Functions Theory Notes

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Functions

Definition 1. function

A **function** is a relation f such that if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.

Let f be a function.
Then f is a relation such that if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.
Since f is a relation, then f is a set of ordered pairs.
Let $domf$ be the domain of f and let $rngf$ be the range of f.
Then $domf = \{a : (\exists b)(a, b) \in f\} = \{a : (\exists b)(f(a) = b)\} = \{a : f(a) \text{ exists}\}$.
Then $rngf = \{b : (\exists a)(a, b) \in f\} = \{b : (\exists a)(f(a) = b)\} = \{f(a) : a \text{ exists}\} = \{f(a) : a \in domf\}$.
Since f is a relation, then for each $a \in domf$, there exists $b \in rngf$ such that $(a, b) \in f$.

Let $a \in domf$.
Then there exists at least one $b \in rngf$ such that $(a, b) \in f$.
Suppose there exists b' such that $(a, b') \in f$.
Since f is a function and $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.
Hence, there exists at most one $b \in rngf$ such that $(a, b) \in f$.
Thus, there exists exactly one $b \in rngf$ such that $(a, b) \in f$, so there is a unique $b \in rngf$ such that $(a, b) \in f$.

Define the statement 'b is the value of the function f at a' by $f(a) = b$ iff $(a, b) \in f$.
Then there is a unique $b \in rngf$ such that $f(a) = b$.
Thus, if $a \in domf$, then there is a unique $b \in rngf$ such that $f(a) = b$.
Therefore, if f is a function, then for each $a \in domf$, there is a unique $b \in rngf$ such that $f(a) = b$.

A relation specified by a listing of its ordered pairs is a function iff no two distinct ordered pairs in the list have the same first element.
A function may be described by specifying its domain and a rule of correspondence $y = f(x)$ for all x in the domain of f.

If $x \in domf$, we say that y is the value of the function f at x and write $y = f(x)$.

A relation f is not a function iff there exists $(a, b) \in f$ and $(a, b') \in f$ and $b \neq b'$.

Proposition 2. ***A function value is unique.***
Let f be a function.
Let $a, b \in domf$.
If $a = b$, then $f(a) = f(b)$.

Let f be a function.
If $a \in domf$, then $f(a)$ is unique.

The negation of $(\forall a, b \in domf)(a = b \to f(a) = f(b))$ is $(\exists a, b \in domf)(a = b \wedge f(a) \neq f(b))$.

Hence, a relation f is not a function iff there exists $a \in domf$ and there exists $b \in domf$ such that $a = b$ and $f(a) \neq f(b)$.

Let $f : A \to B$ be a relation.
To prove f is a function we must show f is **well defined**.
Thus we must prove:
1. **Existence** $(\forall a \in A)(\exists b \in B)[f(a) = b]$.
2. **Uniqueness** $(\forall a, b \in A)(a = b \to f(a) = f(b))$.
If either condition is not satisfied by f, then f is not a function.

Definition 3. image of a function
Let f be a function.
The range of f is called the **image of** f.

Definition 4. equal functions
Let f and g be functions.
Then $f = g$ iff f and g are the same set of ordered pairs.

Theorem 5. ***equality of functions***
Let f and g be functions.
Let $domf$ be the domain of f.
Let $domg$ be the domain of g.
Then $f = g$ iff
1. $domf = domg$.
2. $f(x) = g(x)$ for all $x \in domf \cap domg$.

Let f and g be functions.
Then $f = g$ iff
1. domain of f equals the domain of g.
2. $f(x) = g(x)$ for all x in the common domain.

Definition 6. map from set A to set B

A **map from set A to set B**, denoted $f : A \to B$, consists of a function f such that $dom f = A$ and $rng f \subset B$.

The set A is called the **domain** of f.

The set B is called the **codomain** of f.

Let $f : A \to B$ be a map from set A to set B.

We say that "$f : A \to B$ is a map from A to B" or "f is a function that maps A to B".

Let $f : A \to B$ be a map from set A to set B.

Then f is a function such that $dom f = A$ and $rng f \subset B$.

Since f is a function, then for each $a \in dom f$, there is a unique $b \in rng f$ such that $f(a) = b$.

Thus, for each $a \in A$, there is a unique $b \in rng f$ such that $f(a) = b$.

Since $rng f \subset B$, then for each $a \in A$, there is a unique $b \in B$ such that $f(a) = b$.

Therefore, if $f : A \to B$ is a map from set A to set B, then

1. f is a function.
2. A is the domain of f.
3. B is the codomain of f.
4. $rng f \subset B$.
5. For each $a \in A$, there is a unique $b \in B$ such that $f(a) = b$.

A map $f : A \to B$ assigns a unique $b \in B$ to each $a \in A$.

Definition 7. equal maps

The maps $f : A \to B$ and $g : C \to D$ are **equal** iff $f = g$ and $B = D$.

Theorem 8. ***equality of maps***

The maps $f : A \to B$ and $g : C \to D$ are equal iff

1. $A = C$.

2. $B = D$.

3. $f(x) = g(x)$ for all $x \in A$.

Maps $f : A \to B$ and $g : C \to D$ are equal iff

1. $A = C$ (same domain)
2. $B = D$ (same codomain)
3. $f(x) = g(x)$ for all x in the common domain A.

The restriction of a map is a restriction to some subset of its domain.

Definition 9. restriction of a map

Let $f : A \to B$ be a map.

Let $S \subset A$.

Let $f|_S : S \to B$ be defined by $f|_S(x) = f(x)$ for all $x \in S$.

We call $f|_S$ the **restriction of f to S**.

Proposition 10. ***The restriction of a map is a map.***
Let $f : A \to B$ be a map.
Let $S \subset A$.
Let $f|_S$ be the restriction of f to S.
Then $f|_S : S \to B$ is a map.

Composition of functions

If f and g are functions, then f and g are relations, so the composition of f and g is a relation.

Definition 11. composition of functions
Let f and g be functions.
The **composition of** f **and** g is the relation $g \circ f = \{(a, b) : (\exists c)((a, c) \in f \wedge (c, b) \in g\}$.

Theorem 12. ***Composition of functions is a function.***
Let f and g be functions. Then
1. $g \circ f$ is a function.
2. $dom\ g \circ f = \{x \in dom f : f(x) \in dom g\}$.
3. $(g \circ f)(x) = g(f(x))$ for all $x \in dom\ g \circ f$.

Theorem 13. ***Function composition is associative.***
Let f, g, and h be functions.
Then $(f \circ g) \circ h = f \circ (g \circ h)$.

Proposition 14. ***Composition of maps***
Let $f : A \to B$ and $g : B \to C$ be maps.
Then $g \circ f : A \to C$ is a map and $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

Proposition 15. *Let $f : A \to B$ be a map.*
Let I_A be the identity map on A and I_B be the identity map on B.
Then $f \circ I_A = I_B \circ f = f$.

An injective map preserves distinctness; an injective map maps distinct elements in the domain to distinct elements in the range.

Therefore, a map is injective iff no two distinct ordered pairs have the same second element.

Definition 16. injective map (one to one)
A map $f : A \to B$ is said to be **one to one**, or **injective**, iff the function f is a one to one function; that is, for every $a, b \in A$ if $f(a) = f(b)$, then $a = b$.
Such a map is said to be an **injection** of A into B.

Let $f : A \to B$ be a map.
Then f is injective iff
the function f is one to one iff
$(\forall a, b \in A)(f(a) = f(b) \to a = b)$ iff
$(\forall a, b \in A)(a \neq b \to f(a) \neq f(b))$.

Therefore, f is not injective iff
the function f is not one to one iff
$(\exists a, b \in A)(a \neq b \wedge f(a) = f(b))$.

Theorem 17. ***Left cancellation property of injective maps***
Let $f : X \to Y$ be a map.
Then f is injective iff for every set W and every map $g : W \to X$ and $h : W \to X$ such that $f \circ g = f \circ h$ we have $g = h$.

A surjective map is a map whose range equals its codomain.

Definition 18. surjective map (onto)
A map $f : A \to B$ is said to be **onto**, or a **function that maps** A **onto** B iff $rngf = B$.
We say that such a map is **surjective**, or a **surjection**.

Let $f : A \to B$ be a surjective map.
Then $f(A) = rngf = \{f(a) \in B : a \in A\} = B$.

Proposition 19. *A map $f : A \to B$ is surjective iff $(\forall b \in B)(\exists a \in A)(f(a) = b)$.*

Let $f : A \to B$ be a map.
Then f is surjective iff $(\forall b \in B)(\exists a \in A)[f(a) = b]$.
Therefore, f is not surjective iff $(\exists b \in B)(\forall a \in A)[f(a) \neq b]$.

Theorem 20. ***Right cancellation property of surjective maps***
Let X be a nonempty set.
Let $f : X \to Y$ be a map.
Then f is surjective iff for every set Z and every map $g : Y \to Z$ and $h : Y \to Z$ such that $g \circ f = h \circ f$ we have $g = h$.

Definition 21. bijective map (one to one correspondence)
Let $f : A \to B$ be a map.
Then f is **bijective** iff f is injective and surjective.
A **bijection** is a bijective function.

Let $f : A \to B$ be a map.
Then f is not bijective iff either f is not injective or f is not surjective.

Proposition 22. ***identity map is bijective.***
Let S be a set.
The identity map $I_S : S \to S$ on S is a bijection.

Theorem 23. *Let $f : A \to B$ and $g : B \to C$ be maps.*
1. If f and g are injective, then $g \circ f$ is injective.
A composition of injections is an injection.
2. If f and g are surjective, then $g \circ f$ is surjective.
A composition of surjections is a surjection.
3. If $g \circ f$ is injective, then f is injective.
4. If $g \circ f$ is surjective, then g is surjective.

Corollary 24. *Let $f : A \to B$ and $g : B \to C$ be maps.*

1. If f and g are bijective, then $g \circ f$ is bijective.

A composition of bijections is a bijection.

2. If $g \circ f$ is bijective, then f is injective and g is surjective.

Let f be a function.

Since f is a relation, then the inverse relation f^{-1} exists and is unique.

Inverse functions

Definition 25. inverse of a function

Let f be a function.

The **inverse of** f is the inverse relation $f^{-1} = \{(b, a) : (a, b) \in f\}$.

Let f be a function.

Then f^{-1} is the inverse of f and $f^{-1} = \{(b, a) : (a, b) \in f\}$, so $(b, a) \in f^{-1}$ iff $(a, b) \in f$.

Therefore, $f^{-1}(b) = a$ iff $f(a) = b$ for all $a \in domf$.

Since f is a relation, then $domf^{-1} = rngf$ and $rngf^{-1} = domf$ and $(f^{-1})^{-1} = f$.

Theorem 26. ***existence of inverse function***

Let f be a function.

Then the inverse relation f^{-1} is a function iff f is injective.

Definition 27. invertible map

A map $f : A \to B$ is said to be **invertible** iff there exists a map $g : B \to A$ such that g is an inverse of f.

Theorem 28. ***The inverse of an invertible map is unique.***

Let $f : A \to B$ be an invertible map.

Then the inverse map is unique.

Let $f : A \to B$ be an invertible map.

Then there exists a unique map $g : B \to A$ such that g is an inverse of f.

Since g is the unique inverse of f, we denote g by f^{-1}.

Thus, the inverse map of f is $f^{-1} : B \to A$.

Therefore, $f^{-1}(b) = a$ iff $f(a) = b$ for every $a \in A$ and $b \in B$.

Therefore, a map $f : A \to B$ is invertible iff the inverse map $f^{-1} : B \to A$ exists.

Theorem 29. *Let $f : A \to B$ and $g : B \to A$ be maps.*

Then g is an inverse of f iff

1. $g \circ f = I_A$

2. $f \circ g = I_B$.

Let $f : A \to B$ and $g : B \to A$ be maps such that g is an inverse of f.

Then $g \circ f : A \to A$ and $f \circ g : B \to B$ are maps and $g \circ f = I_A$ and $f \circ g = I_B$.

Since $g \circ f = I_A$, then $(\forall a \in A)[(g \circ f)(a) = a]$.

Since $f \circ g = I_B$, then $(\forall b \in B)[(f \circ g)(b) = b]$.

Corollary 30. *Let $f : A \to B$ be an invertible map. Then*

1. $f^{-1} \circ f = I_A$

2. $f \circ f^{-1} = I_B$.

Let $f : A \to B$ be an invertible map.

Then the inverse map $f^{-1} : B \to A$ exists, so $f^{-1} \circ f : A \to A$ and $f \circ f^{-1} : B \to B$ are maps and $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$.

Since $f^{-1} \circ f = I_A$, then $(\forall a \in A)[(f^{-1} \circ f)(a) = a]$.

Since $f \circ f^{-1} = I_B$, then $(\forall b \in B)[(f \circ f^{-1})(b) = b]$.

Theorem 31. ***An invertible map is bijective.***

Let $f : A \to B$ be a map.

Then f is invertible iff f is bijective.

Lemma 32. *Let $f : A \to B$ be a map.*

If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.

Let $f : A \to B$ be a bijective map.

Since f is bijective, then f is invertible, so the inverse map $f^{-1} : B \to A$ exists.

Therefore,

1) $f^{-1} \circ f = I_A$.

2) $f \circ f^{-1} = I_B$.

Theorem 33. *Let $f : A \to B$ be a bijection. Then*

1. $(f^{-1})^{-1} : A \to B$ *is a bijection.*

2. $(f^{-1})^{-1} = f$.

Theorem 34. *Let $f : A \to B$ and $g : B \to C$ be bijections. Then*

1. $(g \circ f)^{-1} : C \to A$ *is a bijection.*

2. $f^{-1} \circ g^{-1} : C \to A$ *is a bijection.*

3. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Image and inverse image of functions

Definition 35. image of an element of a map

Let $f : A \to B$ be a map.

Let $(a, b) \in f$.

Then $a \in A$ and $b \in B$ and $f(a) = b$.

$f(a) = b$ means f maps a to b

b is the **image** of a under f.

a is the **pre-image** of b under f.

Since f is a function, then

1) $(\forall a \in A)(\exists b \in B)[f(a) = b]$.

Every element in the domain has at least one image.

2) $(\forall a \in A)[f(a)$ is unique$]$.

The image of every element in the domain is unique.

Definition 36. preimage of an element in the codomain of a map

Let $f : A \to B$ be a map.

Let $b \in B$.

The **pre-image of** b is the set $f^{-1}(b) = \{a \in A : f(a) = b\}$.

Therefore $f^{-1}(b) \subset A$.

Proposition 37. *Let $f : A \to B$ be a map.*

1. *Then f is injective iff every $b \in B$ has at most one pre-image.*
2. *Then f is surjective iff every $b \in B$ has at least one pre-image.*
3. *Then f is bijective iff every $b \in B$ has exactly one pre-image.*

Definition 38. image of a subset of the domain of a map

Let $f : A \to B$ be a map.

Let $S \subset A$.

The **image of** S **under** f is the set $f(S) = \{f(x) : x \in S\}$.

Let $f : A \to B$ be a map.

Let $S \subset A$.

Suppose $f(x) \in f(S)$.

Then $x \in S$.

Since $S \subset A$, then $x \in A$.

Since $f : A \to B$ is a map, then $f(x) \in B$.

Hence, $f(S) \subset B$.

Therefore, the image of a subset of the domain of a map is a subset of the codomain of the map.

Let $b \in B$.

Then $b \in f(S)$ iff $b = f(x)$ for some $x \in S$.

Definition 39. inverse image of a subset of the codomain of a map

Let $f : A \to B$ be a map.

Let $T \subset B$.

The **inverse image of** T **under** f is the set $f^{-1}(T) = \{x \in A : f(x) \in T\}$.

Let $f : A \to B$ be a map.

Let $T \subset B$.

Since $f^{-1}(T) = \{x \in A : f(x) \in T\}$, then $f^{-1}(T) \subset A$.

Therefore, the inverse image of a subset of the codomain of a map is a subset of the domain of the map.

Let $x \in A$.
Then $x \in f^{-1}(T)$ iff $f(x) \in T$.

Proposition 40. *Let $f : A \to B$ be a map. Then*
1. $f(\emptyset) = \emptyset$.
The image of the empty set is the empty set.
2. $f^{-1}(\emptyset) = \emptyset$.
The inverse image of the empty set is the empty set.
3. $f(A) = rngf$.
The image of the domain of f is the range of f.
4. $f^{-1}(B) = A$.
The inverse image of the codomain of f is the domain of f.

Definition 41. image of a map(function)
Let $f : A \to B$ be a map.
The **image of** f is the set $f(A) = \{f(x) : x \in A\}$.

Proposition 42. *Let $f : X \to Y$ be a map.*
1. For every subset A and B of X, if $A \subset B$, then $f(A) \subset f(B)$.
2. $f(A \cup B) = f(A) \cup f(B)$ for every subset A and B of X.
The image of a union equals the union of the images.
3. $f(A \cap B) \subset f(A) \cap f(B)$ for every subset A and B of X.
The image of an intersection is a subset of the intersection of the images.
4. $f(A \cap B) = f(A) \cap f(B)$ for every subset A and B of X iff f is injective.

Proposition 43. *Let $f : X \to Y$ be a map.*
1. For every subset C and D of Y, if $C \subset D$, then $f^{-1}(C) \subset f^{-1}(D)$.
2. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ for every subset C and D of Y.
The inverse image of a union equals the union of the inverse images.
3. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$ for every subset C and D of Y.
The inverse image of an intersection equals the intersection of the inverse images.

Proposition 44. ***inverse image of the image of a subset of the domain of a map***
Let $f : A \to B$ be a map. Then
1. $S \subset f^{-1}(f(S))$ for every subset S of A.
2. $f^{-1}(f(S)) = S$ for every subset S of A iff f is injective.

Proposition 45. ***image of the inverse image of a subset of the codomain of a map***
Let $f : A \to B$ be a map. Then
1. $f(f^{-1}(T)) \subset T$ for every subset T of B.
2. $f(f^{-1}(T)) = T$ for every subset T of B iff f is surjective.

Let A, B be finite sets, $|A| = m$ and $|B| = n$.
Let $B^A =$ the set of all functions from A to B.
Then $B^A = \{f : A \to B | f \text{ is a function.}\}$.
Then $|B^A| = |B|^{|A|}$.
Thus, there are n^m different functions from A to B.
There are $P(n, m) = \frac{n!}{(n-m)!}$ different 1-1 functions from A to B.