To prove $f|_X$ is one to one, let $a, b \in X$ such that $f|_X(a) = f|_X(b)$.
Then $f(a) = f(b)$.
Since f is one to one, then $a = b$.
Therefore, $f|_X$ is one to one, as desired.

Conversely, we prove if the restriction of f to the subset $f^{-1}(f(X))$ of A is one to one, then $f^{-1}(f(X)) = X$.

Suppose the restriction of f to the subset $f^{-1}(f(X))$ of A is one to one.

Let $g : f^{-1}(f(X)) \to B$ be the restriction of f to $f^{-1}(f(X))$ defined by $g(x) = f(x)$ for all $x \in f^{-1}(f(X))$.

Then g is one to one.

We first prove $f^{-1}(f(X)) \subset X$.

Let $a \in f^{-1}(f(X))$.

Then $a \in A$ and $f(a) \in f(X)$.

Since $f(a) \in f(X)$, then $f(a) = f(b)$ for some $b \in X$.

Since $b \in X$ and $X \subset A$, then $b \in A$.

Since $f(b) = f(a)$ and $f(a) \in f(X)$, then $f(b) \in f(X)$.

Thus, $b \in A$ and $f(b) \in f(X)$, so $b \in f^{-1}(f(X))$.

Hence, $g(a) = f(a) = f(b) = g(b)$.

Since g is one to one and $g(a) = g(b)$, then $a = b$.

Since $b \in X$, then $a \in X$.

Therefore, $f^{-1}(f(X)) \subset X$.

Since $f^{-1}(f(X)) \subset X$ and $X \subset f^{-1}(f(X))$, then $f^{-1}(f(X)) = X$, as desired. $\square$

Exercise 57. Let $f : A \to B$ be a map.

Let $X \subset A$ and $Y \subset B$. Then

1. $f^{-1}(B - Y) = A - f^{-1}(Y)$.
2. $f(X) \subset Y$ iff $X \subset f^{-1}(Y)$.
3. If f is bijective, then $f(X) = Y$ iff $f^{-1}(Y) = X$.

Proof. We prove 1.

We must prove $f^{-1}(B - Y) = A - f^{-1}(Y)$.

We first prove $f^{-1}(B - Y) \subset A - f^{-1}(Y)$.

Let $x \in f^{-1}(B - Y)$.

Then $x \in A$ and $f(x) \in B - Y$.

Since $f(x) \in B - Y$, then $f(x) \in B$ and $f(x) \notin Y$.

Since $x \in A$ and $f(x) \notin Y$, then $x \notin f^{-1}(Y)$.

Since $x \in A$ and $x \notin f^{-1}(Y)$, then $x \in A - f^{-1}(Y)$.

Thus, $f^{-1}(B - Y) \subset A - f^{-1}(Y)$.

We next prove $A - f^{-1}(Y) \subset f^{-1}(B - Y)$.

Let $y \in A - f^{-1}(Y)$.

Then $y \in A$ and $y \notin f^{-1}(Y)$, so $f(y) \notin Y$.

Since $y \in A$ and $f : A \to B$ is a map, then $f(y) \in B$.

Since $f(y) \in B$ and $f(y) \notin Y$, then $f(y) \in B - Y$.

Hence, $y \in A$ and $f(y) \in B - Y$, so $y \in f^{-1}(B - Y)$.

Thus, $A - f^{-1}(Y) \subset f^{-1}(B - Y)$.

Since $f^{-1}(B - Y) \subset A - f^{-1}(Y)$ and $A - f^{-1}(Y) \subset f^{-1}(B - Y)$, then $f^{-1}(B - Y) = A - f^{-1}(Y)$. $\square$

Proof. We prove 2.
We must prove $f(X) \subset Y$ iff $X \subset f^{-1}(Y)$.

We first prove if $f(X) \subset Y$, then $X \subset f^{-1}(Y)$.
Suppose $f(X) \subset Y$.
To prove $X \subset f^{-1}(Y)$, let $x \in X$.
Then $f(x) \in f(X)$.
Since $f(X) \subset Y$, then $f(x) \in Y$.
Since $x \in X$ and $f(x) \in Y$, then $x \in f^{-1}(Y)$.
Thus, $X \subset f^{-1}(Y)$.

Conversely, we prove if $X \subset f^{-1}(Y)$, then $f(X) \subset Y$.
Suppose $X \subset f^{-1}(Y)$.
To prove $f(X) \subset Y$, let $y \in f(X)$.
Then there exists $x \in X$ such that $y = f(x)$.
Since $x \in X$ and $X \subset f^{-1}(Y)$, then $x \in f^{-1}(Y)$, so $f(x) \in Y$.
Thus, $y \in Y$, so $f(X) \subset Y$. $\square$

Proof. We prove 3.
We prove if f is bijective, then $f(X) = Y$ iff $f^{-1}(Y) = X$.
Suppose f is bijective.
Then f is injective and surjective.
We must prove $f(X) = Y$ iff $f^{-1}(Y) = X$.

We first prove if $f(X) = Y$, then $f^{-1}(Y) = X$.
Suppose $f(X) = Y$.
Then $f(X) \subset Y$, so $X \subset f^{-1}(Y)$.
Let $x \in f^{-1}(Y)$.
Then $x \in A$ and $f(x) \in Y$.
Since $Y = f(X)$, then $f(x) \in f(X)$.
Hence, there exists $a \in X$ such that $f(x) = f(a)$.
Since f is injective, then $x = a$.
Thus, $x \in X$, so $f^{-1}(Y) \subset X$.
Since $f^{-1}(Y) \subset X$ and $X \subset f^{-1}(Y)$, then $f^{-1}(Y) = X$, as desired.

Conversely, we prove if $f^{-1}(Y) = X$, then $f(X) = Y$.
Suppose $f^{-1}(Y) = X$.
Then $X \subset f^{-1}(Y)$, so $f(X) \subset Y$.
Let $y \in Y$.
Since $Y \subset B$, then $y \in B$.
Since f is surjective, then there exists $x \in A$ such that $f(x) = y$, so $f(x) \in Y$.
Since $x \in A$ and $f(x) \in Y$, then $x \in f^{-1}(Y)$.
Since $f^{-1}(Y) = X$, then $x \in X$, so $f(x) \in f(X)$.

Hence, $y \in f(X)$, so $Y \subset f(X)$.
Since $f(X) \subset Y$ and $Y \subset f(X)$, then $f(X) = Y$, as desired. $\square$

Exercise 58. image of a difference

Let $f : X \to Y$ be a map with $A, B \subset X$. Then
1. $f(A) - f(B) \subset f(A - B)$.
2. If f is injective, then $f(A - B) = f(A) - f(B)$.

Proof. We prove 1.
We prove $f(A) - f(B) \subset f(A - B)$.
Let $y \in f(A) - f(B)$.
Then $y \in f(A)$ and $y \notin f(B)$.
Since $y \in f(A)$, then $y = f(a)$ for some $a \in A$.
Since $y \in f(B)$ iff $a \in B$ and $f(a) = y$, then $y \notin f(B)$ iff either $a \notin B$ or $f(a) \neq y$.
Since $y \notin f(B)$, then either $a \notin B$ or $f(a) \neq y$.
Since $f(a) = y$, then we conclude $a \notin B$.
Since $a \in A$ and $a \notin B$, then $a \in A - B$.
Thus, there exists $a \in A - B$ such that $f(a) = y$, so $y \in f(A - B)$.
Therefore, $f(A) - f(B) \subset f(A - B)$. $\square$

Proof. We prove 2.
We prove if f is injective, then $f(A - B) = f(A) - f(B)$.
Suppose f is injective.

We first prove $f(A - B) \subset f(A) - f(B)$.
Let $y \in f(A - B)$.
Then there exists $a \in A - B$ such that $f(a) = y$.
Since $a \in A - B$, then $a \in A$ and $a \notin B$.
Since $y = f(a)$ and $a \in A$, then $y \in f(A)$.
Suppose for the sake of contradiction $y \in f(B)$.
Then there exists $b \in B$ such that $f(b) = y$.
Thus, $f(a) = y = f(b)$.
Since f is injective and $f(a) = f(b)$, then $a = b$.
Since $b \in B$ and $b = a$, then $a \in B$.
Thus, we have $a \in B$ and $a \notin B$, a contradiction.
Hence, $y \notin f(B)$.
Since $y \in f(A)$ and $y \notin f(B)$, then $y \in f(A) - f(B)$.
Therefore, $f(A - B) \subset f(A) - f(B)$.

Since $f(A-B) \subset f(A)-f(B)$ and $f(A)-f(B) \subset f(A-B)$, then $f(A-B) = f(A) - f(B)$. $\square$

Exercise 59. inverse image of a difference equals difference of inverse images

Let $f : X \to Y$ be a map with $C, D \subset Y$.
Then $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.

Proof. We must prove $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.

We first prove $f^{-1}(C - D) \subset f^{-1}(C) - f^{-1}(D)$.
Let $x \in f^{-1}(C - D)$.
Then $x \in X$ and $f(x) \in (C - D)$.
Since $f(x) \in (C - D)$, then $f(x) \in C$ and $f(x) \notin D$.
Since $x \in X$ and $f(x) \in C$, then $x \in f^{-1}(C)$.
Since $x \in X$ and $f(x) \notin D$, then $x \notin f^{-1}(D)$.
Thus, $x \in f^{-1}(C)$ and $x \notin f^{-1}(D)$, so $x \in f^{-1}(C) - f^{-1}(D)$.
Hence, $f^{-1}(C - D) \subset f^{-1}(C) - f^{-1}(D)$.

We next prove $f^{-1}(C) - f^{-1}(D) \subset f^{-1}(C - D)$.
Let $a \in f^{-1}(C) - f^{-1}(D)$.
Then $a \in f^{-1}(C)$ and $a \notin f^{-1}(D)$.
Since $a \in f^{-1}(C)$, then $a \in X$ and $f(a) \in C$.
Since $a \in X$ and $a \notin f^{-1}(D)$, then $f(a) \notin D$.
Thus, $f(a) \in C$ and $f(a) \notin D$, so $f(a) \in (C - D)$.
Since $a \in X$ and $f(a) \in (C - D)$, then $a \in f^{-1}(C - D)$.
Hence, $f^{-1}(C) - f^{-1}(D) \subset f^{-1}(C - D)$.

Since $f^{-1}(C - D) \subset f^{-1}(C) - f^{-1}(D)$ and $f^{-1}(C) - f^{-1}(D) \subset f^{-1}(C - D)$, then $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$. □

Exercise 60. Let $f : A \to B$ and $g : B \to C$ be maps.
Let $S \subset C$.
Then $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$.

Proof. Since $f : A \to B$ and $g : B \to C$ are maps, then $g \circ f : A \to C$ is a map.
Let $x \in (g \circ f)^{-1}(S)$.
Then $x \in A$ and $(g \circ f)(x) \in S$.
Since $x \in A$ and $f : A \to B$ is a map, then $f(x) \in B$.
Since $(g \circ f)(x) = g(f(x))$ and $(g \circ f)(x) \in S$, then $g(f(x)) \in S$.
Since $f(x) \in B$ and $g(f(x)) \in S$, then $f(x) \in g^{-1}(S)$.
Since $x \in A$ and $f(x) \in g^{-1}(S)$, then $x \in f^{-1}(g^{-1}(S))$.
Therefore, $(g \circ f)^{-1}(S) \subset f^{-1}(g^{-1}(S))$.

Let $y \in f^{-1}(g^{-1}(S))$.
Then $y \in A$ and $f(y) \in g^{-1}(S)$.
Since $f(y) \in g^{-1}(S)$, then $f(y) \in B$ and $g(f(y)) \in S$.
Since $g(f(y)) = (g \circ f)(y)$ and $g(f(y)) \in S$, then $(g \circ f)(y) \in S$.
Since $y \in A$ and $(g \circ f)(y) \in S$, then $y \in (g \circ f)^{-1}(S)$.
Therefore, $f^{-1}(g^{-1}(S)) \subset (g \circ f)^{-1}(S)$.

Since $(g \circ f)^{-1}(S) \subset f^{-1}(g^{-1}(S))$ and $f^{-1}(g^{-1}(S)) \subset (g \circ f)^{-1}(S)$, then $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$, as desired. $\square$

Proposition 61. *Let $f : X \mapsto Y$ be a function with $A_1, A_2 \subseteq X$ and $B_1, B_2 \subseteq Y$.*

Then $f^{-1}(Y - B_1) = X - f^{-1}(B_1)$.

Solution. We can use the definition of pre-image under a function.

We know for any $B \subseteq Y$, the pre-image of B under f is the set $f^{-1}(B) = \{a \in X : f(a) \in B\} \subseteq X$.

Thus,

$f^{-1}(B_1) = \{a \in X : f(a) \in B_1\}$

$f^{-1}(Y - B_1) = \{a \in X : f(a) \in Y - B_1\}$

We use the definition of set equality to prove this. $\square$

Proof. We know that $f^{-1}(B_1) = \{a \in X : f(a) \in B_1\}$ and $f^{-1}(Y - B_1) = \{a \in X : f(a) \in Y - B_1\}$.

Observe that

$$\begin{aligned}
f^{-1}(Y - B_1) &= \{a \in X : f(a) \in Y - B_1\} \\
&= \{a \in X : f(a) \in Y \wedge f(a) \notin B_1\} \\
&= \{a \in X : T \wedge f(a) \notin B_1\} \\
&= \{a \in X : f(a) \notin B_1\} \\
&= \{a \in X : a \notin f^{-1}(B_1)\} \\
&= \{a : a \in X \wedge a \notin f^{-1}(B_1)\} \\
&= X - f^{-1}(B_1)
\end{aligned}$$

$\square$