Relations and Functions Examples

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Equivalence Relations

Example 1. Equality relation on a set is an equivalence relation
Let S be a set.
Let $\{(s, s) : s \in S\} = \{(a, b) \in S \times S : a = b\}$ be the equality relation on S.
The equality relation is an equivalence relation on S.

Proof. Since every element of a set equals itself, then if $a \in S$, then $a = a$, so the equality relation is reflexive.

Let $a, b \in S$ such that $a = b$.
Then $b = a$, so the equality relation is symmetric.

Let $a, b, c \in S$ such that $a = b$ and $b = c$.
Then $a = c$, so the equality relation is transitive.

Since the equality relation is reflexive, symmetric, and transitive, then the equality relation is an equivalence relation on S. □

Example 2. total relation on a set is an equivalence relation
Let S be a set.
Let $S \times S = \{(a, b) : a, b \in S\}$ be the total relation relation on S.
Then $S \times S$ is an equivalence relation on S.

Proof. Let $a \in S$.
Then $(a, a) \in S \times S$.
Hence, $S \times S$ is reflexive.

Let $(a, b) \in S \times S$.
Then $a \in S$ and $b \in S$.
Since $b \in S$ and $a \in S$, then $(b, a) \in S \times S$.
Since $(a, b) \in S \times S$ implies $(b, a) \in S \times S$, then $S \times S$ is symmetric.

Let $(a, b) \in S \times S$ and $(b, c) \in S \times S$.
Then $a \in S$ and $b \in S$ and $c \in S$.
Since $a \in S$ and $c \in S$, then $(a, c) \in S \times S$.
Since $(a, b) \in S \times S$ and $(b, c) \in S \times S$ implies $(a, c) \in S \times S$, then $S \times S$ is transitive.
Since $S \times S$ is reflexive, symmetric, and transitive, then $S \times S$ is an equivalence relation on S. $\square$

Example 3. cardinality relation on the power set of a finite set
Let S be a finite set.
Let $\mathscr{P}$ be the power set of S.
Let $R = \{(A, B) \in \mathscr{P} \times \mathscr{P} : A$ and B contain the same number of elements $\} = \{(A, B) \in \mathscr{P} \times \mathscr{P} : |A| = |B|\}$.
Then R is an equivalence relation on $\mathscr{P}$.

Solution. Since $R \subset \mathscr{P} \times \mathscr{P}$, then R is a relation on $\mathscr{P}$.

We prove $domR = rangeR = \mathscr{P}$.
Let $A \in domR$.
Then, by definition of domain, $A \in \mathscr{P}$.
Hence, $A \in domR$ implies $A \in \mathscr{P}$, so $domR \subset \mathscr{P}$.
Let $A \in \mathscr{P}$.
To prove $\mathscr{P} \subset domR$, we must prove there is a set $B \in \mathscr{P}$ such that A contains the same number of elements as B.
Let $B = A$.
Since $A \in \mathscr{P}$, then $B \in \mathscr{P}$.
Each set contains the same number of elements as itself, so A contains the same number of elements as A.
Hence, A contains the same number of elements as B.
Thus, $\mathscr{P} \subset domR$.
Therefore, $domR \subset \mathscr{P}$ and $\mathscr{P} \subset domR$, so $domR = \mathscr{P}$.

Let $B \in rangeR$.
Then, by definition of range, $B \in \mathscr{P}$.
Hence, $B \in rangeR$ implies $B \in \mathscr{P}$, so $rangeR \subset \mathscr{P}$.
Let $B \in \mathscr{P}$.
To prove $\mathscr{P} \subset rangeR$, we must prove there is a set $A \in \mathscr{P}$ such that A and B contain the same number of elements.
Let $A = B$.
Since $B \in \mathscr{P}$, then $A \in \mathscr{P}$.
Each set contains the same number of elements as itself, so B contains the same number of elements as B.
Hence, B and B contain the same number of elements, so A and B contain the same number of elements.
Thus, $\mathscr{P} \subset rangeR$.
Therefore, $rangeR \subset \mathscr{P}$ and $\mathscr{P} \subset rangeR$, so $rangeR = \mathscr{P}$.

Hence, $domR = \mathscr{P} = rangeR$.

We prove R is reflexive.
Let $A \in \mathscr{P}$.
Then $A \subset S$, so A is a set.
Each set contains the same number of elements as itself, so A contains the same number of elements as itself.
Thus, A contains the same number of elements as A, so A and A contain the same number of elements.
Since $(A, A) \in \mathscr{P} \times \mathscr{P}$ and A and A contain the same number of elements, then $(A, A) \in R$.
Therefore, R is reflexive.

We prove R is symmetric.
Let $A, B \in \mathscr{P}$ such that $(A, B) \in R$.
Then A and B contain the same number of elements.
Thus, B and A contain the same number of elements.
Since $(B, A) \in \mathscr{P} \times \mathscr{P}$ and B and A contain the same number of elements, then $(B, A) \in R$.
Therefore, $(A, B) \in R$ implies $(B, A) \in R$, so R is symmetric.

We prove R is transitive.
Let $A, B, C \in \mathscr{P}$ such that $(A, B) \in R$ and $(B, C) \in R$.
Then A and B contain the same number of elements and B and C contain the same number of elements.
Thus, A and C contain the same number of elements.
Since $A \in \mathscr{P}$ and $C \in \mathscr{P}$, then $(A, C) \in \mathscr{P} \times \mathscr{P}$.
Since $(A, C) \in \mathscr{P} \times \mathscr{P}$ and A and C contain the same number of elements, then $(A, C) \in R$.
Therefore, $(A, B) \in R$ and $(B, C) \in R$ implies $(A, C) \in R$, so R is transitive.
Since R is reflexive, symmetric, and transitive, then R is an equivalence relation on $\mathscr{P}$.
Therefore, R is an equivalence relation on the power set of a finite set S.

Let S be a finite set of at least two elements.
Then there exist distinct elements a and b in S.
Hence, $a \in S$ and $b \in S$ and $a \neq b$.
To prove R is not antisymmetric, we must prove there exist sets $A, B \in \mathscr{P}$ such that $(A, B) \in R$ and $(B, A) \in R$ and $A \neq B$.
Thus, we must find sets $A \subset S$ and $B \subset S$ such that A and B contain the same number of elements and B and A contain the same number of elements and $A \neq B$.
Hence, we must find subsets of S, A and B that contain the same number of elements and $A \neq B$.
Let $A = \{a\}$ and $B = \{b\}$.

Then $A \subset S$ and $B \subset S$ and A and B contain 1 element, but $A \neq B$.
Therefore, R is not antisymmetric.

Let $X = \{1, 2, ..., 9, 10\}$.
Then $|X| = 10$ and $R = \{(M, N) \in 2^X \times 2^X : |M| = |N|\}$ is an equivalence relation on 2^X.
Hence, $\frac{2^X}{R} = \{[S] : S \in 2^X\} = \{[S] : S \subset X\}$ is a partition of 2^X.
Since $|2^X| = 2^{10} = 1024$, then there are 1024 subsets of X.
Consider the equivalence class of the subset $S = \{1, 2, 5\} \subset X$.
We have

$$\begin{aligned}
[S] &= \{Y \in 2^X : SRY\} \\
&= \{Y \in 2^X : |S| = |Y|\} \\
&= \{Y \subset X : 3 = |Y|\}.
\end{aligned}$$

Thus, $[S]$ consists of all subsets of X that have 3 elements.
The number of subsets of X that have 3 elements is $\binom{10}{3} = \frac{10!}{7!3!}$.
Let k be an integer between 0 and $|X| = 10$.
Let S be a subset of X that contains k elements.
Then the partition $\frac{2^X}{R}$ consists of 11 cells - a cell consisting of all subsets of X that have $k = 0, 1, 2, ..., 10$ elements.
The number of subsets of X that have k elements is $\binom{10}{k} = \frac{10!}{(10-k)!k!}$.
Thus, $\frac{2^X}{R} = \{[S] : S \in 2^X\} = \{[S] : S \subset X\}$ is a 11 celled partition of 2^X, where $[S] = \{Y \subset X : |S| = |Y|\}$ consists of all subsets of X that have the same number of elements as S and $|[S]| = \binom{10}{k} = \frac{10!}{(10-k)!k!}$ where k is an integer between 0 and 10, inclusive. □

Partial Orderings

Example 4. The interval $(0, 1)$ is bounded in $\mathbb{R}$.

Proof. Let $I = (0, 1)$.
To prove I is bounded in $\mathbb{R}$, we must prove I is bounded above and below in $\mathbb{R}$.
We first prove I is bounded below in $\mathbb{R}$.
Let $x \in I$.
Then $x \in \mathbb{R}$ and $0 < x < 1$.
Since $0 < x < 1$, then $0 < x$.
Since $-0.5 < 0$ and $0 < x$, then $-0.5 < x$, so $-0.5 \leq x$.
Thus, $-0.5 \leq x$ for all $x \in I$.
Since $-0.5 \in \mathbb{R}$ and $-0.5 \leq x$ for all $x \in I$, then -0.5 is a lower bound for I in $\mathbb{R}$.
Therefore, I is bounded below in $\mathbb{R}$.
To prove I is bounded above in $\mathbb{R}$, we must prove there is an upper bound for I in $\mathbb{R}$.

Thus, we must show there exists a real number U such that $x \leq U$ for all $x \in I$.

Let U be the real number 1.02.

Let $x \in I$.

Then $x \in \mathbb{R}$ and $0 < x < 1$, so $x < 1$.

Since $x < 1$ and $1 < 1.02$, then $x < 1.02$, so $x \leq 1.02$.

Thus, $x \leq 1.02$ for all $x \in I$.

Since $1.02 \in \mathbb{R}$ and $x \leq 1.02$ for all $x \in I$, then 1.02 is an upper bound for I in $\mathbb{R}$.

Therefore, I is bounded above in $\mathbb{R}$.

Since I is bounded below and above in $\mathbb{R}$, then I is bounded in $\mathbb{R}$. $\square$

Example 5. The interval $(0, 1]$ is bounded below in $\mathbb{R}$, but not bounded below in $\mathbb{R}^+$.

Proof. Let $I = (0, 1]$.

We must prove I is bounded below in $\mathbb{R}$ and not bounded below in $\mathbb{R}^+$.

We first prove I is bounded below in $\mathbb{R}$.

Let $x \in I$.

Then $x \in \mathbb{R}$ and $0 < x \leq 1$.

Since $0 < x \leq 1$, then $0 < x$.

Thus, $0 < x$ or $0 = x$, so $0 \leq x$.

Hence, $0 \leq x$ for all $x \in I$.

Since $0 \in \mathbb{R}$ and $0 \leq x$ for all $x \in I$, then 0 is a lower bound for I in $\mathbb{R}$.

Therefore, I is bounded below in $\mathbb{R}$.

To prove I is not bounded below in $\mathbb{R}^+$, we must prove there is no lower bound for I in $\mathbb{R}^+$.

Suppose there is a lower bound for I in $\mathbb{R}^+$.

Let L be a lower bound for I in $\mathbb{R}^+$.

Then $L \in \mathbb{R}^+$ and $L \leq x$ for all $x \in I$.

Since $L \in \mathbb{R}^+$, then $L \in \mathbb{R}$ and $L > 0$.

To derive a contradiction, we show the statement $L \leq x$ for all $x \in I$ is false.

Thus, we show there exists $a \in I$ such that $L \not\leq a$.

Hence, we show there exists $a \in I$ such that $a < L$.

Let $a = \frac{L}{2}$.

Since $L \in \mathbb{R}$, then $\frac{L}{2} \in \mathbb{R}$, so $a \in \mathbb{R}$.

Since $0 < L$, then $0 < \frac{L}{2}$, so $0 < a$.

Since $1 \in I$ and $L \leq x$ for all $x \in I$, then $L \leq 1$.

Hence, $\frac{L}{2} \leq \frac{1}{2}$, so $a \leq \frac{1}{2}$.

Since $a \leq \frac{1}{2}$ and $\frac{1}{2} < 1$, then $a < 1$, so $a \leq 1$.

Thus, $0 < a$ and $a \leq 1$, so $0 < a \leq 1$.

Since $a \in \mathbb{R}$ and $0 < a \leq 1$, then $a \in I$.

Since $1 > \frac{1}{2}$ and $L > 0$, then we multiply by L to get $L > \frac{L}{2}$.

Hence, $L > a$, so $a < L$.

Thus, there exists $a \in I$ such that $a < L$.

Therefore, there is no lower bound for I in $\mathbb{R}^+$, so I is not bounded below in $\mathbb{R}^+$. $\square$

Example 6. In the poset $(\mathbb{R}, \leq)$, 0 is a lower bound for the intervals $[0, 1]$ and $(0, 1]$.

Proof. Let $I = [0, 1]$.

To prove 0 is a lower bound for I, we must prove $0 \leq x$ for all $x \in I$.

Let $x \in I$.

Then $x \in \mathbb{R}$ and $0 \leq x \leq 1$.

Since $0 \leq x \leq 1$, then $0 \leq x$.

Hence, $0 \leq x$ for all $x \in I$.

Since $0 \in \mathbb{R}$ and $0 \leq x$ for all $x \in I$, then 0 is a lower bound for I in $\mathbb{R}$.

Let $I = (0, 1]$.

To prove 0 is a lower bound for I, we must prove $0 \leq x$ for all $x \in I$.

Let $x \in I$.

Then $x \in \mathbb{R}$ and $0 < x \leq 1$.

Since $0 < x \leq 1$, then $0 < x$.

Thus, $0 < x$ or $0 = x$, so $0 \leq x$.

Hence, $0 \leq x$ for all $x \in I$.

Since $0 \in \mathbb{R}$ and $0 \leq x$ for all $x \in I$, then 0 is a lower bound for I in $\mathbb{R}$. $\square$

Example 7. poset bounded above need not have a greatest element

In the poset $(\mathbb{R}, \leq)$, the intervals $[0, 1]$ and $(0, 1)$ are bounded above and $[0, 1]$ has a greatest element, but $(0, 1)$ does not have a greatest element.

Proof. To prove $[0, 1]$ is bounded above in $\mathbb{R}$, we will show that the real number 1 is an upper bound for $[0, 1]$ in $\mathbb{R}$.

Let $x \in [0, 1]$.

Then $x \in \mathbb{R}$ and $0 \leq x \leq 1$, so $x \leq 1$.

Since x is arbitrary, then $x \leq 1$ for all $x \in [0, 1]$.

Hence, 1 is an upper bound for $[0, 1]$ in $\mathbb{R}$, so $[0, 1]$ is bounded above in $\mathbb{R}$.

Since $1 \in [0, 1]$ and 1 is an upper bound for $[0, 1]$, then 1 is the greatest element in $[0, 1]$.

To prove $(0, 1)$ is bounded above in $\mathbb{R}$, we will show that the real number 1 is an upper bound for $(0, 1)$ in $\mathbb{R}$.

Let $x \in (0, 1)$.

Then $x \in \mathbb{R}$ and $0 < x < 1$, so $x < 1$.

Hence, $x < 1$ or $x = 1$, so $x \leq 1$.

Since x is arbitrary, then $x \leq 1$ for all $x \in (0, 1)$.

Hence, 1 is an upper bound for $(0, 1)$ in $\mathbb{R}$, so $(0, 1)$ is bounded above in $\mathbb{R}$.

We prove $(0, 1)$ does not have a greatest element.

Suppose $(0, 1)$ has a greatest element.

Let M be a greatest element of $(0, 1)$.

Then $M \in (0, 1)$ and $x \leq M$ for all $x \in (0, 1)$.

Since $M \in (0, 1)$, then $M \in \mathbb{R}$ and $0 < M < 1$, so $0 < M$ and $M < 1$.

To derive a contradiction, we will show that $x \leq M$ for all $x \in (0, 1)$ is false.

That is, we will show that there exists a real number in $(0,1)$ that is not less than or equal to M.

By trichotomy of $\mathbb{R}$, this means we will show there exists a real number in $(0,1)$ that is greater than M.

Since $M \in \mathbb{R}$, then $\frac{M+1}{2} \in \mathbb{R}$.

We show $\frac{M+1}{2} \in (0,1)$.

Since $-1 < 0$ and $0 < M$, then $-1 < M$.

Hence, $0 < M+1$, so $0 < \frac{M+1}{2}$.

Since $M < 1$, then $M+1 < 2$, so $\frac{M+1}{2} < 1$.

Thus, $0 < \frac{M+1}{2}$ and $\frac{M+1}{2} < 1$, so $0 < \frac{M+1}{2} < 1$.

Since $\frac{M+1}{2} \in \mathbb{R}$ and $0 < \frac{M+1}{2} < 1$, then $\frac{M+1}{2} \in (0,1)$.

Since $M < 1$, then $2M < M+1$, so $M < \frac{M+1}{2}$.

Therefore, $\frac{M+1}{2} > M$.

Thus, $\frac{M+1}{2} \in (0,1)$ and $\frac{M+1}{2} > M$, so there exists a real number in $(0,1)$ that is greater than M.

Hence, $(0,1)$ does not have a greatest element. □

Example 8. Let $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$.

In the poset $(\mathbb{R}, \leq)$, 1 is an upper bound for S, but S has no greatest element.

Proof. We prove 1 is an upper bound for S in $\mathbb{R}$.

Let $x \in S$.

Then there is a natural number n such that $x = 1 - \frac{1}{n}$.

Since $n \in \mathbb{N}$, then $n > 0$, so $\frac{1}{n} > 0$.

Hence, $\frac{-1}{n} < 0$, so $1 - \frac{1}{n} < 1$.

Thus, $x < 1$, so $x \leq 1$.

Since x is arbitrary, then $x \leq 1$ for all $x \in S$.

Therefore, 1 is an upper bound for S in $\mathbb{R}$, so S is bounded above in $\mathbb{R}$.

We prove S does not have a greatest element.

Suppose S has a greatest element.

Let M be a greatest element of S.

Then $M \in S$ and $x \leq M$ for all $x \in S$.

To derive a contradiction, we must show there exists an element of S that is not less than or equal to M.

Thus by trichotomy, we must show there exists an element of S that is greater than M.

Since $M \in S$, then there exists a natural number k such that $M = 1 - \frac{1}{k}$.

Since $k \in \mathbb{N}$, then $k+1 \in \mathbb{N}$.

Hence, $1 - \frac{1}{k+1} \in S$.

Since $k \in \mathbb{N}$, then $k > 0$, so $k+1 > 0$.

Thus, $k \neq 0$ and $k+1 \neq 0$, so $\frac{1}{k} \in \mathbb{R}$ and $\frac{1}{k+1} \in \mathbb{R}$.

Observe that $\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}$.

Since $k > 0$ and $k+1 > 0$, then $k(k+1) > 0$, so $\frac{1}{k(k+1)} > 0$.

Hence, $\frac{1}{k} - \frac{1}{k+1} > 0$, so $\frac{1}{k} > \frac{1}{k+1}$.

Thus, $\frac{-1}{k} < \frac{-1}{k+1}$, so $1 - \frac{1}{k} < 1 - \frac{1}{k+1}$.

Therefore, $M < 1 - \frac{1}{k+1}$, so $1 - \frac{1}{k+1} > M$.

Thus, $1 - \frac{1}{k+1} \in S$ and $1 - \frac{1}{k+1} > M$, so there exists an element of S that is greater than M.

Hence, S does not have a greatest element. ☐

Example 9. Let S be a set.

In the poset $(2^S, \subset)$ S is the greatest element and $\emptyset$ is the least element.

Proof. We prove S is the greatest element of 2^S.

Since every set is a subset of itself, then $S \subset S$.

Hence, $S \in 2^S$.

Let $X \in 2^S$.

Then $X \subset S$.

Since X is arbitrary, then $X \subset S$ for all $X \in 2^S$.

Hence, S is an upper bound for S.

Since $S \in 2^S$ and S is an upper bound for 2^S, then S is the greatest element of 2^S.

We prove $\emptyset$ is the least element of 2^S.

Since the empty set is a subset of every set, then $\emptyset \subset S$.

Hence, $\emptyset \in 2^S$.

Let $X \in 2^S$.

Then $X \subset S$.

Since the empty set is a subset of every set, then in particular, $\emptyset \subset X$.

Since X is arbitrary, then $\emptyset \subset X$ for all $X \in 2^S$.

Hence, $\emptyset$ is a lower bound for 2^S.

Since $\emptyset \in 2^S$ and $\emptyset$ is a lower bound for 2^S, then $\emptyset$ is the least element of 2^S. ☐

Example 10. In the poset $(\mathbb{Z}^+ \cup \{0\}, |)$ 0 is the greatest element and 1 is the least element.

Proof. Let $S = \mathbb{Z}^+ \cup \{0\}$.

Then $(S, |)$ is a poset.

We prove 0 is the greatest element of S.

Clearly, $0 \in S$.

Let $n \in S$.

Since $S \subset \mathbb{Z}$, then $n \in \mathbb{Z}$.

Every integer divides zero, so $n|0$.

Since n is arbitrary, then $n|0$ for all $n \in S$.

Hence, 0 is an upper bound for S.

Since $0 \in S$ and 0 is an upper bound for S, then 0 is the greatest element of S.

We prove 1 is the least element of S.

Clearly, $1 \in S$.

Let $n \in S$.

Since $S \subset \mathbb{Z}$, then $n \in \mathbb{Z}$.

The number 1 divides every integer, so $1|n$.

Since n is arbitrary, then $1|n$ for all $n \in S$.
Hence, 1 is a lower bound for S.
Since $1 \in S$ and 1 is a lower bound for S, then 1 is the least element of S. ☐

Example 11. In the poset $(\mathbb{R}, \leq)$ let $S = (0,1)$ and $T = [0,1]$.
Then $\sup(S) = 1 = \sup(T)$ and $\sup(S) \notin S$ and $\sup(T) \in T$ and there is no greatest element in S and 1 is the greatest element of T.

Solution. We sketch the intervals on the real number line.
We proved in a previous example that there is no greatest element of $(0,1)$.
Let B be the set of all upper bounds of S in $\mathbb{R}$.
Then $B = \{u \in \mathbb{R} : u \text{ is an upper bound of } S\}$ and $B \subset \mathbb{R}$.
Clearly, $B = [1,\infty) = \{x \in \mathbb{R} : 1 \leq x\}$.
We formally prove $B = [1,\infty) = \{x \in \mathbb{R} : 1 \leq x\}$.
Let $x \in [1,\infty)$.
Then $x \in \mathbb{R}$ and $1 \leq x$.
Let $s \in S$.
Then $s \in \mathbb{R}$ and $0 < s < 1$, so $s < 1$.
Hence, $s \leq 1$.
Since $s \leq 1$ and $1 \leq x$, then $s \leq x$.
Thus, $s \leq x$ for all $s \in S$, so x is an upper bound for S in $\mathbb{R}$.
Since $x \in \mathbb{R}$ and x is an upper bound for S, then $x \in B$.
Therefore, $x \in [1,\infty)$ implies $x \in B$, so $[1,\infty) \subset B$.
Let $b \in B$.
Then $b \in \mathbb{R}$ and b is an upper bound of S.
Hence, $x \leq b$ for all $x \in S$.
Let $x \in S$.
Then $x \in \mathbb{R}$ and $0 < x < 1$ and $x \leq b$.
Thus, $0 < x$ and $x < 1$ and $x \leq b$.
Since $0 < x$ and $x \leq b$, then $0 < b$.
To prove $b \in [1,\infty)$, we must prove $b \in \mathbb{R}$ and $1 \leq b$.
To prove $1 \leq b$, we must prove either $1 < b$ or $1 = b$.
Suppose $1 \neq b$.
We must prove $1 < b$.
Since $1 \neq b$, then by trichotomy, either $1 < b$ or $1 > b$.
Suppose for the sake of contradiction $1 > b$.
Then $b < 1$.
To derive a contradiction, we prove there exists $c \in S$ such that $c \not\leq b$.
I.e, we prove there exists $c \in S$ such that $c > b$.
Let $c = \frac{b+1}{2}$.
Since $b \in \mathbb{R}$, then $\frac{b+1}{2} \in \mathbb{R}$, so $c \in \mathbb{R}$.
Since $-1 < 0$ and $0 < b$, then $-1 < b$, so $0 < b + 1$.
Hence, $0 < \frac{b+1}{2}$, so $0 < c$.
Since $b < 1$, then $b + 1 < 2$, so $\frac{b+1}{2} < 1$.
Hence, $c < 1$.

Since $0 < c$ and $c < 1$, then $0 < c < 1$.
Since $c \in \mathbb{R}$ and $0 < c < 1$, then $c \in S$.
Since $1 > b$, then $b + 1 > 2b$, so $\frac{b+1}{2} > b$.
Therefore, $c > b$.
Thus, there exists $c \in S$ such that $c > b$, so not every element of S is less than or equal to b.
Therefore, $1 > b$ is false.
Since either $1 < b$ or $1 > b$ and $1 \not> b$, then $1 < b$.
Since $1 \neq b$ implies $1 < b$, then either $1 = b$ or $1 < b$, so $1 \leq b$.
Since $b \in \mathbb{R}$ and $1 \leq b$, then $b \in [1, \infty)$.
Thus, $b \in B$ implies $b \in [1, \infty)$, so $B \subset [1, \infty)$.
Since $B \subset [1, \infty)$ and $[1, \infty) \subset B$, then $B = [1, \infty)$, as desired.
The least element of B is the least upper bound of S.
Since $1 \in B$ and for all $x \in B, 1 \leq x$, then 1 is the least element of S.
Therefore, $lub(S) = \sup(S) = 1$.
Since $1 \notin S$, then $\sup(S) \notin S$.
Let B' be the set of all upper bounds of T in $\mathbb{R}$.
Then $B' = \{u \in \mathbb{R} : u \text{ is an upper bound of } T\}$ and $B' \subset \mathbb{R}$.
Clearly, all real numbers greater than 1 are upper bounds of T.
Is 1 an upper bound of T?
Let $x \in T$.
Then $0 \leq x \leq 1$, so $x \leq 1$.
Hence, $x \leq 1$ for all $x \in T$.
Therefore, 1 is an upper bound for T.
Thus, $B' = [1, \infty) = B$.
The least element of B' is the least upper bound of T.
Since $1 \in B'$ and $1 \leq x$ for all $x \in B'$, then 1 is the least element of T.
Therefore, $lub(T) = \sup(T) = 1$.
Since $1 \in T$, then $\sup(T) \in T$.
We proved in a previous example that the greatest element of $[0, 1]$ is 1. □

Example 12. Let $S \subset \mathbb{R}$.
Then $(S, \leq)$ is a total order.
Therefore, any subset of $\mathbb{R}$ is linearly ordered under the relation $\leq$.

Proof. Since $(\mathbb{R}, \leq)$ is a total order, then $\leq$ is reflexive, antisymmetric, transitive and any two elements are comparable.

Thus, for any $x \in \mathbb{R}$, $x \leq x$ and for any $x, y \in \mathbb{R}$, if $x \leq y$ and $y \leq x$, then $x = y$ and for any $x, y, z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$ and for any $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$.

Let $a, b, c \in S$.

Then $a \leq a$ and if $a \leq b$ and $b \leq a$, then $a = b$ and if $a \leq b$ and $b \leq c$, then $a \leq c$ and either $a \leq b$ or $b \leq a$.

Therefore, $\leq$ is reflexive, antisymmetric, transitive, and any two elements of S are comparable.

Hence, $(S, \leq)$ is a total order. □

Functions

Example 13. The inverse of the identity map on a set is the identity map on the set.

Let I_S be the identity map on a set S.

Then $I_S^{-1} = I_S$.

Proof. Let $g : S \to S$ be the map defined by $g(x) = x$ for all $x \in S$.

Then $g = I_S$.

Since $g \circ I_S = g = I_S$ and $I_S \circ g = g = I_S$, then g is the inverse of I_S.

Since $g = I_S$, then I_S is the inverse of I_S.

Therefore, $I_S^{-1} = I_S$. □