Relations and Functions Exercises

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Relations

Exercise 1. Let $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$.
Let $R = \{(1, x), (2, y), (3, z)\}$.
Analyze R.

Solution. Observe that $R \subset A \times B$, so R is a relation from A to B.
The domain of R is the set $\{1, 2, 3\} = A$.
The range of R is the set $\{x, y, z\} \subset B$.
The inverse of R is $R^{-1} = \{(x, 1), (y, 2), (z, 3)\} \subset B \times A$.
The domain of R^{-1} is the set $\{x, y, z\}$.
The range of R^{-1} is the set $\{1, 2, 3\} = A$. $\square$

Exercise 2. Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 3 < x \leq 8, -2 \leq y < 4\}$.
The relation R is a rectangle in the xy plane that is open on the left and top and closed on the right and bottom.

Exercise 3. Let H be the set of all living human beings.
Let $R = \{(x, y) \in H \times H : y \text{ is a biological parent of } x\}$.
Then $R \subset H \times H$, so R is a relation on H.

Exercise 4. Let $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$.
Let $R = \{(2, w), (2, x), (2, y), (2, z)\}$.
Analyze R.

Solution. Since $R \subset A \times B$, then R is a relation from A to B.
Observe that $R = \{(2, b) : b \in B\}$.
The domain of R is the set $\{2\} \subset A$.
The range of R is the set $\{w, x, y, z\} = B$.
The inverse of R is $R^{-1} = \{(w, 2), (x, 2), (y, 2), (z, 2)\} \subset B \times A$.
Thus, $R^{-1} = \{(b, 2) : b \in B\}$.
The domain of R^{-1} is the set $\{w, x, y, z\} = B$.
The range of R^{-1} is the set $\{2\}$. $\square$

Exercise 5. Let $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$.
Let $R = \{(1, z), (2, z), (3, z)\}$.
Analyze R.

Solution. Since $R \subset A \times B$, then R is a relation from A to B.

Observe that $R = \{(a, z) : a \in A\}$.

The domain of R is the set $\{1, 2, 3\} = A$.

The range of R is the set $\{z\}$.

The inverse of R is $R^{-1} = \{(z, 1), (z, 2), (z, 3)\} \subset B \times A$.

Thus, $R^{-1} = \{(z, a) : a \in A\}$.

The domain of R^{-1} is the set $\{z\}$.

The range of R^{-1} is the set $\{1, 2, 3\} = A$. $\square$

Exercise 6. Let $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$.

Let $R = \{(x, 1), (x, 3)\}$.

Analyze R.

Solution. Since $R \subset B \times A$, then R is a relation from B to A.

The domain of R is the set $\{x\} \subset B$.

The range of R is the set $\{1, 3\} \subset A$.

The inverse of R is $R^{-1} = \{(1, x), (3, x)\} \subset A \times B$.

The domain of R^{-1} is the set $\{1, 3\}$.

The range of R^{-1} is the set $\{x\}$. $\square$

Exercise 7. Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 3 < x \leq 8, -2 \leq y < 4\}$.

Analyze R.

Solution. Since $R \subset \mathbb{R} \times \mathbb{R}$, then R is a relation on $\mathbb{R}$.

The relation R is a rectangle in the xy plane that is open on the left and top and closed on the right and bottom.

The domain of R is the set $\{x \in \mathbb{R} : (\exists y \in \mathbb{R})((x, y) \in R\} = \{x \in \mathbb{R} : 3 < x \leq 8\} = (3, 8]$.

The range of R is the set $\{y \in \mathbb{R} : (\exists x \in \mathbb{R})((x, y) \in R\} = \{y \in \mathbb{R} : -2 \leq y < 4\} = [-2, 4)$.

Is R reflexive?

Observe that $1 \in \mathbb{R}$ and $(1, 1) \in \mathbb{R} \times \mathbb{R}$, but $(1, 1) \notin R$.

Therefore, R is not reflexive.

Is R symmetric?

Observe that $(5, 3) \in R$, but $(3, 5) \notin R$.

Therefore, R is not symmetric.

Is R transitive?

Let $x, y, z \in \mathbb{R}$ such that $(x, y) \in R$ and $(y, z) \in R$.

Since $(x, y) \in R$, then $3 < x \leq 8$.

Since $(y, z) \in R$, then $-2 \leq z < 4$.

Since $(x, z) \in \mathbb{R} \times \mathbb{R}$ and $3 < x \leq 8$ and $-2 \leq z < 4$, then $(x, z) \in R$.

Therefore, R is transitive.

Is R antisymmetric?

Observe that $3.1, 3.4 \in \mathbb{R}$ and $(3.1, 3.4) \in R$ and $(3.4, 3.1) \in R$, but $3.1 \neq 3.4$.

Therefore, R is not antisymmetric.

The inverse of R is $R^{-1} = \{(y, x) : (x, y) \in R\} = \{(y, x) : -2 \leq y < 4, 3 < x \leq 8\}$.

The inverse R^{-1} is a rectangle in the xy plane that is open on the right and bottom and closed on the left and top.

The domain of R^{-1} is the set $\{y \in \mathbb{R} : (\exists x)((y,x) \in R^{-1})\} = \{y \in \mathbb{R} : -2 \leq y < 4\} = [-2, 4)$.

The range of R^{-1} is the set $\{x \in \mathbb{R} : (\exists y)((y,x) \in R^{-1})\} = \{x \in \mathbb{R} : 3 < x \leq 8\} = (3, 8]$. $\square$

Exercise 8. Let H be the set of all living human beings.

Let $R = \{(x,y) \in H \times H : x \text{ is older than } y \text{ or is the same age as } y\}$.

Analyze R.

Solution. Observe that $R \subset H \times H$, so R is a relation on H.

By definition, the domain of R is a subset of H, so $domR \subset H$.

We prove $H \subset domR$.

Let $a \in H$.

To prove $a \in domR$, we must prove there exists $b \in H$ such that $(a,b) \in R$.

Hence, we must find a human b such that a is older than b or is the same age as b.

Let $b = a$.

Each person is the same age as himself, so b is the same age as b.

Therefore, $a \in domR$.

Therefore, $H \subset domR$.

Since $domR \subset H$ and $H \subset domR$, then $domR = H$.

By definition, the range of R is a subset of H, so $rangeR \subset H$.

Let $b \in H$.

Then b is the same age as b.

Thus, there exists $b \in H$ such that $(b,b) \in H$.

Hence, $b \in rangeR$.

Therefore, $H \subset rangeR$.

Since $rangeR \subset H$ and $H \subset rangeR$, then $rangeR = H$.

We consider which properties of relations hold for the relation R on H.

Define predicates $p(x,y) : x$ is older than y and

$q(x,y) : x$ is the same age as y over $H \times H$.

Then $R = \{(x,y) \in H \times H : p(x,y) \vee q(x,y)\}$. $\square$

Proof. We prove R is reflexive.

Let a be an arbitrary living human being.

Each human is the same age as himself, so a is the same age as a.

Hence, a is the same age as a or a is older than a, so a is older than a or a is the same age as a.

Therefore, every living human is older than himself or is the same age as himself.

Hence, R is reflexive.

We prove R is not symmetric.

Let a be a human who is older than b.

Then b is younger than a.

Thus, b is not older than a and b is not the same age as a.

Thus, neither b is older than a nor b is the same age as a.

Hence, R is not symmetric.

We prove R is transitive.

Let a, b, and c be arbitrary humans such that $(a, b) \in R$ and $(b, c) \in R$.

Then a is older than b or a is the same age as b and b is older than c or b is the same age as c.

Thus, there are 4 cases to consider.

Case 1: Suppose a is older than b and b is older than c.

Then a is older than c.

Case 2: Suppose a is older than b and b is the same age as c.

Then a is older than c.

Case 3: Suppose a is the same age as b and b is older than c.

Then a is older than c.

Case 4: Suppose a is the same age as b and b is the same age as c.

Then a is the same age as c.

Thus, in all cases, a is older than c or a is the same age as c.

Since $(a, c) \in H \times H$ and a is older than c or a is the same age as c, then $(a, c) \in R$.

Therefore, $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$, so R is transitive.

We prove R is not antisymmetric.

Let a and b be different people who have the same age.

Then $a, b \in H$ and $a \neq b$ and a is the same age as b.

Thus, $(a, b) \in H \times H$ and $(b, a) \in H \times H$.

Since a is the same age as b, then a is the same age as b or a is older than b.

Hence, a is older than b or a is the same age as b.

Since $(a, b) \in H \times H$, then this implies $(a, b) \in R$.

Since a is the same age as b, then b is the same age as a.

Thus, b is the same age as a or b is older than a.

Hence, b is older than a or b is the same age as a.

Since $(b, a) \in H \times H$, then this implies $(b, a) \in R$.

Thus, we have $(a, b) \in R$ and $(b, a) \in R$, but $a \neq b$.

Therefore, R is not antisymmetric. □

Exercise 9. Let H be the set of all living human beings.

Let $R = \{(x, y) \in H \times H : x \text{ and } y \text{ are both male or both female } \}$.

Analyze R.

Solution. Define predicate $p(x) : x$ is male over H.

Then $R = \{(x, y) \in H \times H : (p(x) \wedge p(y)) \vee (\neg p(x) \wedge \neg p(y))\}$.

Since $R \subset H \times H$, then R is a relation on H.

By definition, the domain of R is a subset of H, so $domR \subset H$.

Let $a \in H$.

Then $(a, a) \in H \times H$ and either a is male or a is not male.

Thus, either a is male or a is female.

Hence, either a is male and a is male, or a is female and a is female, so either a and a are both male or a and a are both female.

Thus, there exists $a \in H$ such that $(a, a) \in R$.
Therefore, $a \in domR$.
Thus, $a \in H$ implies $a \in domR$, so $H \subset domR$.
Since $domR \subset H$ and $H \subset domR$, then $domR = H$.
By definition, the range of R is a subset of H, so $rangeR \subset H$.
Let $b \in H$.
Then $(b, b) \in H \times H$ and either b is male or b is female.
Hence, either b and b are male or b and b are female, so either b and b are both male or both female.
Hence, there exists $b \in H$ such that $(b, b) \in R$.
Therefore, $b \in rangeR$.
Thus, $b \in H$ implies $b \in rangeR$, so $H \subset rangeR$.
Since $rangeR \subset H$ and $H \subset rangeR$, then $rangeR = H$.
We prove R is reflexive.
Let $a \in H$.
Then $(a, a) \in H \times H$ and either a is male or a is not male.
Thus, either a is male or a is female.
Hence, either a and a are male or a and a are female, so either a and a are both male or a and a are both female.
Therefore, $(a, a) \in R$, so R is reflexive.
We prove R is symmetric.
Let $a, b \in H$ such that $(a, b) \in R$.
Then $(a, b) \in H \times H$ and $(b, a) \in H \times H$ and either a and b are both male or a and b are both female.
Thus, either b and a are both male or b and a are both female.
Since $(b, a) \in H \times H$, then this implies $(b, a) \in R$.
Hence, $(a, b) \in R$ implies $(b, a) \in R$, so R is symmetric.
We prove R is transitive.
Let $a, b, c \in H$ such that $(a, b) \in R$ and $(b, c) \in R$.
Then $(a, c) \in H \times H$ and a and b are both male or both female and b and c are both male or both female.
Either a is male or a is not male.
We consider these cases separately.
Case 1: Suppose a is male.
Suppose a and b are both female.
Then a is female, so a is not male.
Thus, we have a is male and a is not male, a contradiction.
Therefore, a and b cannot be both female.
Since either a and b are both male or both female, then this implies a and b are both male.
Hence, b is male.
Suppose b and c are both female.
Then b is female, so b is not male.
Hence, b is male and b is not male, a contradiction.
Therefore, b and c cannot be both female.

Since either b and c are both male or both female, then this implies b and c are both male.

Hence, c is male.

Therefore, a is male and c is male, so a and c are both male.

Since $(a, c) \in H \times H$, then this implies $(a, c) \in R$.

Case 2: Suppose a is not male.

Suppose a and b are both male.

Then a is male.

Thus, we have a is not male and a is male, a contradiction.

Therefore, a and b cannot be both male.

Since either a and b are both male or both female, then this implies a and b are both female.

Hence, b is female.

Suppose b and c are both male.

Then b is male, so b is not female.

Hence, b is female and b is not female, a contradiction.

Therefore, b and c cannot be both male.

Since either b and c are both male or both female, then this implies b and c are both female.

Hence, c is female.

Since a is not male, then a is female.

Since a is female and c is female, then a and c are both female.

Since $(a, c) \in H \times H$, then this implies $(a, c) \in R$.

Thus, in either case, $(a, c) \in R$.

Therefore, $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$, so R is transitive.

Since R is reflexive, symmetric, and transitive, then R is an equivalence relation on H.

Since R is an equivalence relation, then the collection of equivalence classes $\frac{H}{R}$ is a partition of H.

Let $h \in H$.

Then either h is male or h is female.

If h is male, then $[h] = \{y \in H : hRy\} = \{y \in H : (h, y) \in R\} = \{y \in H :$ h and y are both male or both female $\} = \{y \in H : h$ and y are both male $\} = \{y \in H : y$ is male $\}$.

Thus, $[h]$ is the set of all living male humans.

On the other hand, if h is female, then $[h] = \{y \in H : hRy\} = \{y \in H : (h, y) \in R\} = \{y \in H : h$ and y are both male or both female $\} = \{y \in H : h$ and y are both female $\} = \{y \in H : y$ is female $\}$.

Thus, $[h]$ is the set of all living female humans.

Therefore, quotient set $\frac{H}{R} = \{[h] : h \in H\}$ consists of the set of all living male humans and the set of all living female humans and is a 2 celled partition of H.

We prove R is not antisymmetric.

Let a and b be two different males.

Then $a, b \in H$ and $a \neq b$ and a is male and b is male and $(a, b) \in H \times H$ and $(b, a) \in H \times H$.

Since a is male and b is male, then a and b are both male.

Hence, a and b are both male or a and b are both female.

Since $(a, b) \in H \times H$, then this implies $(a, b) \in R$.

Since a is male and b is male, then b is male and a is male.

Hence, b and a are both male.

Thus, b and a are both male or b and a are both female.

Since $(b, a) \in H \times H$, then this implies $(b, a) \in R$.

Thus, there exist $a, b \in H$ such that $(a, b) \in R$ and $(b, a) \in R$, but a and b are different humans.

Therefore, R is not antisymmetric. $\square$

Exercise 10. Let H be the set of all living human beings.

Let $R = \{(x, y) \in H \times H : y \text{ is a biological parent of } x\}$.

Analyze R.

Solution. Define predicate $p(x, y) : y$ is a biological parent of x over $H \times H$.

Then the truth set of $p(x, y)$ is the relation R, a subset of $H \times H$ and $R = \{(x, y) : p(x, y)\}$.

By definition, the domain of R is a subset of H, so $domR \subset H$.

Let $a \in H$.

Every human has biological parents.

Thus, there is at least one biological parent of a.

Hence, there exists $b \in H$ such that b is a biological parent of a.

Since $(a, b) \in H \times H$, then this implies $(a, b) \in R$.

Thus, there exists $b \in H$ such that $(a, b) \in R$, so $a \in domR$.

Therefore, $a \in H$ implies $a \in domR$, so $H \subset domR$.

Since $domR \subset H$ and $H \subset domR$, then $domR = H$.

By definition, the range of R is a subset of H, so $rangeR \subset H$.

Is $H \subset rangeR$?

Since not everyone is a parent, then $H \not\subset rangeR$.

Therefore, $rangeR \neq H$.

The inverse of R is $R^{-1} = \{(y, x) : (x, y) \in R\} = \{(y, x) : y \text{ is a biological parent of } x\}$.

Thus, $R^{-1} = \{(y, x) : y \text{ is a biological parent of } x\}$. $\square$

Proof. We prove R is not reflexive.

No human is his own biological parent.

Hence, there is no human h such that h is the parent of h.

Thus, for each human h, h is not the parent of h.

Since there are living humans, then this implies there is a human a such that a is not the parent of a.

Thus, there exists $(a, a) \in H \times H$ such that a is not the parent of a.

Hence, there exists $a \in H$ such that $(a, a) \notin R$.

Therefore, R is not reflexive.

We prove R is not symmetric.

Let a be a biological child of b.

Then b is a biological parent of a and a is not a biological parent of b.

Since $(a, b) \in H \times H$ and b is a biological parent of a, then $(a, b) \in R$.

Since $(b, a) \in H \times H$ and a is not a biological parent of b, then $(b, a) \notin R$.

Thus, there exist $a, b \in H$ such that $(a, b) \in R$ and $(b, a) \notin R$, so R is not symmetric.

We prove R is not transitive.

Let b be a parent of a and let c be a parent of b.

Thus, c is a grandparent of a, so c is not a parent of a.

Since $(a, b) \in H \times H$ and b is a parent of a, then $(a, b) \in R$.

Since $(b, c) \in H \times H$ and c is a parent of b, then $(b, c) \in R$.

Since $(a, c) \in H \times H$ and c is not a parent of a, then $(a, c) \notin R$.

Thus, $(a, b) \in R$ and $(b, c) \in R$, but $(a, c) \notin R$, so R is not transitive.

We prove R is antisymmetric.

Let $a, b \in H$.

Either a and b are the same person or a and b are different people.

Thus, there are 2 cases to consider.

Case 1: Suppose a and b are the same person.

Since no person is his own parent, then a is not his own biological parent.

Thus, a is a biological parent of a is false.

Hence, a is a biological parent of a and a is a biological parent of a is false.

Thus, b is a biological parent of a and a is a biological parent of b is false.

Hence, $(a, b) \in R$ and $(b, a) \in R$ is false.

Case 2: Suppose a and b are two different people.

Since it is impossible for two different people to be each other's biological parents, then the statement b is a biological parent of a and a is a biological parent of b is false.

Hence, $(a, b) \in R$ and $(b, a) \in R$ is false.

Thus, in either case, the statement $(a, b) \in R$ and $(b, a) \in R$ is false.

Hence, the conditional $(a, b) \in R$ and $(b, a) \in R$ implies a and b are the same person is vacuously true.

Therefore, R is antisymmetric. $\square$

Exercise 11. identity relation on $\mathbb{Z}$

Let $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m = n\}$.

Analyze R.

Solution. Since $R \subset \mathbb{Z} \times \mathbb{Z}$, then R is a relation on $\mathbb{Z}$.

By definition of domain, the domain of R is a subset of $\mathbb{Z}$, so $domR \subset \mathbb{Z}$.

We prove $\mathbb{Z} \subset domR$.

Let $a \in \mathbb{Z}$.

To prove $\mathbb{Z} \subset domR$, we must find an integer b such that $a = b$.

Let $b = a$.

Then b is an integer and $a = b$.

Thus, $\mathbb{Z} \subset domR$.

Therefore, $domR \subset \mathbb{Z}$ and $\mathbb{Z} \subset domR$, so $domR = \mathbb{Z}$.

The range of R is a subset of $\mathbb{Z}$, by definition of range.

Thus, $rangeR \subset \mathbb{Z}$.

We prove $\mathbb{Z} \subset rangeR$.

Let $b \in \mathbb{Z}$.

To prove $\mathbb{Z} \subset rangeR$, we must find an integer a such that $a = b$.

Let $a = b$.

Then a is an integer and $a - b$.

Thus, $\mathbb{Z} \subset rangeR$.

Therefore, $rangeR \subset \mathbb{Z}$ and $\mathbb{Z} \subset rangeR$, so $rangeR = \mathbb{Z}$.

The inverse of R is $R^{-1} = \{(n,m) \in \mathbb{Z} \times \mathbb{Z} : (m,n) \in R\} = \{(n,m) \in \mathbb{Z} \times \mathbb{Z} : m = n\} = \{(n,m) \in \mathbb{Z} \times \mathbb{Z} : n = m\} = R$.

Therefore, $R^{-1} = R$.

We prove R is reflexive.

Let $a \in \mathbb{Z}$.

Since every integer equals itself, then in particular, $a = a$.

Since $(a,a) \in \mathbb{Z} \times \mathbb{Z}$ and $a = a$, then $(a,a) \in R$.

Therefore, R is reflexive.

We prove R is symmetric.

Let a and b be arbitrary integers such that $(a,b) \in R$.

Then $a = b$, so $b = a$.

Since $(b,a) \in \mathbb{Z} \times \mathbb{Z}$ and $b = a$, then $(b,a) \in R$.

Therefore, $(a,b) \in R$ implies $(b,a) \in R$, so R is symmetric.

We prove R is transitive.

Let $a, b, c \in \mathbb{Z}$ such that $(a,b) \in R$ and $(b,c) \in R$.

Then $a = b$ and $b = c$, so $a = c$.

Since $(a,c) \in \mathbb{Z} \times \mathbb{Z}$ and $a = c$, then $(a,c) \in R$.

Hence, $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$, so R is transitive.

Since R is reflexive, symmetric, and transitive, then R is an equivalence relation on $\mathbb{Z}$.

Since R is an equivalence relation, then the collection of equivalence classes $\frac{\mathbb{Z}}{R}$ is a partition of $\mathbb{Z}$.

Let $n \in \mathbb{Z}$.

Then $[n] = \{y \in \mathbb{Z} : n = y\} = \{n\}$.

Thus, the equivalence class of any integer n is the singleton set consisting of n itself.

Therefore, $\frac{\mathbb{Z}}{R} = \{[n] : n \in \mathbb{Z}\} = \{\{n\} : n \in \mathbb{Z}\}$, a collection of all singleton sets of all integers, is an infinitely many celled partition of $\mathbb{Z}$.

We prove R is antisymmetric.

Let a and b be arbitrary integers such that $(a,b) \in R$ and $(b,a) \in R$.

Then $a = b$ and $b = a$, so $a = b$.

Thus, $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$, so R is antisymmetric. $\square$

Exercise 12. Let $R = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m \text{ and } n \text{ are both even or both odd } \}$.

Analyze R.

Solution. Define predicates $p(x) : x$ is even and $q(x) : x$ is odd over $\mathbb{Z}$.

Then $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : (p(m) \wedge p(n)) \vee (q(m) \wedge q(n))\}$.

Since $R \subset \mathbb{Z} \times \mathbb{Z}$, then R is a relation on $\mathbb{Z}$.

By definition, the domain of R is a subset of $\mathbb{Z}$, so $domR \subset \mathbb{Z}$.
Let $a \in \mathbb{Z}$.
Let b be the integer $a + 2$.
Then $(a, b) \in \mathbb{Z} \times \mathbb{Z}$.
Either a is even or a is not even.
We consider these cases separately.
Case 1: Suppose a is even.
Then $a = 2k$ for some integer k.
Thus, $b = a + 2 = 2k + 2 = 2(k + 1)$.
Hence, b is even.
Thus, a and b are even, so $(a, b) \in R$.
Case 2: Suppose a is not even.
Then a is odd, so $a = 2m + 1$ for some integer m.
Thus, $b = a + 2 = (2m + 1) + 2 = 2m + 3 = 2(m + 1) + 1$.
Hence, b is odd, so a and b are odd.
Therefore, $(a, b) \in R$.
Thus, in either case, there is an integer b such that $(a, b) \in R$.
Hence, $a \in domR$.
Therefore, $a \in \mathbb{Z}$ implies $a \in domR$, so $\mathbb{Z} \subset domR$.
Since $domR \subset \mathbb{Z}$ and $\mathbb{Z} \subset domR$, then $domR = \mathbb{Z}$.
By definition, the range of R is a subset of $\mathbb{Z}$, so $rangeR \subset \mathbb{Z}$.
We prove $\mathbb{Z} \subset rangeR$.
Let $b \in \mathbb{Z}$.
Let a be the integer $b + 2$.
Then $(a, b) \in \mathbb{Z} \times \mathbb{Z}$.
Either b is even or b is not even.
We consider these cases separately.
Case 1: Suppose b is even.
Then $b = 2k$ for some integer k.
Thus, $a = b + 2 = 2k + 2 = 2(k + 1)$.
Hence, a is even.
Thus, a and b are even, so $(a, b) \in R$.
Case 2: Suppose b is not even.
Then b is odd, so $b = 2m + 1$ for some integer m.
Thus, $a = b + 2 = (2m + 1) + 2 = 2m + 3 = 2(m + 1) + 1$.
Hence, a is odd, so a and b are odd.
Therefore, $(a, b) \in R$.
Thus, in either case, there is an integer a such that $(a, b) \in R$.
Hence, $b \in rangeR$.
Therefore, $b \in \mathbb{Z}$ implies $b \in rangeR$, so $\mathbb{Z} \subset rangeR$.
Since $rangeR \subset \mathbb{Z}$ and $\mathbb{Z} \subset rangeR$, then $rangeR = \mathbb{Z}$.
We prove R is reflexive.
Let $a \in \mathbb{Z}$.
Then $(a, a) \in \mathbb{Z} \times \mathbb{Z}$ and either a is even or a is not even.
Thus, either a is even or a is odd.

Hence, either a is even and a is even or a is odd and a is odd, so either a and a are both even or a and a are both odd.

Thus, a and a are both even or both odd.

Hence, $(a, a) \in R$.

Therefore, R is reflexive.

We prove R is symmetric.

Let $a, b \in \mathbb{Z}$ such that $(a, b) \in R$.

Then $(b, a) \in \mathbb{Z} \times \mathbb{Z}$ and a and b are both even or both odd.

Thus, either a is even and b is even or a is odd and b is odd.

Hence, either b is even and a is even or b is odd and a is odd.

Thus, b and a are both even or b and a are both odd, so b and a are both even or both odd.

Therefore, $(b, a) \in R$.

Thus, $(a, b) \in R$ implies $(b, a) \in R$, so R is symmetric.

We prove R is transitive.

Let $a, b, c \in \mathbb{Z}$ such that $(a, b) \in R$ and $(b, c) \in R$.

Then $(a, c) \in \mathbb{Z} \times \mathbb{Z}$ and a and b are both even or both odd and b and c are both even or both odd.

Either a is even or a is not even.

We consider these cases separately.

Case 1: Suppose a is even.

Suppose a and b are both odd.

Then a is odd, so a is not even.

Thus, we have a is even and a is not even, a contradiction.

Therefore, a and b cannot be both odd.

Since either a and b are both even or both odd, then this implies a and b are both even.

Hence, b is even.

Suppose b and c are both odd.

Then b is odd, so b is not even.

Hence, b is even and b is not even, a contradiction.

Therefore, b and c cannot be both odd.

Since either b and c are both even or both odd, then this implies b and c are both even.

Hence, c is even.

Therefore, a is even and c is even, so a and c are both even.

Thus, $(a, c) \in R$.

Case 2: Suppose a is not even.

Suppose a and b are both even.

Then a is even.

Thus, we have a is not even and a is even, a contradiction.

Therefore, a and b cannot be both even.

Since either a and b are both even or both odd, then this implies a and b are both odd.

Hence, b is odd.

Suppose b and c are both even.

Then b is even, so b is not odd.
Hence, b is odd and b is not odd, a contradiction.
Therefore, b and c cannot be both even.
Since either b and c are both even or both odd, then this implies b and c are both odd.
Hence, c is odd.
Since a is not even, then a is odd.
Thus, a is odd and c is odd, so a and c are both odd.
Hence, $(a, c) \in R$.
Thus, in either case, $(a, c) \in R$.
Therefore, $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$, so R is transitive.
Since R is reflexive, symmetric, and transitive, then R is an equivalence relation on $\mathbb{Z}$.
Since R is an equivalence relation, then the collection of equivalence classes $\frac{\mathbb{Z}}{R}$ is a partition of $\mathbb{Z}$.
Let $n \in \mathbb{Z}$.
Then either n is even or n is odd.
If n is even, then $[n] = \{y \in \mathbb{Z} : nRy\} = \{y \in \mathbb{Z} : (n, y) \in R\} = \{y \in \mathbb{Z} :$ n and y are both even or both odd $\} = \{y \in \mathbb{Z} : n$ and y are both even $\} = \{y \in \mathbb{Z} : y$ is even $\}$.
Thus, $[n]$ is the set of all even integers.
On the other hand, if n is odd, then $[n] = \{y \in \mathbb{Z} : nRy\} = \{y \in \mathbb{Z} : (n, y) \in R\} = \{y \in \mathbb{Z} : n$ and y are both even or both odd $\} = \{y \in \mathbb{Z} : n$ and y are both odd $\} = \{y \in \mathbb{Z} : y$ is odd $\}$.
Thus, $[n]$ is the set of all odd integers.
Therefore, quotient set $\frac{\mathbb{Z}}{R} = \{[n] : n \in \mathbb{Z}\}$ consists of the set of all even integers and the set of all odd integers and is a 2 celled partition of $\mathbb{Z}$.
We prove R is not antisymmetric.
Since 2 and 6 are both even, then $(2, 6) \in R$.
Since 6 and 2 are both even, then $(6, 2) \in R$.
Thus, $(2, 6) \in R$ and $(6, 2) \in R$, but $2 \neq 6$.
Therefore, R is not antisymmetric. $\square$

Exercise 13. less than or equals to relation on $\mathbb{Z}$
Let $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m \leq n\}$.
Analyze R.

Solution. Since $R \subset \mathbb{Z} \times \mathbb{Z}$, then R is a relation on $\mathbb{Z}$.
By definition of domain, the domain of R is a subset of $\mathbb{Z}$, so $domR \subset \mathbb{Z}$.
We prove $\mathbb{Z} \subset domR$.
Let $a \in \mathbb{Z}$.
To prove $\mathbb{Z} \subset domR$, we must find an integer b such that $a \leq b$.
Let $b = a$.
Then b is an integer and $a = b$, so $a = b$ or $a < b$.
Hence, $a < b$ or $a = b$, so $a \leq b$.
Since $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, then this implies $(a, b) \in R$.

Thus, there exists an integer b such that $(a, b) \in R$, so $a \in domR$.
Therefore, $a \in \mathbb{Z}$ implies $a \in domR$, so $\mathbb{Z} \subset domR$.
Since $domR \subset \mathbb{Z}$ and $\mathbb{Z} \subset domR$, then $domR = \mathbb{Z}$.
The range of R is a subset of $\mathbb{Z}$, by definition of range.
Thus, $rangeR \subset \mathbb{Z}$.
We prove $\mathbb{Z} \subset rangeR$.
Let $b \in \mathbb{Z}$.
To prove $\mathbb{Z} \subset rangeR$, we must find an integer a such that $a \leq b$.
Let $a = b$.
Then a is an integer and $a = b$ or $a < b$.
Thus, $a < b$ or $a = b$, so $a \leq b$.
Hence, $b \in rangeR$.
Therefore, $b \in \mathbb{Z}$ implies $b \in rangeR$, so $\mathbb{Z} \subset rangeR$.
Thus, $rangeR \subset \mathbb{Z}$ and $\mathbb{Z} \subset rangeR$, so $rangeR = \mathbb{Z}$.

The graph of R is the half plane of integers to the left of and on the line $y = x$.

The inverse of R is the set $R^{-1} = \{(n, m) : (m, n) \in R\} = \{(n, m) : m \leq n\} = \{(n, m) : n \geq m\}$.

Therefore, $R^{-1} = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} : n \geq m\}$.
We prove R is reflexive.
Let a be an arbitrary integer.
Then $(a, a) \in \mathbb{Z} \times \mathbb{Z}$.
Since every integer equals itself, then $a = a$.
Thus, $a = a$ or $a < a$, so $a < a$ or $a = a$.
Hence, $a \leq a$, so $(a, a) \in R$.
Therefore, R is reflexive.
We prove R is not symmetric.
Observe that 2 and 3 are integers and $2 \leq 3$ but $3 \not\leq 2$.
Thus, $(2, 3) \in R$ but $(3, 2) \notin R$, so R is not symmetric.
We prove R is transitive.
Let $x, y, z \in \mathbb{Z}$ such that $(x, y) \in R$ and $(y, z) \in R$.
Then $x \leq y$ and $y \leq z$, so $x < y$ or $x = y$ and $y < z$ or $y = z$.

Hence, either both $x < y$ or $x = y$ and $y < z$, or both $x < y$ or $x = y$ and $y = z$.

Thus, either $x < y$ and $y < z$ or $x = y$ and $y < z$ or $x < y$ and $y = z$ or $x = y$ and $y = z$.

Therefore, there are 4 cases to consider.
Case 1: Suppose $x < y$ and $y < z$.
Then by transitivity of $<$, $x < z$.
Case 2: Suppose $x < y$ and $y = z$.
Then $x < z$.
Case 3: Suppose $x = y$ and $y < z$.
Then $x < z$.
Case 4: Suppose $x = y$ and $y = z$.
Then $x = z$.
Thus, in all cases, either $x < z$ or $x = z$, so $x \leq z$.

Since $(x,z) \in \mathbb{Z} \times \mathbb{Z}$ and $x \leq z$, then $(x,z) \in R$.
Therefore, $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$, so R is transitive.
We prove R is antisymmetric.
Let $a,b \in \mathbb{Z}$ such that $(a,b) \in R$ and $(b,a) \in R$.
Then $a \leq b$ and $b \leq a$.
Thus, $a < b$ or $a = b$ and $b < a$ or $b = a$.
Hence, $a = b$ or $a < b$ and $a = b$ or $b < a$.
Therefore, either $a = b$ or $a < b$ and $b < a$.
By trichotomy, the statement $a < b$ and $b < a$ is false.
Hence, $a = b$.
Therefore, $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$, so R is antisymmetric. $\square$

Exercise 14. divides relation on $\mathbb{Z}$
Let $R = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m|n\}$.
Analyze R.

Solution. Since $R \subset \mathbb{Z} \times \mathbb{Z}$, then R is a relation on $\mathbb{Z}$.
By definition of domain, the domain of R is a subset of $\mathbb{Z}$, so $domR \subset \mathbb{Z}$.
We prove $\mathbb{Z} \subset domR$.
Let $a \in \mathbb{Z}$.
To prove $\mathbb{Z} \subset domR$, we must find an integer b such that $a|b$.
Let $b = a$.
Then b is an integer.
Since every integer divides itself, then $b|b$.
Hence, $a|b$, so $\mathbb{Z} \subset domR$.
Therefore, $domR \subset \mathbb{Z}$ and $\mathbb{Z} \subset domR$, so $domR = \mathbb{Z}$.
The range of R is a subset of $\mathbb{Z}$, by definition of range.
Thus, $rangeR \subset \mathbb{Z}$.
We prove $\mathbb{Z} \subset rangeR$.
Let $b \in \mathbb{Z}$.
To prove $\mathbb{Z} \subset rangeR$, we must find an integer a such that $a|b$.
Let $a = b$.
Then a is an integer.
Since every integer divides itself, then $a|a$.
Hence, $a|b$, so $\mathbb{Z} \subset rangeR$.
Thus, $rangeR \subset \mathbb{Z}$ and $\mathbb{Z} \subset rangeR$, so $rangeR = \mathbb{Z}$.
The inverse of R is $R^{-1} = \{(n,m) \in \mathbb{Z} \times \mathbb{Z} : (m,n) \in R\} = \{(n,m) \in \mathbb{Z} \times \mathbb{Z} : m|n\}$.
We prove R is reflexive.
Let $a \in \mathbb{Z}$.
Then $(a,a) \in \mathbb{Z} \times \mathbb{Z}$.
Since every integer divides itself, then in particular, $a|a$.
Since $(a,a) \in \mathbb{Z} \times \mathbb{Z}$ and $a|a$, then $(a,a) \in R$.
Therefore, R is reflexive.
We prove R is not symmetric.

Observe that 3 and 6 are integers and $(3,6) \in \mathbb{Z} \times \mathbb{Z}$ and $(6,3) \in \mathbb{Z} \times \mathbb{Z}$ and $3|6$, but $6 \not| 3$.

Thus, $(3,6) \in R$, but $(6,3) \notin R$.

Therefore, R is not symmetric.

We prove R is transitive.

Let $a,b,c \in \mathbb{Z}$ such that $(a,b) \in R$ and $(b,c) \in R$.

Then $(a,c) \in \mathbb{Z} \times \mathbb{Z}$ and $a|b$ and $b|c$.

Thus, $b = ak_1$ for some integer k_1 and $c = bk_2$ for some integer k_2.

Hence, $c = (ak_1)k_2 = a(k_1k_2)$.

Since k_1k_2 is an integer, then $a|c$.

Thus, $(a,c) \in \mathbb{Z} \times \mathbb{Z}$ and $a|c$, so $(a,c) \in R$.

Therefore, $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$, so R is transitive.

To prove R is not antisymmetric, we must find integers a and b such that $a|b$ and $b|a$ and $a \neq b$.

Let $a = 1$ and $b = -1$.

Then $1| - 1$ and $-1|1$, but $1 \neq -1$.

Therefore, R is not antisymmetric. ☐

Exercise 15. Let $R = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m \geq 0, n \geq 0\}$.

Analyze R.

Solution. Since $R \subset \mathbb{Z} \times \mathbb{Z}$, then R is a relation on $\mathbb{Z}$.

The domain of R is the set $\{x \in \mathbb{Z} : (\exists y \in \mathbb{Z})((x,y) \in R)\} = \{x \in \mathbb{Z} : (\exists y \in \mathbb{Z})(x \geq 0 \wedge y \geq 0)\} = \{x \in \mathbb{Z} : x \geq 0\} = \{x \in \mathbb{Z} : x > 0 \vee x = 0\} = \{x \in \mathbb{Z} : x > 0\} \cup \{0\} = \mathbb{Z}^+ \cup \{0\}$.

The range of R is the set $\{y \in \mathbb{Z} : (\exists x \in \mathbb{Z})((x,y) \in R)\} = \{y \in \mathbb{Z} : (\exists x \in \mathbb{Z})(x \geq 0, y \geq 0)\} = \{y \in \mathbb{Z} : y \geq 0\} = \mathbb{Z}^+ \cup \{0\} = domR$.

The graph of R is the grid of integers in quadrant I of the xy plane, including the positive x and y axes and $(0,0)$.

We prove R is not reflexive.

Since -1 is an integer, then $(-1,-1) \in \mathbb{Z} \times \mathbb{Z}$ and $-1 < 0$.

Thus, $-1 < 0$ or $-1 < 0$, so $(-1,-1) \notin R$.

Therefore, R is not reflexive.

We prove R is symmetric.

Let $x,y \in \mathbb{Z}$ such that $(x,y) \in R$.

Then $(y,x) \in \mathbb{Z} \times \mathbb{Z}$ and $x \geq 0$ and $y \geq 0$.

Thus, $y \geq 0$ and $x \geq 0$.

Hence, $(y,x) \in R$.

Therefore, $(x,y) \in R$ implies $(y,x) \in R$, so R is symmetric.

We prove R is transitive.

Let $x,y,z \in \mathbb{Z}$ such that $(x,y) \in R$ and $(y,z) \in R$.

Since x and z are integers, then $(x,z) \in \mathbb{Z} \times \mathbb{Z}$.

Since $(x,y) \in R$, then $x \geq 0$.

Since $(y,z) \in R$, then $z \geq 0$.

Thus, $(x,z) \in \mathbb{Z} \times \mathbb{Z}$ and $x \geq 0$ and $z \geq 0$, so $(x,z) \in R$.

Therefore, R is transitive.

We prove R is not antisymmetric.

Since 2 and 3 are integers and $2 \geq 0$ and $3 \geq 0$, then $(2,3) \in R$ and $(3,2) \in R$, but $2 \neq 3$.

Therefore, R is not antisymmetric. $\square$

Exercise 16. subset relation on the power set of a set

Let X be a finite set.

Let $R = \{(M,N) \in 2^X \times 2^X : M \subset N\}$.

Analyze R.

Solution. Since $R \subset 2^X \times 2^X$, then R is a relation on 2^X, the power set of X.

By definition of domain, the domain of R is a subset of 2^X, so $domR \subset 2^X$.

We prove $2^X \subset domR$.

Let $A \in 2^X$.

Then $A \subset X$.

To prove $2^X \subset domR$, we must find a set $B \in 2^X$ such that $A \subset B$.

Let $B = X$.

Since $A \subset X$ and $X = B$, then $A \subset B$.

Since every set is a subset of itself, then in particular, $X \subset X$.

Hence, $X \in 2^X$, so $B \in 2^X$.

Thus, there is a set $B \in 2^X$ such that $A \subset B$, so $2^X \subset domR$.

Therefore, $domR \subset 2^X$ and $2^X \subset domR$, so $domR = 2^X$.

The range of R is a subset of 2^X, by definition of range.

Thus, $rangeR \subset 2^X$.

We prove $2^X \subset rangeR$.

Let $B \in 2^X$.

Then $B \subset X$, so B is a set.

To prove $2^X \subset rangeR$, we must find a set $A \in 2^X$ such that $A \subset B$.

Let $A = \emptyset$.

Since the empty set is a subset of every set, then in particular, $\emptyset \subset X$ and $\emptyset \subset B$.

Since $\emptyset \subset X$, then $A \subset X$, so $A \in 2^X$.

Since $\emptyset \subset B$, then $A \subset B$.

Thus, there exists a set $A \in 2^X$ such that $A \subset B$, so $2^X \subset rangeR$.

Therefore, $rangeR \subset 2^X$ and $2^X \subset rangeR$, so $rangeR = 2^X$.

We prove R is reflexive.

Let $A \in 2^X$.

Then $A \subset X$, so A is a set.

Every set is a subset of itself, so $A \subset A$.

Since $(A,A) \in 2^X \times 2^X$ and $A \subset A$, then $(A,A) \in R$.

Therefore, R is reflexive.

We prove R is not symmetric.

Let X be a nonempty finite set.

Then $X \neq \emptyset$.

Let $A = \emptyset$ and $B = X$.

Since the empty set is a subset of every set, then $\emptyset \subset X$.

Hence, $A \subset B$.
Since X is not empty, then there is an element in X.
Let $x \in X$.
Suppose $X \subset \emptyset$.
Then $x \in \emptyset$.
But, $\emptyset$ contains no elements, so $x \notin \emptyset$.
Hence, $X \not\subset \emptyset$, so $B \not\subset A$.
Since the empty set is a subset of every set, then $\emptyset \subset X$, so $A \subset X$.
Hence, $A \in 2^X$.
Since every set is a subset of itself, then $X \subset X$, so $X \in 2^X$.
Hence, $B \in 2^X$.
Since $(A, B) \in 2^X \times 2^X$ and $A \subset B$, then $(A, B) \in R$.
Since $(B, A) \in 2^X \times 2^X$ and $B \not\subset A$, then $(B, A) \notin R$.
Thus, $(A, B) \in R$ but $(B, A) \notin R$, so R is not symmetric.
We prove R is transitive.
Let $A, B, C \in 2^X$ such that $(A, B) \in R$ and $(B, C) \in R$.
Then $A \subset B$ and $B \subset C$.
Thus, by transitivity of the subset relation, $A \subset C$.
Since $(A, C) \in 2^X \times 2^X$ and $A \subset C$, then $(A, C) \in R$.
Hence, $(A, B) \in R$ and $(B, C) \in R$ implies $(A, C) \in R$, so R is transitive.
We prove R is antisymmetric.
Let $A, B \in 2^X$ such that $(A, B) \in R$ and $(B, A) \in R$.
Then $A \subset B$ and $B \subset A$.
Thus, $A = B$.
Since $(A, B) \in R$ and $(B, A) \in R$ implies $A = B$, then R is antisymmetric.
□

Exercise 17. Let X be a finite set.
Let $R = \{(M, N) \in 2^X \times 2^X : M \cap N = \emptyset\}$.
Analyze R.

Solution. Since $R \subset 2^X \times 2^X$, then R is a relation on 2^X, the power set of X.
We prove $domR = rangeR$.
Let $A \in domR$.
Then $A \in 2^X$ and there is a set $B \in 2^X$ such that $A \cap B = \emptyset$.
Hence, $A \in 2^X$ and there is a set $B \in 2^X$ such that $B \cap A = \emptyset$.
Thus, $A \in rangeR$.
Therefore, $A \in domR$ implies $A \in rangeR$, so $domR \subset rangeR$.
Let $B \in rangeR$.
Then $B \in 2^X$ and there is a set $A \in 2^X$ such that $A \cap B = \emptyset$.
Hence, $B \in 2^X$ and there is a set $A \in 2^X$ such that $B \cap A = \emptyset$.
Thus, $B \in domR$.
Therefore, $B \in rangeR$ implies $B \in domR$, so $rangeR \subset domR$.
Since $domR \subset rangeR$ and $rangeR \subset domR$, then $domR = rangeR$.
We prove $domR = 2^X$.
Let $A \in domR$.

Then $A \in 2^X$, by definition of domain.

Hence, $A \in domR$ implies $A \in 2^X$, so $domR \subset 2^X$.

Let $B \in 2^X$.

Then $B \subset X$.

To prove $B \in domR$, we must prove $B \in 2^X$ and there is a set $A \in 2^X$ such that $B \cap A = \emptyset$.

Since $B \in 2^X$, we need only prove there is a set $A \in 2^X$ such that $B \cap A = \emptyset$.

Let $A = \emptyset$.

Since the empty set is a subset of every set, then in particular, $A \subset X$.

Hence, $A \in 2^X$.

Observe that $\emptyset = B \cap \emptyset = B \cap A$.

Therefore, $B \in domR$.

Thus, $B \in 2^X$ implies $B \in domR$, so $2^X \subset domR$.

Since $domR \subset 2^X$ and $2^X \subset domR$, then $domR = 2^X$.

Therefore, $2^X = domR = rangeR$.

Let X be a finite set of at least two elements.

Then $X \neq \emptyset$.

To prove R is not reflexive, we must find a set $A \in 2^X$ such that $A \cap A \neq \emptyset$.

Let $A = X$.

Since every set is a subset of itself, then $X \subset X$.

Hence, $A \subset X$, so $A \in 2^X$.

Thus, $A \cap A = A = X \neq \emptyset$, so $A \cap A \neq \emptyset$.

Therefore, R is not reflexive.

We prove R is symmetric.

Let $A, B \in 2^X$ such that $(A, B) \in R$.

Then $A \cap B = \emptyset$.

Thus, $\emptyset = A \cap B = B \cap A$.

Since $(B, A) \in 2^X \times 2^X$ and $B \cap A = \emptyset$, then $(B, A) \in R$.

Therefore, $(A, B) \in R$ implies $(B, A) \in R$, so R is symmetric.

To prove R is not transitive, we must find sets $A, B, C \in 2^X$ such that $(A, B) \in R$ and $(B, C) \in R$ and $(A, C) \notin R$.

Let $X = \{1, 2, ...\}$.

Let $A = \{1\}$ and $B = \emptyset$ and $C = \{1, 2\}$.

Then $A \cap B = \emptyset$ and $B \cap C = \emptyset$ and $A \cap C = \{1\} \neq \emptyset$.

Since $A \subset X$ and $B \subset X$ and $C \subset X$, then $A \in 2^X$ and $B \in 2^X$ and $C \in 2^X$.

Since $(A, B) \in 2^X \times 2^X$ and $A \cap B = \emptyset$, then $(A, B) \in R$.

Since $(B, C) \in 2^X \times 2^X$ and $B \cap C = \emptyset$, then $(B, C) \in R$.

Since $(A, C) \in 2^X \times 2^X$ and $A \cap C \neq \emptyset$, then $(A, C) \notin R$.

Therefore, $(A, B) \in R$ and $(B, C) \in R$, but $(A, C) \notin R$, so R is not transitive.

To prove R is not antisymmetric, we must prove there exist sets $A, B \in 2^X$ such that $(A, B) \in R$ and $(B, A) \in R$ and $A \neq B$. Thus, we must find sets $A \subset X$ and $B \subset X$ such that $A \cap B = \emptyset$ and $B \cap A = \emptyset$ and $A \neq B$.

Let $A = \emptyset$ and $B = \{1\}$.

Clearly, $A \subset X$ and $B \subset X$.

Observe that $A \cap B = \emptyset$ and $B \cap A = \emptyset$, but $A \neq B$.

Therefore, R is not antisymmetric. $\square$

Exercise 18. Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{1}{x}\}$.
Analyze R.

Solution. Since $R \subset \mathbb{R} \times \mathbb{R}$, then R is a relation on $\mathbb{R}$.

The domain of R is the set $\{x \in \mathbb{R} : (\exists y \in \mathbb{R})((x, y) \in R)\} = (-\infty, 0) \cup (0, \infty)$.

The range of R is the set $\{y \in \mathbb{R} : (\exists x \in \mathbb{R})((x, y) \in R)\} = (-\infty, 0) \cup (0, \infty)$.

Therefore, $domR = (-\infty, 0) \cup (0, \infty) = rangeR$.

The graph of R is the hyperbola in the xy plane with asymptotes the x and y axes.

We prove R is not reflexive.

Since 2 is a real number, then $(2, 2) \in \mathbb{R} \times \mathbb{R}$, but $(2, 2) \notin R$.

Therefore, R is not reflexive.

We prove R is symmetric.

Let $x, y \in \mathbb{R}$ such that $(x, y) \in R$.

Then $y = \frac{1}{x}$, so $\frac{1}{x} \in \mathbb{R}$.

Since $\frac{1}{x} \in \mathbb{R}$ iff $x \neq 0$, then $x \neq 0$.

Since $x \neq 0$, then $\frac{1}{x} \neq 0$, so $y \neq 0$.

Since $y = \frac{1}{x}$, then $xy = 1$, so $x = \frac{1}{y}$.

Since $(y, x) \in \mathbb{R} \times \mathbb{R}$ and $x = \frac{1}{y}$, then $(y, x) \in R$.

Therefore, $(x, y) \in R$ implies $(y, x) \in R$, so R is symmetric.

We prove R is not transitive.

Observe that $(3, \frac{1}{3}) \in R$ and $(\frac{1}{3}, 1/\frac{1}{3}) = (\frac{1}{3}, 3) \in R$.

Since $3 \neq \frac{1}{3}$, then $(3, 3) \notin R$.

Therefore, $(3, \frac{1}{3}) \in R$ and $(\frac{1}{3}, 1/\frac{1}{3}) \in R$ but $(3, 3) \notin R$, so R is not transitive.

We prove R is not antisymmetric.

Observe that $(4, \frac{1}{4}) \in R$ and $(\frac{1}{4}, 1/\frac{1}{4}) = (\frac{1}{4}, 4) \in R$, but $4 \neq \frac{1}{4}$.

Therefore, $(4, \frac{1}{4}) \in R$ and $(\frac{1}{4}, 4) \in R$, but $4 \neq \frac{1}{4}$, so R is not antisymmetric.

$\square$

Exercise 19. Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| \leq 1\}$.
Analyze R.

Solution. Since $R \subset \mathbb{R} \times \mathbb{R}$, then R is a relation on $\mathbb{R}$.

The domain of R is the set $\{x \in \mathbb{R} : (\exists y \in \mathbb{R})((x, y) \in R)\} = \{x \in \mathbb{R} : (\exists y \in \mathbb{R})(|x - y| \leq 1)\} = \mathbb{R}$.

The range of R is the set $\{y \in \mathbb{R} : (\exists x \in \mathbb{R})((x, y) \in R)\} = \{y \in \mathbb{R} : (\exists x \in \mathbb{R})(|x - y| \leq 1)\} = \mathbb{R}$.

Let $(x, y) \in R$.

Then $x, y \in \mathbb{R}$ and $|x - y| \leq 1$.

Thus, either $|x - y| < 1$ or $|x - y| = 1$.

Case 1 : Suppose $|x - y| = 1$.

Then either $x - y = 1$ or $y - x = 1$.

Thus, either $x - 1 = y$ or $y = x + 1$.

Thus, either (x, y) is on the line $y = x - 1$ or (x, y) is on the line $y = x + 1$.

Case 2: Suppose $|x - y| < 1$.

Then $-1 < x - y < 1$.

Thus, $-1 < x - y$ and $x - y < 1$.

Hence, $y - 1 < x$ and $x - 1 < y$.

Therefore, $y < x + 1$ and $x - 1 < y$.

Thus, $x - 1 < y < x + 1$.

Consequently, y is between the lines $y = x - 1$ and $y = x + 1$.

Therefore, the graph of R is the diagonal strip between and including the parallel lines $y = x + 1$ and $y = x - 1$ in the xy plane.

We prove R is reflexive.

For any $x \in R$, $|x - x| = 0 \leq 1$.

Therefore, R is reflexive.

We prove R is symmetric.

If $x, y \in R$ such that $|x - y| \leq 1$, then $|y - x| = |x - y| \leq 1$.

Therefore, R is symmetric.

We prove R is not transitive.

Since $|1 - 2| = 1 \leq 1$, then $(1, 2) \in R$.

Since $|2 - 3| = 1 \leq 1$, then $(2, 3) \in R$.

Since $|1 - 3| = 2 > 1$, then $(1, 3) \notin R$.

Hence, $(1, 2) \in R$ and $(2, 3) \in R$, but $(1, 3) \notin R$.

Therefore, R is not transitive.

We prove R is not antisymmetric.

Since $|2 - 2.5| = 0.5 \leq 1$, then $(2, 2.5) \in R$.

Since $|2.5 - 2| = 0.5 \leq 1$, then $(2.5, 2) \in R$.

Therefore, $(2, 2.5) \in R$ and $(2.5, 2) \in R$, but $2 \neq 2.5$.

Therefore, R is not antisymmetric. $\square$

Exercise 20. Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \neq y\}$.

Analyze R.

Solution. Since $R \subset \mathbb{R} \times \mathbb{R}$, then R is a relation on $\mathbb{R}$.

Thus, $R = \mathbb{R} \times \mathbb{R} - \{(x, y) : y = x\} = \mathbb{R} \times \mathbb{R} - \{(x, x)\}$.

Hence, the graph of R consists of the plane except for all points on the line $y = x$.

The domain of R is $\mathbb{R}$ and the range of R is $\mathbb{R}$.

We formally prove $domR = rangeR = \mathbb{R}$.

By definition of domain, $domR \subset \mathbb{R}$.

Let $x \in \mathbb{R}$.

To prove $x \in domR$, we must prove there exists a real number y such that $(x, y) \in R$.

Thus, we must find a real number y such that $x \neq y$.

Let $y = x + 1$.

Then $y \in \mathbb{R}$.

Since $x = x + 1$ iff $0 = 1$ and $0 \neq 1$, then $x \neq x + 1$.

Hence, $x \neq y$.

Thus, $x \in domR$.

Therefore, $x \in \mathbb{R}$ implies $x \in domR$, so $\mathbb{R} \subset domR$.

Since $domR \subset \mathbb{R}$ and $\mathbb{R} \subset domR$, then $domR = \mathbb{R}$.
We prove $rangeR = \mathbb{R}$.
By definition of range, $rangeR \subset \mathbb{R}$.
We prove $\mathbb{R} \subset rangeR$.
Let $y \in \mathbb{R}$.
To prove $y \in rangeR$, we must prove there exists $x \in \mathbb{R}$ such that $(x, y) \in R$.
Thus, we must find a real number x such that $x \neq y$.
Let $x = y + 1$.
Then $x \in \mathbb{R}$.
Since $y = y + 1$ iff $0 = 1$ and $0 \neq 1$, then $y \neq y + 1$.
Hence, $y \neq x$, so $x \neq y$.
Thus, $y \in rangeR$.
Therefore, $y \in \mathbb{R}$ implies $y \in rangeR$, so $\mathbb{R} \subset rangeR$.
Since $rangeR \subset \mathbb{R}$ and $\mathbb{R} \subset rangeR$, then $rangeR = \mathbb{R}$.
We prove R is not reflexive.
Since 3 is a real number, then $(3, 3) \in \mathbb{R} \times \mathbb{R}$.
Every real number equals itself, so $3 = 3$.
Hence, $3 \neq 3$ is false, so $(3, 3) \notin R$.
Therefore, R is not reflexive.
We prove R is symmetric.
Let $x, y \in \mathbb{R}$ such that $(x, y) \in R$.
Then $x \neq y$, so either $x < y$ or $x > y$.
Hence, $y > x$ or $y < x$, so $y \neq x$.
Since $(y, x) \in \mathbb{R} \times \mathbb{R}$ and $y \neq x$, then $(y, x) \in R$.
Therefore, $(x, y) \in R$ implies $(y, x) \in R$, so R is symmetric.
We prove R is not transitive.
Since $2 \neq 3$, then $(2, 3) \in R$.
Since $3 \neq 2$, then $(3, 2) \in R$.
Thus, $(2, 3) \in R$ and $(3, 2) \in R$, but $2 = 2$, so R is not transitive.
We prove R is not antisymmetric.
Since 4 and 5 are real numbers and $4 \neq 5$, then $(4, 5) \in R$.
Since 5 and 4 are real numbers and $5 \neq 4$, then $(5, 4) \in R$.
Thus, $(4, 5) \in R$ and $(5, 4) \in R$, but $4 \neq 5$.
Therefore, R is not antisymmetric. □

Exercise 21. The empty relation on a nonempty set is symmetric and transitive, but not reflexive.

Proof. Let S be a nonempty set.
The empty set is a subset of every set, so $\emptyset \subset S \times S$.
Thus, $\emptyset$ is a relation on S.
Since S is not empty, then there is an element in S.
Let x be an element of S.
We prove $\emptyset$ is reflexive.
Since $\emptyset$ is empty, then there is no element in $\emptyset$.
Thus, $(x, x) \notin \emptyset$.

Since there exists $x \in S$ such that $(x, x) \notin \emptyset$, then $\emptyset$ is reflexive.
We prove $\emptyset$ is symmetric.
Let $a, b \in S$ be arbitrary.
Since $\emptyset$ is empty, then there is no element in $\emptyset$.
Hence, $(a, b) \in \emptyset$ is false.
Thus, the conditional $(a, b) \in \emptyset$ implies $(b, a) \in \emptyset$ is vacuously true.
Therefore, $\emptyset$ is symmetric.
We prove $\emptyset$ is transitive.
Let $a, b, c \in S$ be arbitrary.
Since $\emptyset$ is empty, then there is no element in $\emptyset$.
Hence, $(a, b) \in \emptyset$ is false and $(b, c) \in \emptyset$ is false.
Thus, the conditional $(a, b) \in \emptyset$ and $(b, c) \in \emptyset$ implies $(a, c) \in \emptyset$ is vacuously true. Therefore, $\emptyset$ is transitive. $\square$

Exercise 22. Give an example of a nonempty relation on a set which is symmetric and transitive, but not reflexive.

Solution. Let $S = \{1, 2, 3\}$.
Let $R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.
Then $R \subset S \times S$, so R is a nonempty relation on S.
Since $(3, 3) \notin R$, then R is not reflexive.
Since $(1, 2) \in R$ and $(2, 1) \in R$, then R is symmetric.
Observe that $(1, 2) \in R$ and $(2, 1) \in R$ and $(1, 1) \in R$.
Observe that $(1, 2) \in R$ and $(2, 2) \in R$.
Observe that $(2, 1) \in R$ and $(1, 2) \in R$ and $(2, 2) \in R$.
Observe that $(2, 1) \in R$ and $(1, 1) \in R$.
Therefore, R is transitive. $\square$

Exercise 23. Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \sqrt{x} - 5\} = \{(x, \sqrt{x} - 5) : x \in \mathbb{R}\}$. What is the domain and range of R?

Solution. Since R is a subset of $\mathbb{R} \times \mathbb{R}$, then R is a relation on $\mathbb{R}$.
The graph of R is the square root function shifted vertically down by 5 units.
The domain is $[0, \infty)$ and the range is $[-5, \infty)$. $\square$

Exercise 24. Let R be a nonempty relation from set A to set B.
Then $R = R^{-1}$ iff R is symmetric and $A = B$.

Proof. We prove $R = R^{-1}$ implies R is symmetric and $A = B$.
Suppose $R = R^{-1}$.
We first prove R is symmetric.
Let $(a, b) \in R$.
Then $(b, a) \in R^{-1}$, by definition of R^{-1}.
Since $R^{-1} = R$, then $(b, a) \in R$.
Thus, $(a, b) \in R$ implies $(b, a) \in R$, so R is symmetric.
We next prove $A = B$.
Let $(b, a) \in R^{-1}$.
Since $R^{-1} \subset B \times A$, then $(b, a) \in B \times A$.

Hence, $b \in B$ and $a \in A$.
Since $(b, a) \in R^{-1}$ and $R^{-1} = R$, then $(b, a) \in R$.
Since $R \subset A \times B$, then $(b, a) \in A \times B$.
Thus, $b \in A$ and $a \in B$.
Since $a \in A$ and $a \in B$, then $a \in A$ implies $a \in B$.
Hence, $A \subset B$.
Since $b \in B$ and $b \in A$, then $b \in B$ implies $b \in A$.
Hence, $B \subset A$.
Thus, $A \subset B$ and $B \subset A$, so $A = B$.
Conversely, we prove R is symmetric and $A = B$ implies $R = R^{-1}$.
Suppose R is symmetric and $A = B$.
Let $(a, b) \in R$.
Since R is symmetric, then $(b, a) \in R$.
Since $(b, a) \in R$ iff $(a, b) \in R^{-1}$, then $(a, b) \in R^{-1}$.
Hence, $(a, b) \in R$ implies $(a, b) \in R^{-1}$, so $R \subset R^{-1}$.
Let $(b, a) \in R^{-1}$.
By definition of R^{-1}, $(b, a) \in R^{-1}$ iff $(a, b) \in R$.
Hence, $(a, b) \in R$.
Since R is symmetric, then $(b, a) \in R$.
Hence, $(b, a) \in R^{-1}$ implies $(b, a) \in R$, so $R^{-1} \subset R$.
Therefore, $R \subset R^{-1}$ and $R^{-1} \subset R$, so $R = R^{-1}$. □

Exercise 25. Let R be a nonempty relation from A to B.
If $R \cap R^{-1} = I_A$, then $A \subset B$ and R is antisymmetric.

Proof. Suppose $R \cap R^{-1} = I_A$.
We first prove $A \subset B$.
Let $x \in A$.
Then $(x, x) \in A \times A$, so $(x, x) \in I_A$.
Since $I_A = R \cap R^{-1}$, then $(x, x) \in R \cap R^{-1}$, so $(x, x) \in R$.
Since R is a relation from A to B, then $x \in B$.
Therefore, $A \subset B$.
We prove R is antisymmetric.
Let $(x, y) \in R$ and $(y, x) \in R$.
Since $(y, x) \in R$ iff $(x, y) \in R^{-1}$, then $(x, y) \in R^{-1}$.
Thus, $(x, y) \in R$ and $(x, y) \in R^{-1}$, so $(x, y) \in R \cap R^{-1} = I_A$.
Hence, $(x, y) \in I_A$, so $x = y$.
Therefore, R is antisymmetric. □

Exercise 26. Let R be a nonempty relation on a set A.
If R is antisymmetric, then $R \cap R^{-1} \subset I_A$.

Proof. Suppose R is antisymmetric.
Let $(a, b) \in R \cap R^{-1}$.
Then $a, b \in A$ and $(a, b) \in R$ and $(a, b) \in R^{-1}$.
Since $(a, b) \in R^{-1}$, then $(b, a) \in R$.
Thus, $(a, b) \in R$ and $(b, a) \in R$.

Since R is antisymmetric, then $a = b$.
Hence, $(a, b) = (a, a) \in I_A$.
Since $(a, b) \in R \cap R^{-1}$ implies $(a, b) \in I_A$, then $R \cap R^{-1} \subset I_A$. $\square$

Exercise 27. If R is a nonempty relation on a set A, then $R \cap R^{-1}$ and $R \cup R^{-1}$ are symmetric.

Proof. Let R be a nonempty relation on a set A.
We first prove $R \cap R^{-1}$ is symmetric.
Let $(a, b) \in R \cap R^{-1}$.
Then $(a, b) \in R$ and $(a, b) \in R^{-1}$.
Thus, $(a, b) \in R$ and $(b, a) \in R$.
Hence, $(b, a) \in R$ and $(a, b) \in R$, so $(b, a) \in R$ and $(b, a) \in R^{-1}$.
Therefore, $(b, a) \in R \cap R^{-1}$.
Thus, $(a, b) \in R \cap R^{-1}$ implies $(b, a) \in R \cap R^{-1}$, so $R \cap R^{-1}$ is symmetric.
We next prove $R \cup R^{-1}$ is symmetric.
Let $(a, b) \in R \cup R^{-1}$.
Then either $(a, b) \in R$ or $(a, b) \in R^{-1}$, so either $(a, b) \in R$ or $(b, a) \in R$.
Thus, either $(b, a) \in R$ or $(a, b) \in R$, so either $(b, a) \in R$ or $(b, a) \in R^{-1}$.
Hence, $(b, a) \in R \cup R^{-1}$.
Thus, $(a, b) \in R \cup R^{-1}$ implies $(b, a) \in R \cup R^{-1}$, so $R \cup R^{-1}$ is symmetric. $\square$

Exercise 28. If a relation R on a nonempty set A is reflexive, symmetric, and antisymmetric, then $R = I_A$.

Proof. Let A be a nonempty set.
Let R be a relation on A.
Suppose R is reflexive, symmetric, and antisymmetric.
Since R is reflexive iff $I_A \subset R$ and R is reflexive, then $I_A \subset R$.
Let $(a, b) \in R$ be arbitrary.
Since R is symmetric, then $(b, a) \in R$.
Thus, $(a, b) \in R$ and $(b, a) \in R$.
Since R is antisymmetric, then $a = b$.
Hence, $(a, b) = (a, a) \in I_A$.
Therefore, $(a, b) \in R$ implies $(a, b) \in I_A$, so $R \subset I_A$.
Since $R \subset I_A$ and $I_A \subset R$, then $R = I_A$. $\square$

Equivalence Relations

Exercise 29. Define a relation R on the set S of all students in math class by $R = \{(x, y) \in S \times S : x$ and y achieve the same numerical grade on a given test $\}$.
Then R is an equivalence relation on S.

Solution. We prove R is reflexive.
Each student achieves the same grade as himself.
Therefore, R is reflexive.

If a and b achieve the same numerical grade, then b and a achieve the same numerical grade.

Therefore, R is symmetric.

Let a and b and c be arbitrary students that take the given test.

If a and b achieve the same grade and b and c achieve the same numerical grade, then certainly a and c achieve the same numerical grade.

Therefore, R is transitive.

Since R is reflexive, symmetric, and transitive, then R is an equivalence relation on S. $\square$

Exercise 30. Let S be the set of all living people.

Let $R = \{(x, y) \in S \times S : x \text{ and } y \text{ have the same biological parents } \}$.

Analyze R.

Solution. Since $R \subset S \times S$, then R is a relation over S.

Reflexive property: $(\forall x \in S)((x, x) \in R)$ means for each living person x, x and x have the same biological parents.

Since each person has the same biological parents as himself or herself, then R is reflexive.

Symmetric property: $(\forall x, y \in S)((x, y) \in R \rightarrow (y, x) \in R)$ means if x and y are living people such that x and y have the same biological parents, then y and x have the same biological parents.

If x and y have the same biological parents, then it is obviously true that y and x have the same biological parents.

Hence, R is symmetric.

Transitive property: $(\forall x, y, z \in S)((x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R)$ means if x, y, z are living people such that x and y have the same biological parents and y and z have the same biological parents, then x and z have the same biological parents.

Suppose x, y, z are people such that x and y have the same biological parents and y and z have the same biological parents.

Then it is obvious that x and z have the same biological parents.

Hence, R is transitive.

Therefore, R is an equivalence relation on S. $\square$

Exercise 31. Let S be the set of all living people.

Let $R = \{(x, y) \in S \times S : x \text{ and } y \text{ are the same weight } \}$.

Analyze R.

Solution. Since $R \subset S \times S$, then R is a relation over S.

Reflexive property: $(\forall x \in S)((x, x) \in R)$ means for each living person x, x and x have the same weight.

Since each person has the same weight as himself or herself, then R is reflexive.

Symmetric property: $(\forall x, y \in S)((x, y) \in R \rightarrow (y, x) \in R)$ means if x and y are living people such that x and y have the same weight, then y and x have the same weight.

If x and y have the same weight, then it is obvious that y and x have the same weight.

Hence, R is symmetric.

Transitive property: $(\forall x, y, z \in S)((x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R)$ means if x, y, z are living people such that x and y have the same weight and y and z have the same weight, then x and z have the same weight.

Suppose x, y, z are people such that x and y have the same weight and y and z have the same weight.

Then it is obvious that x and z have the same weight.

Hence, R is transitive.

Therefore, R is an equivalence relation on S. $\square$

Exercise 32. Let S be the set of all living people.

Let T be the set of all current major league baseball players.

Let $R = \{(x, y) \in T \times T : x$ and y have the same number of home runs during the current season $\}$.

Analyze R.

Solution. Observe that $T \subset S$.

Since $R \subset T \times T$, then R is a relation over T.

Reflexive property: $(\forall x \in T)((x, x) \in R)$ means for each major league baseball player x, x and x have the same number of home runs. That is, R is reflexive iff each major baseball player has the same number of home runs as himself.

Since each major baseball player has the same number of home runs as himself, then R is reflexive.

Symmetric property: $(\forall x, y \in T)((x, y) \in R \rightarrow (y, x) \in R)$ means if x and y are major league baseball players such that x and y have the same number of home runs, then y and x have the same number of home runs.

If x and y are major league baseball players such that x and y have the same number of home runs, then y and x have the same number of home runs.

Hence, R is symmetric.

Transitive property: $(\forall x, y, z \in T)((x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R)$ means if x, y, z are major league baseball players such that x and y have the same number of home runs and y and z have the same number of home runs, then x and z have the same number of home runs.

Suppose x, y, z are major league baseball players such that x and y have the same number of home runs and y and z have the same number of home runs.

Then x and z have the same number of home runs.

Hence, R is transitive.

Therefore, R is an equivalence relation on T. $\square$

Exercise 33. Let $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| \leq 4\}$.

Analyze A.

Solution. Since $A \subset \mathbb{R} \times \mathbb{R}$, then A is a relation on $\mathbb{R}$.

Let $x \in \mathbb{R}$.

Since $|x - x| = |0| = 0 < 4$, then $(x, x) \in A$, so A is reflexive.

Let $x, y \in \mathbb{R}$ such that $(x, y) \in A$.
Then $|x - y| \leq 4$.
Since $4 \geq |x - y| = |y - x|$, then $|y - x| \leq 4$, so $(y, x) \in A$.
Therefore, A is symmetric.

Let $x = 1, y = 4, z = 7$.
Since $|x - y| = |1 - 4| = 3 < 4$, then $(x, y) \in A$.
Since $|y - z| = |4 - 7| = 3 < 4$, then $(y, z) \in A$.
Since $|x - z| = |1 - 7| = 6 > 4$, then $(x, z) \notin A$.
Since $(x, y) \in A$ and $(y, z) \in A$, but $(x, z) \notin A$, then A is not transitive, so A is not an equivalence relation. $\square$

Exercise 34. Let $E = \{(x, y) \in \mathbb{R}^* \times \mathbb{R}^* : xy > 0\}$ and $N = \{(x, y) \in \mathbb{R}^* \times \mathbb{R}^* : xy < 0\}$.

Then E is an equivalence relation on $\mathbb{R}^*$, but N is symmetric only and hence is not an equivalence relation.

Proof. We prove E is an equivalence relation on $\mathbb{R}^*$.
Observe that $E \subset \mathbb{R}^* \times \mathbb{R}^*$, so E is a relation on $\mathbb{R}^*$.
Let a be a nonzero real number.
The square of any real number is nonnegative, so $a^2 \geq 0$.
Hence, either $a^2 > 0$ or $a^2 = 0$.
Since $a^2 = 0$ iff $a = 0$ and $a \neq 0$, then $a^2 \neq 0$.
Thus, $a^2 > 0$, so $(a, a) \in E$.
Therefore, E is reflexive.

Let a and b be nonzero real numbers such that $(a, b) \in E$.
Then $ab > 0$.
Since $ab = ba$, then $ba > 0$.
Hence, $(b, a) \in R$, so E is symmetric.

Let a, b, c be nonzero real numbers such that $(a, b) \in E$ and $(b, c) \in E$.
Then $ab > 0$ and $bc > 0$.
The product of two positive real numbers is positive, so $(ab)(bc) > 0$.
Thus, $ab^2c > 0$.
Since $b^2 \geq 0$ and $b \neq 0$, then $b^2 > 0$.
Since $ab^2c > 0$, then we divide by positive b^2 to obtain $ac > 0$.
Thus, $(a, c) \in E$.
Since $(a, b) \in E$ and $(b, c) \in E$ implies $(a, c) \in E$, then E is transitive.
Since E is reflexive, symmetric, and transitive, then E is an equivalence relation on $\mathbb{R}^*$.

We prove N is not reflexive.
Observe that $4^2 = 16 > 0$.
Hence, 4^2 is not less than 0, so N is not reflexive.

Let a, b be nonzero real numbers such that $(a, b) \in N$.
Then $ab < 0$, so $ba < 0$.
Thus, $(b, a) \in N$, so N is symmetric.

We prove N is not transitive.
Observe that $2 * (-3) = -6 < 0$ and $(-3)(5) = -15 < 0$ and $2 * 5 = 10 > 0$.
Thus, $2(-3) < 0$ and $(-3)(5) < 0$, but $2 * 5 \not< 0$.
Hence, N is not transitive.
Since N is not reflexive, then N is not an equivalence relation. $\square$

Exercise 35. Give examples of a relation that is reflexive, symmetric, and not transitive.

Solution. Let $S = \{1, 2, 3\}$.
Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\}$.
Clearly, R is a relation on S and R is reflexive and symmetric.
However, R is not transitive since $(2, 1) \in R$ and $(1, 3) \in R$, but $(2, 3) \notin R$. $\square$

Exercise 36. Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : xy \geq 0\}$.
Analyze R.

Solution. Since $R \subset \mathbb{R} \times \mathbb{R}$, then R is a relation on $\mathbb{R}$.
For any $x \in \mathbb{R}, xx = x^2 \geq 0$.
Hence, R is reflexive.
If $x, y \in \mathbb{R}$ such that $xy \geq 0$, then $yx \geq 0$.
Hence, R is symmetric.
Observe that $3 * 0 \geq 0$ and $0(-1) \geq 0$, but $3(-1) = -3 < 0$.
Hence, R is not transitive.
Therefore, R is not an equivalence relation. $\square$

Exercise 37. Define a relation $\sim$ over $\mathbb{R}$ by the rule $x \sim y$ iff $x - y \in \mathbb{Z}$.
Analyze the relation.

Solution. For any $x \in \mathbb{R}, x - x = 0 \in \mathbb{Z}$.
Therefore, $\sim$ is reflexive.
Let $x, y \in \mathbb{Z}$ such that $x - y \in \mathbb{Z}$.
Then $-(x - y) = -x + y = y - x \in \mathbb{Z}$.
Therefore, $\sim$ is symmetric.
Let $x, y, z \in \mathbb{Z}$ such that $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$.
Then $(x - y) + (y - z) = x - z \in \mathbb{Z}$.
Therefore, $\sim$ is transitive.
Since $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is an equivalence relation on $\mathbb{R}$.
Since $\sim$ is an equivalence relation, then the collection of equivalence classes $\frac{\mathbb{R}}{\sim} = \{[x] : x \in \mathbb{R}\}$ is a partition of $\mathbb{R}$.
For example, if $x = 3.5$, then $[3.5] = \{y \in \mathbb{R} : 3.5 - y \in \mathbb{Z}\}$.

Thus, elements of $[3.5]$ include $-5.5, -4.5, -3.5, -2.5, -1.5, -0.5, 0.5, 1.5, 2.5, 3.5, 4.5, 5.5$ and infinitely many others.

Let $x \in \mathbb{R}$.

Then $[x] = \{y \in \mathbb{R} : x \sim y\} = \{y \in \mathbb{R} : x - y \in \mathbb{Z}\}$.

Thus, the equivalence class of a real number x consists of all real numbers y for which the difference $x - y$ is an integer.

Therefore, $\frac{\mathbb{R}}{\sim} = \{[x] : x \in \mathbb{R}\} = \{\{y \in \mathbb{R} : x - y \in \mathbb{Z}\} : x \in \mathbb{R}\}$ is an infinitely many celled partition of $\mathbb{R}$. $\square$

Exercise 38. Define a relation $\sim$ over $\mathbb{R}$ by the rule $x \sim y$ iff $x - y \in \mathbb{Q}$.

Analyze the relation.

Solution. For any $x \in \mathbb{R}, x - x = 0 = \frac{0}{1} \in \mathbb{Q}$.

Therefore, $\sim$ is reflexive.

Let $x, y \in \mathbb{R}$ such that $x - y \in \mathbb{Q}$.

Then $y - x = -x + y = -(x - y) \in \mathbb{Q}$.

Therefore, $\sim$ is symmetric.

Let $x, y, z \in \mathbb{R}$ such that $x - y \in \mathbb{Q}$ and $y - z \in \mathbb{Q}$.

Then $x - z = (x - y) + (y - z) \in \mathbb{Q}$.

Therefore, $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is an equivalence relation on $\mathbb{R}$.

Since $\sim$ is an equivalence relation, then the collection of equivalence classes $\frac{\mathbb{R}}{\sim} = \{[x] : x \in \mathbb{R}\}$ is a partition of $\mathbb{R}$.

For example, if $x = 3.7$, then $[3.7] = \{y \in \mathbb{R} : 3.7 - y \in \mathbb{Q}\}$.

Thus, elements of $[3.7]$ include $1.7, 1.8, 1.9, 1.10, 1.11, 2.34, 2.35, 2.36, 3.7$ and infinitely many others.

Let $x \in \mathbb{R}$.

Then $[x] = \{y \in \mathbb{R} : x \sim y\} = \{y \in \mathbb{R} : x - y \in \mathbb{Q}\}$.

Thus, the equivalence class of a real number x consists of all real numbers y for which the difference $x - y$ is a rational number.

Therefore, $\frac{\mathbb{R}}{\sim} = \{[x] : x \in \mathbb{R}\} = \{\{y \in \mathbb{R} : x - y \in \mathbb{Q}\} : x \in \mathbb{R}\}$ is an infinitely many celled partition of $\mathbb{R}$. $\square$

Exercise 39. is the same distance from zero is an equivalence relation on $\mathbb{R}$

Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 = y^2\}$.

Analyze R.

Solution. Since $R \subset \mathbb{R} \times \mathbb{R}$, then R is a relation on $\mathbb{R}$.

Since $a^2 = |a|$ for any real number a, then $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| = |y|\}$.

The graph of R consists of the lines $y = x$ and $y = -x$ in the xy plane.

Thus, the domain of R is $\mathbb{R}$ and the range of R is $\mathbb{R}$.

We formally prove $domR = rangeR = \mathbb{R}$.

By definition of domain, $domR \subset \mathbb{R}$.

Let $x \in \mathbb{R}$.

To prove $x \in domR$, we must prove there exists a real number y such that $(x, y) \in R$.

Thus, we must find a real number y such that $x^2 = y^2$.

Let $y = -x$.

Then $y \in \mathbb{R}$ and $y^2 = (-x)^2 = x^2$.

Since $(x, y) \in \mathbb{R} \times \mathbb{R}$ and $x^2 = y^2$, then $(x, y) \in R$.

Hence, $x \in domR$.

Therefore, $x \in \mathbb{R}$ implies $x \in domR$, so $\mathbb{R} \subset domR$.

Since $domR \subset \mathbb{R}$ and $\mathbb{R} \subset domR$, then $domR = \mathbb{R}$.

We prove $rangeR = \mathbb{R}$.

By definition of range, $rangeR \subset \mathbb{R}$.

We prove $\mathbb{R} \subset rangeR$.

Let $y \in \mathbb{R}$.

To prove $y \in rangeR$, we must prove there exists $x \in \mathbb{R}$ such that $(x, y) \in R$.

Thus, we must find a real number x such that $x^2 = y^2$.

Let $x = -y$.

Then $x \in \mathbb{R}$ and $x^2 = (-y)^2 = y^2$.

Since $(x, y) \in \mathbb{R} \times \mathbb{R}$ and $x^2 = y^2$, then $(x, y) \in R$.

Hence, $y \in rangeR$.

Therefore, $y \in \mathbb{R}$ implies $y \in rangeR$, so $\mathbb{R} \subset rangeR$.

Since $rangeR \subset \mathbb{R}$ and $\mathbb{R} \subset rangeR$, then $rangeR = \mathbb{R}$.

We prove R is reflexive.

Let $x \in \mathbb{R}$.

Then $(x, x) \in \mathbb{R} \times \mathbb{R}$ and $x^2 = x^2$.

Therefore, $(x, x) \in R$, so R is reflexive.

We prove R is symmetric.

Let $x, y \in \mathbb{R}$ such that $(x, y) \in R$.

Then $x^2 = y^2$, so $y^2 = x^2$.

Since $(y, x) \in \mathbb{R} \times \mathbb{R}$ and $y^2 = x^2$, then $(y, x) \in R$.

Therefore, $(x, y) \in R$ implies $(y, x) \in R$, so R is symmetric.

We prove R is transitive.

Let $x, y, z \in \mathbb{R}$ such that $(x, y) \in R$ and $(y, z) \in R$.

Then $x^2 = y^2$ and $y^2 = z^2$, so $x^2 = z^2$.

Since $(x, z) \in \mathbb{R} \times \mathbb{R}$ and $x^2 = z^2$, then $(x, z) \in R$.

Therefore, $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$, so R is transitive.

Since R is reflexive, symmetric, and transitive, then R is an equivalence relation on $\mathbb{R}$.

Since R is an equivalence relation, then the collection of equivalence classes $\frac{\mathbb{R}}{R} = \{[x] : x \in \mathbb{R}\}$ is a partition of $\mathbb{R}$.

Let $x \in \mathbb{R}$.

Then $[x] = \{y \in \mathbb{R} : xRy\} = \{y \in \mathbb{R} : |x| = |y|\} = \{x, -x\}$.

Thus, the equivalence class of a real number x consists of x itself and its additive inverse $-x$.

Therefore, $\frac{\mathbb{R}}{R} = \{[x] : x \in \mathbb{R}\} = \{\{x, -x\} : x \in \mathbb{R}\}$ is an infinitely many celled partition of $\mathbb{R}$.

We prove R is not antisymmetric.

Since $3^2 = 9 = (-3)^2$, then $(3, -3) \in R$.
Since $(-3)^2 = 9 = 3^2$, then $(-3, 3) \in R$.
Thus, $(3, -3) \in R$ and $(-3, 3) \in R$, but $3 \neq -3$.
Therefore, R is not antisymmetric. $\square$

Exercise 40. congruence modulo 5 on $\mathbb{Z}$
Let $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : 5|(m - n)\}$.
Then R is an equivalence relation on $\mathbb{Z}$.

Solution. Since $R \subset \mathbb{Z} \times \mathbb{Z}$, then R is a relation on $\mathbb{Z}$.
By definition of domain, the domain of R is a subset of $\mathbb{Z}$, so $domR \subset \mathbb{Z}$.
We prove $\mathbb{Z} \subset domR$.
Let $a \in \mathbb{Z}$.
To prove $\mathbb{Z} \subset domR$, we must find an integer b such that $5|(a - b)$.
Let $b = a - 5k$ for some integer k.
Then b is an integer and $a - b = 5k$.
Thus, $5|(a - b)$, so $\mathbb{Z} \subset domR$.
Therefore, $domR \subset \mathbb{Z}$ and $\mathbb{Z} \subset domR$, so $domR = \mathbb{Z}$.

The range of R is a subset of $\mathbb{Z}$, by definition of range.
Thus, $rangeR \subset \mathbb{Z}$.
We prove $\mathbb{Z} \subset rangeR$.
Let $b \in \mathbb{Z}$.
To prove $\mathbb{Z} \subset rangeR$, we must find an integer a such that $5|(a - b)$.
Let $a = 5k + b$ for some integer k.
Then a is an integer and $a - b = 5k$.
Thus, $5|(a - b)$, so $\mathbb{Z} \subset rangeR$.
Therefore, $rangeR \subset \mathbb{Z}$ and $\mathbb{Z} \subset rangeR$, so $rangeR = \mathbb{Z}$.

We prove R is reflexive.
Let $a \in \mathbb{Z}$.
Then $(a, a) \in \mathbb{Z} \times \mathbb{Z}$.
Since every integer divides zero, then in particular, $5|0$.
Thus, $5|(a - a)$, so $(a, a) \in R$.
Therefore, R is reflexive.

We prove R is symmetric.
Let $(a, b) \in R$.
Then $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ and $5|(a - b)$.
Since $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, then $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$.
Since $5|(a - b)$, then $a - b = 5k$ for some integer k, so $b - a = -5k = 5(-k)$.
Since $-k$ is an integer, then $5|(b - a)$.
Since $b \in \mathbb{Z}$ and $a \in \mathbb{Z}$, then $(b, a) \in \mathbb{Z} \times \mathbb{Z}$.
Since $(b, a) \in \mathbb{Z} \times \mathbb{Z}$ and $5|(b - a)$, then $(b, a) \in R$.
Therefore, $(a, b) \in R$ implies $(b, a) \in R$, so R is symmetric.

We prove R is transitive.

Let $(a, b) \in R$ and $(b, c) \in R$.

Since $(a, b) \in R$, then $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ and $5|(a - b)$.

Since $(b, c) \in R$, then $(b, c) \in \mathbb{Z} \times \mathbb{Z}$ and $5|(b - c)$.

Since $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ and $(b, c) \in \mathbb{Z} \times \mathbb{Z}$, then $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ and $c \in \mathbb{Z}$.

Since $5|(a - b)$ and $5|(b - c)$, then 5 is a common divisor of $a - b$ and $b - c$, so 5 divides the sum $(a - b) + (b - c) = a - c$.

Hence, $5|(a - c)$.

Since $a \in \mathbb{Z}$ and $c \in \mathbb{Z}$, then $(a, c) \in \mathbb{Z} \times \mathbb{Z}$.

Since $(a, c) \in \mathbb{Z} \times \mathbb{Z}$ and $5|(a - c)$, then $(a, c) \in R$.

Thus, $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$, so R is transitive.

Since R is reflexive, symmetric, and transitive, then R is an equivalence relation on $\mathbb{Z}$.

To prove R is not antisymmetric, we must find integers a and b such that $5|(a - b)$ and $5|(b - a)$ and $a \neq b$.

Let $a = 4$ and $b = 9$.

Then $5|(4 - 9)$ since $5| - 5$ and $5|(9 - 4)$ since $5|5$ and $4 \neq 9$.

Therefore, R is not antisymmetric. $\square$

Exercise 41. Let f and g be differentiable functions on $\mathbb{R}$.

Define a relation $\sim$ by the rule $f \sim g$ iff $f' = g'$.

Then $\sim$ is an equivalence relation on the class of differentiable functions on $\mathbb{R}$.

Proof. Let f be a differentiable function on $\mathbb{R}$.

Let f' be the derivative of f.

Since $f' = f'$, then $f \sim f$, so $\sim$ is reflexive.

Let f and g be differentiable functions on $\mathbb{R}$ such that $f \sim g$.

Then $f' = g'$, so $f = g + c$ for some constant $c \in \mathbb{R}$.

Hence, $g = f - c$, so taking the derivative we obtain $g' = f' - 0 = f'$.

Since $g' = f'$, then $g \sim f$, so $\sim$ is symmetric.

Let f, g, h be differentiable functions on $\mathbb{R}$ such that $f \sim g$ and $g \sim h$.

Then $f' = g'$ and $g' = h'$.

Since $f' = g'$, then $f = g + c_1$ for some constant $c_1 \in \mathbb{R}$.

Since $g' = h'$, then $g = h + c_2$ for some constant $c_2 \in \mathbb{R}$.

Thus, $f - g = c_1$ and $g - h = c_2$, so adding these equations we obtain $f - h = c_1 + c_2$.

Taking the derivative, we obtain $f' - h' = 0$, so $f' = h'$.

Hence, $f \sim h$, so $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is an equivalence relation.

Two differentiable functions are in the same equivalence class whenever they differ by a constant. ☐

Exercise 42. Define a relation $\sim$ over $\mathbb{R}^2$ by the rule $(a,b) \sim (c,d)$ iff $a^2+b^2 = c^2+d^2$ for all $(a,b) \in \mathbb{R}^2$ and $(c,d) \in \mathbb{R}^2$.

Then $\sim$ is an equivalence relation on $\mathbb{R}^2$.

Proof. Let $(a,b) \in \mathbb{R}^2$.

Then $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Since $a^2+b^2 = a^2+b^2$, then $(a,b) \sim (a,b)$, so $\sim$ is reflexive.

Let $(a,b) \in \mathbb{R}^2$ and $(c,d) \in \mathbb{R}^2$ such that $(a,b) \sim (c,d)$.

Then $a^2+b^2 = c^2+d^2$, so $c^2+d^2 = a^2+b^2$.

Therefore, $(c,d) \sim (a,b)$, so $\sim$ is symmetric.

Let $(a,b) \in \mathbb{R}^2$ and $(c,d) \in \mathbb{R}^2$ and $(e,f) \in \mathbb{R}^2$ such that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$.

Then $a^2+b^2 = c^2+d^2$ and $c^2+d^2 = e^2+f^2$, so $a^2+b^2 = e^2+f^2$.

Therefore, $(a,b) \sim (e,f)$, so $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is an equivalence relation over $\mathbb{R}^2$.

Two ordered pairs of real numbers are in the same equivalence class if they lie on the same circle centered at the origin. ☐

Exercise 43. similar matrices

Let A and B be 2×2 matrices with entries in $\mathbb{R}$.

Define a relation $\sim$ on the set of all such 2×2 matrices by $A \sim B$ iff there exists an invertible matrix P such that $PAP^{-1} = B$.

Then $\sim$ is an equivalence relation.

Proof. Let I be the 2×2 identity matrix defined by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

.

Observe that $IAI^{-1} = AI^{-1} = AI = A$.

Therefore, $A \sim A$, so $\sim$ is reflexive.

Let A and B be 2×2 matrices with entries in $\mathbb{R}$ such that $A \sim B$.

Then there exists an invertible matrix P such that $PAP^{-1} = B$.

Since P is invertible, then the matrix P^{-1} exists and $PP^{-1} = P^{-1}P = I$, so P^{-1} is invertible.

Observe that

$$\begin{aligned}
P^{-1}B(P^{-1})^{-1} &= P^{-1}BP \\
&= P^{-1}(PAP^{-1})P \\
&= (P^{-1}P)A(P^{-1}P) \\
&= IAI \\
&= A.
\end{aligned}$$

Since P^{-1} is an invertible matrix and $P^{-1}B(P^{-1})^{-1} = A$, then $B \sim A$, so $\sim$ is symmetric.

Let A, B and C be 2×2 matrices with entries in $\mathbb{R}$ such that $A \sim B$ and $B \sim C$.

Then there exists an invertible matrix P such that $PAP^{-1} = B$ and there exists an invertible matrix C such that $QBQ^{-1} = C$.

Since P and Q are invertible matrices, then the matrix QP is invertible.

Observe that

$$\begin{aligned}
(QP)A(QP)^{-1} &= (QP)A(P^{-1}Q^{-1}) \\
&= Q(PAP^{-1})Q^{-1} \\
&= QBQ^{-1} \\
&= C.
\end{aligned}$$

Since QP is an invertible matrix and $(QP)A(QP)^{-1} = C$, then $A \sim C$, so $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is an equivalence relation. $\square$

Solution. We show examples of similar matrices.

Let A be the 2×2 matrix defined by

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

.

Let B be the 2×2 matrix defined by

$$B = \begin{bmatrix} -18 & 33 \\ -11 & 20 \end{bmatrix}$$

.

Matrix A is similar to matrix B since the matrix P defined by

$$P = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

has inverse P^{-1} defined by

$$P^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

and $PAP^{-1} = B$.

Matrix B is similar to matrix A since the matrix Q defined by

$$Q = \begin{bmatrix} -11 & 21 \\ -11 & 18 \end{bmatrix}$$

has inverse Q^{-1} defined by

$$Q^{-1} = \begin{bmatrix} \frac{6}{11} & \frac{-7}{11} \\ \frac{1}{3} & \frac{-1}{3} \end{bmatrix}$$

and $QBQ^{-1} = A$.

$\square$

Exercise 44. Let f be a real valued function with domain $\mathbb{R}$.

Define a relation $\sim$ over $\mathbb{R}$ by the rule $x \sim y$ iff $f(x) = f(y)$.

Then $\sim$ is an equivalence relation on $\mathbb{R}$.

Proof. For any $x \in \mathbb{R}, f(x) = f(x)$, so $x \sim x$.

Therefore, $\sim$ is reflexive.

Let $x, y \in \mathbb{R}$ such that $f(x) = f(y)$.

Then $f(y) = f(x)$, so $\sim$ is symmetric.

Let $x, y, z \in \mathbb{R}$ such that $f(x) = f(y)$ and $f(y) = f(z)$.

Then $f(x) = f(z)$.

Therefore, $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is an equivalence relation.

Therefore, the collection of equivalence classes $\frac{\mathbb{R}}{\sim} = \{[x] : x \in \mathbb{R}\}$ is a partition of $\mathbb{R}$.

Let $x \in \mathbb{R}$.

Then $[x] = \{y \in \mathbb{R} : x \sim y\} = \{y \in \mathbb{R} : f(x) = f(y)\}$.

Thus, the equivalence class of a real number x consists of all real numbers y for which $f(x) = f(y)$.

Therefore, $\frac{\mathbb{R}}{\sim} = \{[x] : x \in \mathbb{R}\} = \{\{y \in \mathbb{R} : f(x) = f(y)\} : x \in \mathbb{R}\}$. $\square$

Exercise 45. Let F be the set of all real valued functions with domain $\mathbb{R}$.

Define a relation $\sim$ over F by the rule $f \sim g$ iff the set $\{x : f(x) \neq g(x)\}$ is finite.

Then $\sim$ is an equivalence relation on F.

Proof. Reflexive:

Let $f \in F$.

Then f is a real valued function with domain $\mathbb{R}$.

Let $S = \{x \in \mathbb{R} : f(x) \neq f(x)\}$.

Suppose there is an element in S.

Let x be some element of S.

Then $x \in \mathbb{R}$ and $f(x) \neq f(x)$.

Hence, $f(x) \neq f(x)$, a contradiction.
Thus, there is no element in S.
Therefore, S is empty, so $S = \emptyset$.
Since the empty set is finite, then S is finite. Hence, $f \sim f$, so $\sim$ is reflexive.
Symmetric:
Let $f, g \in F$ such that $f \sim g$.
Then f and g are real valued functions with domain $\mathbb{R}$ and the set $\{x \in \mathbb{R} : f(x) \neq g(x)\}$ is finite.
To prove $g \sim f$, we must prove the set $\{x \in \mathbb{R} : g(x) \neq f(x)\}$ is finite.
Let $S = \{x \in \mathbb{R} : f(x) \neq g(x)\}$ and $T = \{x \in \mathbb{R} : g(x) \neq f(x)\}$.
Then S is finite.
To prove T is finite, we first prove $T = S$.
Observe that

$$\begin{aligned} x \in T &\Leftrightarrow x \in \mathbb{R} \wedge g(x) \neq f(x) \\ &\Leftrightarrow x \in \mathbb{R} \wedge f(x) \neq g(x) \\ &\Leftrightarrow x \in S. \end{aligned}$$

Hence, $T = S$.
Since $T = S$ and S is finite, then T is finite.
Therefore, $g \sim f$, so $\sim$ is symmetric.
Transitive:
Let $f, g, h \in F$ such that $f \sim g$ and $g \sim h$.
Then f, g and h are real valued functions with domain $\mathbb{R}$ and the sets $\{x \in \mathbb{R} : f(x) \neq g(x)\}$ and $\{x \in \mathbb{R} : g(x) \neq h(x)\}$ are finite.
To prove $f \sim h$, we must prove the set $\{x \in \mathbb{R} : f(x) \neq h(x)\}$ is finite.
Let $A = \{x \in \mathbb{R} : f(x) \neq g(x)\}$ and $B = \{x \in \mathbb{R} : g(x) \neq h(x)\}$ and $C = \{x \in \mathbb{R} : f(x) \neq h(x)\}$.
Then A and B are finite.
We must prove C is finite.
Since f, g and h are real valued functions of a real variable and $A = \{x \in \mathbb{R} : f(x) \neq g(x)\}$ and $B = \{x \in \mathbb{R} : g(x) \neq h(x)\}$ and $C = \{x \in \mathbb{R} : f(x) \neq h(x)\}$, then $C \subset A \cup B$.
The union of finite sets is finite.
Thus, $A \cup B$ is finite since A and B are finite.
A subset of a finite set is finite.
Since $C \subset A \cup B$, then C is finite.
Hence, $f \sim h$, so $\sim$ is transitive.
Therefore, $\sim$ is an equivalence relation on F.
Since $\sim$ is an equivalence relation, then the collection of equivalence classes $\frac{F}{\sim} = \{[f] : f \in F\}$ is a partition of F.
Let $f \in F$.
Then $[f] = \{g \in F : f \sim g\}$.
Thus, the equivalence class of a real valued function f with domain $\mathbb{R}$ consists of all real valued functions g with domain $\mathbb{R}$ such that the set $\{x : f(x) \neq g(x)\}$ is finite. $\square$

Exercise 46. Give examples of a relation that is not reflexive, not symmetric, and transitive.

Solution. Let $S = \{1, 2, 3\}$.

Let $R = \{(1, 1), (2, 2), (1, 2), (1, 3)\}$.

Clearly, R is a relation on S.

Since $(3, 3) \notin R$, then R is not reflexive.

Since $(1, 2) \in R$, but $(2, 1) \notin R$, then R is not symmetric.

However, R is transitive.

Let R be the relation is taller than on the set H of all living humans.

Then $R = \{(x, y) \in H \times H : x \text{ is taller than } y\}$.

A person is not taller than himself, so R is not reflexive.

If a person is taller than a second person, then the second person is not taller than the first person.

Hence, R is not symmetric.

If a person is taller than a second person and the second person is taller than a third person, then the first person is taller than the third person.

Therefore, R is transitive. $\square$

Exercise 47. Give examples of a relation that is reflexive, not symmetric, and transitive.

Solution. Let $S = \{1, 2, 3\}$.

Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3)\}$.

Clearly, R is a relation on S.

Since $(1, 1), (2, 2), (3, 3) \in R$, then R is reflexive.

Since $(1, 2) \in R$, but $(2, 1) \notin R$, then R is not symmetric.

However, R is transitive.

Let R be the relation is greater than or equal to on the set $\mathbb{R}$.

Then $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq y\}$.

For each $x \in \mathbb{R}$, $x = x$, so $x \geq x$.

Hence, R is reflexive.

Since 4.5 and 5.7 are real numbers and $5.7 \geq 4.5$, but $4.5 \ngeq 5.7$, then R is not symmetric.

Since $\geq$ relation on $\mathbb{R}$ is transitive, then R is transitive. $\square$

Exercise 48. Let S be the set of all lines in $\mathbb{R}^3$.

Define a relation R on S by $l \sim m$ iff $l = m$ or l is parallel to m for all lines $l, m \in S$.

Analyze R.

Solution. Let $R = \{(l, m) \in S \times S : l = m \text{ or } l \text{ is parallel to } m\}$.

Since $R \subset S \times S$, then R is a relation over S.

Reflexive property: $(\forall x \in S)((x, x) \in S)$ means for each line l in S, $l = l$ or l is parallel to l.

Thus, R is reflexive iff each line in S equals itself or is parallel to itself.

Each line in S equals itself or is parallel to itself, so R is reflexive.

Symmetric property: $(\forall l, m \in S)((l,m) \in R \to (m,l) \in R)$ means if l and m are any lines in S such that $l = m$ or l is parallel to m, then $m = l$ or m is parallel to l.

Let l and m be arbitrary lines in S such that $l = m$ or l is parallel to m.

Then $m = l$ or l is parallel to m, so $m = l$ or m is parallel to l.

Hence, R is symmetric.

Transitive property: $(\forall l, m, n \in S)((l,m) \in R \wedge (m,n) \in R \to (l,n) \in R)$ means if $l, m, n \in S$ such that both $l = m$ or l is parallel to m and $m = n$ or m is parallel to n, then $l = n$ or l is parallel to n.

Suppose l, m, and n are arbitrary lines in S such that both $l = m$ or l is parallel to m and $m = n$ or m is parallel to n.

Then either $l = m$ and $m = n$ or $l = m$ and m is parallel to n or l is parallel to m and $m = n$ or l is parallel to m and m is parallel to n.

Hence, we consider these 4 cases separately.

Case 1: Suppose $l = m$ and $m = n$.

Then $l = n$.

Case 2: Suppose $l = m$ and m is parallel to n.

Then l is parallel to n.

Case 3: Suppose l is parallel to m and $m = n$.

Then l is parallel to n.

Case 4: Suppose l is parallel to m and m is parallel to n.

Then l is parallel to n.

Hence, in all cases either $l = n$ or l is parallel to n.

Thus, R is transitive.

Therefore, R is an equivalence relation on S. □

Exercise 49. Let S be the set of all lines in $\mathbb{R}^3$.

Define a relation R on S by $l \sim m$ iff l is perpendicular to m for all lines $l, m \in S$.

Analyze R.

Solution. Let $R = \{(l,m) \in S \times S : l \perp m\}$.

Since $R \subset S \times S$, then R is a relation over S.

Reflexive property: $(\forall l \in S)((l,l) \in S)$ means for each line l in S, $l \perp l$.

Thus, R is reflexive iff each line in S is perpendicular to itself.

Hence, R is not reflexive iff there is a line in S that is not perpendicular to itself.

No line is perpendicular to itself.

Thus, every line is not perpendicular to itself.

Since S is not empty, let l be a line in S.

Then l is not perpendicular to itself.

Hence, there is a line in S that is not perpendicular to itself.

Therefore, R is not reflexive.

Symmetric property: $(\forall l, m \in S)((l,m) \in R \to (m,l) \in R)$ means if l and m are any lines in S such that $l \perp m$, then $m \perp l$.

Let l and m be arbitrary lines in S such that $l \perp m$.

Then $m \perp l$.

Hence, R is symmetric.

Transitive property: $(\forall l, m, n \in S)((l, m) \in R \wedge (m, n) \in R \rightarrow (l, n) \in R)$ means if $l, m, n \in S$ such that both $l \perp m$ and $m \perp n$, then $l \perp n$.

Let $l, m, n \in S$ such that l is the x axis and m is the y axis and n is some line perpendicular to m such that n is not the z axis in $\mathbb{R}^3$.

Then $l \perp m$ and $m \perp n$, but $l \not\perp n$.

Hence, R is not transitive.

Thus, R is not an equivalence relation on S. $\square$

Exercise 50. Let S be the set of all lines in $\mathbb{R}^3$.

Define a relation R on S by $l \sim m$ iff l and m are coplanar for all lines $l, m \in S$.

Analyze R.

Solution. Let $R = \{(l, m) \in S \times S : l \text{ and } m \text{ are coplanar } \}$.

Since $R \subset S \times S$, then R is a relation over S.

Reflexive property: $(\forall l \in S)((l, l) \in S)$ means for each line l in S, l and l are coplanar.

Thus, R is reflexive iff each line in S is coplanar to itself.

Each line lies in the same plane as itself since each line intersects itself.

Hence, each line is coplanar to itself.

Therefore, R is reflexive.

Symmetric property: $(\forall l, m \in S)((l, m) \in R \rightarrow (m, l) \in R)$ means if l and m are any lines in S such that l and m are coplanar, then m and l are coplanar.

Let l and m be arbitrary lines in S such that l and m are coplanar.

Then m and l are coplanar.

Hence, R is symmetric.

Transitive property: $(\forall l, m, n \in S)((l, m) \in R \wedge (m, n) \in R \rightarrow (l, n) \in R)$ means if $l, m, n \in S$ such that both l and m are coplanar and m and n are coplanar, then l and n are coplanar.

Let $l, m, n \in S$ such that l is the x axis and m is the y axis and n is some line that crosses the y axis not at the origin and is parallel to the z axis in $\mathbb{R}^3$.

Then l and m are coplanar since l and m intersect at a point and m and n are coplanar since m and n intersect at a point, but l and n are not coplanar since l and n do not intersect and l and n are not parallel.

Therefore, R is not transitive.

Thus, R is not an equivalence relation on S. $\square$

Exercise 51. Let S be the set of all lines in $\mathbb{R}^3$.

Define a relation R on S by $l \sim m$ iff l and m are skew for all lines $l, m \in S$.

Analyze R.

Solution. Let $R = \{(l, m) \in S \times S : l \text{ and } m \text{ are skew } \}$.

By definition, two lines are skew iff they do not intersect and are not parallel.

Hence, two lines are not skew iff either they do intersect or they are parallel.

Since $R \subset S \times S$, then R is a relation over S.

Reflexive property: $(\forall l \in S)((l,l) \in S)$ means for each line l in S, l and l are skew.

Thus, R is reflexive iff each line in S is skew to itself.

Hence, R is not reflexive iff there is a line in S that is not skew to itself.

Therefore, R is not reflexive iff there is a line in S that does intersect itself or is parallel to itself.

Each line intersects itself and is parallel to itself.

Since S is not empty, let l be a line in S.

Then l intersects itself and l is parallel to itself.

Hence, l intersects itself or l is parallel to itself.

Therefore, l and l are not skew, so R is not reflexive.

Symmetric property: $(\forall l, m \in S)((l,m) \in R \to (m,l) \in R)$ means if l and m are any lines in S such that l and m are skew, then m and l are skew.

Let l and m be arbitrary lines in S such that l and m are skew.

Then l and m do not intersect and l and m are not parallel.

Hence, m and l do not intersect and m and l are not parallel.

Thus, m and l are skew, so R is symmetric.

Transitive property: $(\forall l, m, n \in S)((l,m) \in R \wedge (m,n) \in R \to (l,n) \in R)$ means if $l, m, n \in S$ such that both l and m are skew and m and n are skew, then l and n are skew.

Let $l, m, n \in S$ such that l is a line parallel to the x axis and below the xy plane and m is the y axis and n is a line above the xy plane and is parallel to the x axis in $\mathbb{R}^3$.

Then l and m are skew since l and m do not intersect and l and m are not parallel and m and n are skew since m and n do not intersect and m and n are not parallel.

However, l and n are not skew since l and n are parallel (since each line is parallel to the x axis).

Therefore, R is not transitive.

Thus, R is not an equivalence relation on S. $\square$

Exercise 52. Let S be the set of all lines in $\mathbb{R}^3$.

Define a relation R on S by $l \sim m$ iff l and m intersect at a point for all lines $l, m \in S$.

Analyze R.

Solution. Let $R = \{(l,m) \in S \times S : l \text{ and } m \text{ intersect at a point } \}$.

Since $R \subset S \times S$, then R is a relation over S.

Reflexive property: $(\forall l \in S)((l,l) \in S)$ means for each line l in S, l and l intersect at a point.

Thus, R is reflexive iff each line in S intersects itself at a point.

Hence, R is not reflexive iff there is a line in S that does not intersect itself at a point.

The intersection of a line and itself is the line itself.

Since S is not empty, let l be a line in S.

Then $l \cap l = l$, so l does not intersect itself at a point.

Hence, there is a line in S that does not intersect itself at a point.

Therefore, R is not reflexive.

Symmetric property: $(\forall l, m \in S)((l,m) \in R \rightarrow (m,l) \in R)$ means if l and m are any lines in S such that l and m intersect at a point, then m and l intersect at a point.

Let l and m be arbitrary lines in S such that l and m intersect at a point.

Then there is a point p such that $l \cap m = p$.

Hence, there is a point p such that $m \cap l = p$.

Thus, m and l intersect at a point.

Therefore, R is symmetric.

Transitive property: $(\forall l, m, n \in S)((l,m) \in R \wedge (m,n) \in R \rightarrow (l,n) \in R)$ means if $l, m, n \in S$ such that both l and m intersect at a point and m and n intersect at a point, then l and n intersect at a point.

Let $l, m, n \in S$ such that l is the x axis and m is the y axis and n is a line parallel to the x axis and crosses the y axis not at the origin.

Then $l \cap m$ is the origin, so l and m intersect at a point and there exists a point p distinct from the origin such that $m \cap n = p$, so m and n intersect at a point.

Since $l \parallel n$, then $l \cap n = \emptyset$, so l and n do not intersect.

Therefore, R is not transitive.

Thus, R is not an equivalence relation on S. $\square$

Exercise 53. Let S be the set of all triangles in a plane $\mathbb{R}^2$.

Define a relation R on S by $X \sim Y$ iff X and Y are congruent for all triangles $X, Y \in S$.

Analyze R.

Solution. Let $R = \{(X,Y) \in S \times S : X \cong Y\}$.

By definition, two triangles are congruent iff they have the same size and same shape. Hence, two triangles are not congruent iff either they do not have the same size or they do not have the same shape.

Since $R \subset S \times S$, then R is a relation over S.

Reflexive property: $(\forall X \in S)((X,X) \in S)$ means for each triangle X in S, $X \cong X$.

Thus, R is reflexive iff each triangle in S is congruent to itself.

Each triangle has the same size and same shape as itself, so each triangle is congruent to itself.

Therefore, R is reflexive.

Symmetric property: $(\forall X, Y \in S)((X,Y) \in R \rightarrow (Y,X) \in R)$ means if X and Y are any triangles in S such that $X \cong Y$, then $Y \cong X$.

Let X and Y be arbitrary triangles in S such that $X \cong Y$.

Then X and Y have the same size and same shape.

Thus, Y and X have the same size and same shape, so $Y \cong X$.

Hence, $X \cong Y$ implies $Y \cong X$, so R is symmetric.

Transitive property: $(\forall X, Y, Z \in S)((X,Y) \in R \wedge (Y,Z) \in R \rightarrow (X,Z) \in R)$ means if $X, Y, Z \in S$ such that $X \cong Y$ and $Y \cong Z$, then $X \cong Z$.

Let X, Y, and Z be arbitrary triangles in S such that $X \cong Y$ and $Y \cong Z$.
Then X has the same size and shape as Y and Y has the same size and shape as Z.
Hence, X has the same size and shape as Z, so $X \cong Z$.
Thus, $X \cong Y$ and $Y \cong Z$ implies $X \cong Z$, so R is transitive.
Therefore, R is an equivalence relation on S. $\square$

Exercise 54. Let S be the set of all triangles in a plane $\mathbb{R}^2$.
Define a relation R on S by $X \sim Y$ iff X and Y are similar for all triangles $X, Y \in S$.
Analyze R.

Solution. Let $R = \{(X,Y) \in S \times S : X \sim Y\}$.
By definition, two triangles are similar iff they have the same shape.
Hence, two triangles are not similar iff they do not have the same shape.
Since $R \subset S \times S$, then R is a relation over S.
Reflexive property: $(\forall X \in S)((X,X) \in S)$ means for each triangle X in S, $X \sim X$.
Thus, R is reflexive iff each triangle in S is similar to itself.
Each triangle has the same shape as itself, so each triangle is similar to itself.
Therefore, R is reflexive.
Symmetric property: $(\forall X, Y \in S)((X,Y) \in R \rightarrow (Y,X) \in R)$ means if X and Y are any triangles in S such that $X \sim Y$, then $Y \sim X$.
Let X and Y be arbitrary triangles in S such that $X \sim Y$.
Then X and Y have the same shape.
Thus, Y and X have the same shape, so $Y \sim X$.
Hence, $X \sim Y$ implies $Y \sim X$, so R is symmetric.
Transitive property: $(\forall X, Y, Z \in S)((X,Y) \in R \wedge (Y,Z) \in R \rightarrow (X,Z) \in R)$ means if $X, Y, Z \in S$ such that $X \sim Y$ and $Y \sim Z$, then $X \sim Z$.
Let X, Y, and Z be arbitrary triangles in S such that $X \sim Y$ and $Y \sim Z$.
Then X has the shape as Y and Y has the shape as Z.
Hence, X has the same shape as Z, so $X \sim Z$.
Thus, $X \sim Y$ and $Y \sim Z$ implies $X \sim Z$, so R is transitive.
Therefore, R is an equivalence relation on S. $\square$

Exercise 55. Compute all equivalence classes $[n]$ for $n = -9$ to $n = 10$ corresponding to the equivalence relation congruence modulo 5.
Describe the set $\frac{\mathbb{Z}}{\equiv_5}$ of all equivalence classes.

Solution. Congruence modulo 5 relation on $\mathbb{Z}$ is defined by $a \equiv b \pmod 5$ iff $5|(a-b)$ for all $a, b \in \mathbb{Z}$.
The equivalence classes are
$[-9] = \{y \in \mathbb{Z} : -9 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(-9-y)\} = \{..., -14, -9, -4, 1, 6, 11, ...\}$.
$[-8] = \{y \in \mathbb{Z} : -8 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(-8-y)\} = \{..., -13, -8, -3, 2, 7, 12, ...\}$.
$[-7] = \{y \in \mathbb{Z} : -7 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(-7-y)\} = \{..., -12, -7, -2, 3, 8, 13, ...\}$.
$[-6] = \{y \in \mathbb{Z} : -6 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(-6-y)\} = \{..., -11, -6, -1, 4, 9, 14, ...\}$.
$[-5] = \{y \in \mathbb{Z} : -5 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(-5-y)\} = \{..., -10, -5, 0, 5, 10, 15, ...\}$.

$[-4] = \{y \in \mathbb{Z} : -4 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(-4-y)\} = \{..., -9, -4, 1, 6, 11, 16, ...\}$.
$[-3] = \{y \in \mathbb{Z} : -3 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(-3-y)\} = \{..., -8, -3, 2, 7, 12, 17, ...\}$.
$[-2] = \{y \in \mathbb{Z} : -2 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(-2-y)\} = \{..., -7, -2, 3, 8, 13, 18, ...\}$.
$[-1] = \{y \in \mathbb{Z} : -1 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(-1-y)\} = \{..., -6, -1, 4, 9, 14, 19, ...\}$.
$[0] = \{y \in \mathbb{Z} : 0 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|-y\} = \{..., -5, 0, 5, 10, 15, 20, ...\}$.
$[1] = \{y \in \mathbb{Z} : 1 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(1-y)\} = \{..., -4, 1, 6, 11, 16, 21, ...\}$.
$[2] = \{y \in \mathbb{Z} : 2 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(2-y)\} = \{..., -3, 2, 7, 12, 17, 22, ...\}$.
$[3] = \{y \in \mathbb{Z} : 3 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(3-y)\} = \{..., -2, 3, 8, 13, 18, 23, ...\}$.
$[4] = \{y \in \mathbb{Z} : 4 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(4-y)\} = \{..., -1, 4, 9, 14, 19, 24, ...\}$.
$[5] = \{y \in \mathbb{Z} : 5 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(5-y)\} = \{..., 0, 5, 10, 15, 20, 25, ...\}$.
$[6] = \{y \in \mathbb{Z} : 6 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(6-y)\} = \{..., 1, 6, 11, 16, 21, 26, ...\}$.
$[7] = \{y \in \mathbb{Z} : 7 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(7-y)\} = \{..., 2, 7, 12, 17, 22, 27, ...\}$.
$[8] = \{y \in \mathbb{Z} : 8 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(8-y)\} = \{..., 3, 8, 13, 18, 23, 28, ...\}$.
$[9] = \{y \in \mathbb{Z} : 9 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(9-y)\} = \{..., 4, 9, 14, 19, 24, 29, ...\}$.
$[10] = \{y \in \mathbb{Z} : 10 \equiv y \pmod 5\} = \{y \in \mathbb{Z} : 5|(10-y)\} = \{..., 5, 10, 15, 20, 25, 30, ...\}$.
The quotient set of $\mathbb{Z}$ by $\equiv_5$ is $\frac{\mathbb{Z}}{\equiv_5} = \{[x] : x \in \mathbb{Z}\}$.
Thus, the following sets are elements of $\frac{\mathbb{Z}}{\equiv_5}$:
$\{..., -14, -9, -4, 1, 6, 11, ...\}$
$\{..., -13, -8, -3, 2, 7, 12, ...\}$
$\{..., -12, -7, -2, 3, 8, 13, ...\}$
$\{..., -11, -6, -1, 4, 9, 14, ...\}$
$\{..., -10, -5, 0, 5, 10, 15, ...\}$. □

Exercise 56. Let $S = \{a, b, c, d, e, f\}$.
Let $\sim$ be the relation consisting of the ordered pairs
$(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (a, d), (d, a), (d, f), (f, d), (a, f), (f, a)$.
Analyze $\sim$.

Solution. Since $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is an equivalence relation on S.

Observe that $[a] = \{x \in S : a \sim x\} = \{a, d, f\}$ and $[b] = \{x \in S : b \sim x\} = \{b\}$ and $[c] = \{x \in S : c \sim x\} = \{c\}$ and $[d] = \{x \in S : d \sim x\} = \{d, a, f\}$ and $[e] = \{x \in S : e \sim x\} = \{e\}$ and $[f] = \{x \in S : f \sim x\} = \{f, d, a\}$, so $[a] = [d] = [f] = \{a, d, f\}$ and $[b] = \{b\}$ and $[c] = \{c\}$ and $[e] = \{e\}$.

Therefore, $\frac{S}{\sim} = \{[x] : x \in S\} = \{\{a, d, f\}, \{b\}, \{c\}, \{e\}\}$ is a 4 celled partition of S. □

Exercise 57. Let $S = \{a, b, c, d, e, f\}$.
Let $\sim$ be the relation consisting of the ordered pairs
$(a, a), (b, b), (c, c), (d, d), (e, e), (f, f)$.
Analyze $\sim$.

Solution. Observe that $\sim$ is the equality(i.e., identity) relation over S.

Therefore, $\sim$ is an equivalence relation on S.

Observe that $[a] = \{x \in S : a \sim x\} = \{a\}$ and $[b] = \{x \in S : b \sim x\} = \{b\}$ and $[c] = \{x \in S : c \sim x\} = \{c\}$ and $[d] = \{x \in S : d \sim x\} = \{d\}$ and $[e] = \{x \in S : e \sim x\} = \{e\}$ and $[f] = \{x \in S : f \sim x\} = \{f\}$.

Therefore, $\frac{S}{\sim} = \{[x] : x \in S\} = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}\}$ is a 6 celled partition of S. $\square$

Exercise 58. Let $S = \{a, b, c, d, e, f\}$.

Let $\sim$ be the relation consisting of the ordered pairs
$(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (a, b), (b, a), (c, e), (e, c), (d, f), (f, d)$.
Analyze $\sim$.

Solution. Since $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is an equivalence relation on S.

Observe that $[a] = \{x \in S : a \sim x\} = \{a, b\}$ and $[b] = \{x \in S : b \sim x\} = \{b, a\}$ and $[c] = \{x \in S : c \sim x\} = \{c, e\}$ and $[d] = \{x \in S : d \sim x\} = \{d, f\}$ and $[e] = \{x \in S : e \sim x\} = \{e, c\}$ and $[f] = \{x \in S : f \sim x\} = \{f, d\}$, so $[a] = [b] = \{a, b\}$ and $[c] = [e] = \{c, e\}$ and $[d] = [f] = \{d, f\}$.

Therefore, $\frac{S}{\sim} = \{[x] : x \in S\} = \{\{a, b\}, \{c, e\}, \{d, f\}\}$ is a 3 celled partition of S. $\square$

Exercise 59. Let $S = \{a, b, c, d, e, f\}$.

Analyze the relation $S \times S$.

Solution. Observe that $S \times S = \{(x, y) : x, y \in S\}$ consists of the ordered pairs
$(a, a), (a, b), (a, c), (a, d), (a, e), (a, f)$
$(b, a), (b, b), (b, c), (b, d), (b, e), (b, f)$
$(c, a), (c, b), (c, c), (c, d), (c, e), (c, f)$
$(d, a), (d, b), (d, c), (d, d), (d, e), (d, f)$
$(e, a), (e, b), (e, c), (e, d), (e, e), (e, f)$
$(f, a), (f, b), (f, c), (f, d), (f, e), (f, f)$.

Since $S \times S$ is the total relation on S, then $S \times S$ is an equivalence relation over S.

Observe that $[a] = \{x \in S : a \sim x\} = \{a, b, c, d, e, f\}$ and $[b] = \{x \in S : b \sim x\} = \{a, b, c, d, e, f\}$ and $[c] = \{x \in S : c \sim x\} = \{a, b, c, d, e, f\}$ and $[d] = \{x \in S : d \sim x\} = \{a, b, c, d, e, f\}$ and $[e] = \{x \in S : e \sim x\} = \{a, b, c, d, e, f\}$ and $[f] = \{x \in S : f \sim x\} = \{a, b, c, d, e, f\}$, so $[a] = [b] = [c] = [d] = [e] = [f] = \{a, b, c, d, e, f\} = S$.

Therefore, $\frac{S}{S \times S} = \{[x] : x \in S\} = \{\{a, b, c, d, e, f\}\} = \{S\}$ is a 1 celled partition of S. $\square$

Exercise 60. Let $S = \{a, b, c, d, e, f\}$.

Let $\sim$ be the relation consisting of the ordered pairs
$(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (c, d), (d, c), (c, e), (e, c), (c, f)$
$(f, c), (d, e), (e, d), (d, f), (f, d), (e, f), (f, e)$.
Analyze $\sim$.

Solution. Since $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is an equivalence relation on S.

Observe that $[a] = \{x \in S : a \sim x\} = \{a\}$ and $[b] = \{x \in S : b \sim x\} = \{b\}$ and $[c] = \{x \in S : c \sim x\} = \{c, d, e, f\}$ and $[d] = \{x \in S : d \sim x\} = \{d, c, e, f\}$

and $[e] = \{x \in S : e \sim x\} = \{e, c, d, f\}$ and $[f] = \{x \in S : f \sim x\} = \{f, c, d, e\}$, so $[a] = \{a\}$ and $[b] = \{b\}$ and $[c] = [d] = [e] = [f] = \{c, d, e, f\}$.

Therefore, $\frac{S}{\sim} = \{[x] : x \in S\} = \{[a], [b], [c]\} = \{\{a\}, \{b\}, \{c, d, e, f\}\}$ is a 3 celled partition of S. $\square$

Exercise 61. Let $S = \{a, b, c, d, e, f\}$.

Let $P = \{\{a, c, e\}, \{b, d, f\}\}$.

What can be concluded?

Solution. Observe that P is a 2 celled partition of S.

Hence, P induces an equivalence relation $\frac{S}{P}$ over S defined by $a \sim b$ iff there exists a cell $T \in P$ such that $a \in T$ and $b \in T$.

Thus, $\frac{S}{P}$ consists of the ordered pairs

$(a, a), (c, c), (e, e), (a, c), (c, a), (a, e), (e, a), (c, e), (e, c)$

$(b, b), (d, d), (f, f), (b, d), (d, b), (b, f), (f, b), (d, f), (f, d)$. $\square$

Exercise 62. Let $S = \{a, b, c, d, e, f\}$.

Let $P = \{\{a\}, \{b\}, \{c, d, f\}, \{e\}\}$.

What can be concluded?

Solution. Observe that P is a 4 celled partition of S.

Hence, P induces an equivalence relation $\frac{S}{P}$ over S defined by $a \sim b$ iff there exists a cell $T \in P$ such that $a \in T$ and $b \in T$.

Thus, $\frac{S}{P}$ consists of the ordered pairs

$(a, a), (b, b), (c, c), (d, d), (f, f), (c, d), (d, c), (c, f), (f, c), (d, f), (f, d), (e, e)$. $\square$

Exercise 63. Let $S = \{a, b, c, d, e, f\}$.

Let $P = \{\{a, b, c, d, e, f\}\}$.

What can be concluded?

Solution. Observe that P is a 1 celled partition of S.

Hence, P induces an equivalence relation $\frac{S}{P}$ over S defined by $a \sim b$ iff there exists a cell $T \in P$ such that $a \in T$ and $b \in T$.

Thus, $\frac{S}{P}$ consists of the ordered pairs

$(a, a), (a, b), (a, c), (a, d), (a, e), (a, f)$

$(b, a), (b, b), (b, c), (b, d), (b, e), (b, f)$

$(c, a), (c, b), (c, c), (c, d), (c, e), (c, f)$

$(d, a), (d, b), (d, c), (d, d), (d, e), (d, f)$

$(e, a), (e, b), (e, c), (e, d), (e, e), (e, f)$

$(f, a), (f, b), (f, c), (f, d), (f, e), (f, f)$.

Therefore, $\frac{S}{P} = S \times S$, the total relation over S. $\square$

Exercise 64. Let $S = \{a, b, c, d, e, f\}$.

Let $P = \{\{a, c\}, \{e, f\}, \{b\}, \{d\}\}$.

What can be concluded?

Solution. Observe that P is a 4 celled partition of S.

Hence, P induces an equivalence relation $\frac{S}{P}$ over S defined by $a \sim b$ iff there exists a cell $T \in P$ such that $a \in T$ and $b \in T$.

Thus, $\frac{S}{P}$ consists of the ordered pairs
$(a,a),(c,c),(a,c),(c,a),(e,e),(f,f),(e,f),(f,e),(b,b),(d,d)$. $\square$

Exercise 65. Let $S = \{a,b,c,d,e,f\}$.
Let $P = \{\{a\},\{b\},\{c\},\{d\},\{e\},\{f\}\}$.
What can be concluded?

Solution. Observe that P is a 6 celled partition of S.

Hence, P induces an equivalence relation $\frac{S}{P}$ over S defined by $a \sim b$ iff there exists a cell $T \in P$ such that $a \in T$ and $b \in T$.

Thus, $\frac{S}{P}$ consists of the ordered pairs
$(a,a),(b,b),(c,c),(d,d),(e,e),(f,f)$.
Therefore, $\frac{S}{P}$ is the equality (i.e. identity) relation on S. $\square$

Exercise 66. Let $S = \{1,2,3,4,5\}$.
Let E_1 be a relation consisting of the ordered pairs:
$(1,1),(2,2),(3,3),(4,4),(5,5),(2,5),(5,2),(3,5),(5,3),(2,3),(3,2)$.
Let E_2 be a relation consisting of the ordered pairs:
$(1,1),(1,2),(2,2),(2,1),(3,3),(4,4),(5,5),(3,4),(4,3),(3,5),(5,3),(4,5),(5,4)$.
What can be concluded about $E_1 \cup E_2$?

Solution. Consider the relation E_1.

Since E_1 is reflexive, symmetric, and transitive, then E_1 is an equivalence relation over S.

Equivalently, observe that $[1] = \{1\}$ and $[2] = \{2,3,5\}$ and $[3] = \{3,5,2\}$ and $[4] = \{4\}$ and $[5] = \{5,2,3\}$, so $[2] = [3] = [5] = \{2,3,5\}$.

Hence, the set $P = \{\{1\},\{2,3,5\},\{4\}\}$ is a 3 celled partition of S, so $\frac{S}{P}$ is an equivalence relation on S defined by $a \sim b$ iff there is a cell $T \in P$ such that $a \in T$ and $b \in T$ for each $a,b \in S$. Since $\frac{S}{P} = E_1$, then E_1 is an equivalence relation on S.

Consider the relation E_2.

Since E_2 is reflexive, symmetric, and transitive, then E_2 is an equivalence relation over S.

Equivalently, observe that $[1] = \{1,2\}$ and $[2] = \{2,1\}$ and $[3] = \{3,4,5\}$ and $[4] = \{4,3,5\}$ and $[5] = \{5,3,4\}$, so $[1] = [2] = \{1,2\}$ and $[3] = [4] = [5] = \{3,4,5\}$.

Hence, the set $P = \{\{1,2\},\{3,4,5\}$ is a 2 celled partition of S, so $\frac{S}{P}$ is an equivalence relation on S defined by $a \sim b$ iff there is a cell $T \in P$ such that $a \in T$ and $b \in T$ for each $a,b \in S$. Since $\frac{S}{P} = E_2$, then E_2 is an equivalence relation on S.

Is $E_1 \cup E_2$ an equivalence relation?
Observe that $E_1 \cup E_2 = \{(x,y) \in S \times S : (x,y) \in E_1 \vee (x,y) \in E_2\}$.
Hence, $E_1 \cup E_2 \subset S \times S$, so $E_1 \cup E_2$ is a relation over S.
Observe that $E_1 \cup E_2$ consists of the ordered pairs:
$(1,1),(2,2),(3,3),(4,4),(5,5),(2,5),(5,2),(3,5),(5,3),(2,3),(3,2)$
$(1,2),(2,1),(3,4),(4,3),(4,5),(5,4)$.
Clearly, $E_1 \cup E_2$ is reflexive and symmetric.

However, observe that $(4,5) \in E_1 \cup E_2$ and $(5,2) \in E_1 \cup E_2$, but $(4,2) \notin E_1 \cup E_2$.

Hence, $E_1 \cup E_2$ is not transitive.

Therefore, $E_1 \cup E_2$ is not an equivalence relation.

Thus, the union of two equivalence relations is not necessarily an equivalence relation. $\square$

Example 67. Define a relation $\sim$ on $\mathbb{R}^2$ by $(a,b) \sim (c,d)$ iff $a^2+b^2 \leq c^2+d^2$.

Show that $\sim$ is reflexive and transitive, but not symmetric.

Proof. We prove $\sim$ is reflexive.

Let $(a,b) \in \mathbb{R}^2$.

Then $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Since $a^2+b^2 \leq a^2+b^2$, then $(a,b) \sim (a,b)$, so $\sim$ is reflexive. $\square$

Proof. Observe that $(1,1) \in \mathbb{R}^2$ and $(1,2) \in \mathbb{R}^2$.

Since $1^1+1^2 = 2 < 5 = 1^2+2^2$, then $(1,1) \sim (1,2)$.

Since $1^2+2^2 = 5 \not\leq 2 = 1^1+1^2$, then $(1,2) \not\sim (1,1)$.

Since $(1,1) \sim (1,2)$ but $(1,2) \not\sim (1,1)$, then $\sim$ is not symmetric. $\square$

Proof. Let $(a,b),(c,d),(e,f) \in \mathbb{R}^2$ such that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$.

Since $(a,b),(c,d),(e,f) \in \mathbb{R}^2$, then $a,b,c,d,e,f \in \mathbb{R}$.

Since $(a,b) \sim (c,d)$, then $a^2+b^2 \leq c^2+d^2$.

Since $(c,d) \sim (e,f)$, then $c^2+d^2 \leq e^2+f^2$.

Thus, by transitivity of $\leq$, we have $a^2+b^2 \leq e^2+f^2$, so $(a,b) \sim (e,f)$.

Therefore, $\sim$ is transitive. $\square$

Example 68. real projective line $\mathbb{P}(\mathbb{R})$ in geometry

Define a relation $\sim$ on $\mathbb{R}^2 - \{(0,0)\}$ by $(x_1,y_1) \sim (x_2,y_2)$ iff there exists a nonzero real number λ such that $(x_1,y_1) = (\lambda x_2, \lambda y_2)$.

Then $\sim$ is an equivalence relation on $\mathbb{R}^2 - \{(0,0)\}$.

Proof. Let $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$.

Then $(x,y) \in \mathbb{R}^2$ and $(x,y) \notin \{(0,0)\}$, so $x,y \in \mathbb{R}$ and $x \neq 0$ and $y \neq 0$.

Let $\lambda = 1$.

Since 1 is a nonzero real number, then λ is a nonzero real number.

Since $x = 1 \cdot x$ and $y = 1 \cdot y$, then $(x,y) = (1 \cdot x, 1 \cdot y) = (\lambda x, \lambda y)$.

Since λ is a nonzero real number and $(x,y) = (\lambda x, \lambda y)$, then $(x,y) \sim (x,y)$, so $\sim$ is reflexive. $\square$

Proof. Let $(a,b),(c,d) \in \mathbb{R}^2 - \{(0,0)\}$ such that $(a,b) \sim (c,d)$.

Since $(a,b),(c,d) \in \mathbb{R}^2 - \{(0,0)\}$, then $a,b,c,d \in \mathbb{R}$ and $a \neq 0, b \neq 0, c \neq 0, d \neq 0$.

Since $(a,b) \sim (c,d)$, then there exists $\lambda_1 \in \mathbb{R}$ with $\lambda_1 \neq 0$ such that $(a,b) = (\lambda_1 c, \lambda_1 d)$, so $a = \lambda_1 c$ and $b = \lambda_1 d$.

Since $c \neq 0$ and $d \neq 0$, then we divide to obtain $\lambda_1 = \frac{a}{c} = \frac{b}{d}$.

Let $\lambda = \frac{1}{\lambda_1}$.

Then $\lambda = \frac{c}{a} = \frac{d}{b}$, since a,b,c,d are all nonzero real numbers.

Hence, λ is a nonzero real number.

Since $c = c \cdot 1 = c \cdot \frac{a}{a} = \frac{c}{a} \cdot a = \lambda a$ and $d = d \cdot 1 = d \cdot \frac{b}{b} = \frac{d}{b} \cdot b = \lambda b$, then $(c, d) = (\lambda a, \lambda b)$.

Since λ is a non-zero real number and $(c, d) = (\lambda a, \lambda b)$, then $(c, d) \sim (a, b)$, so $\sim$ is symmetric. $\square$

Proof. Let $(a, b), (c, d), (e, f) \in \mathbb{R}^2 - \{(0, 0)\}$ such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.

Since $(a, b), (c, d), (e, f) \in \mathbb{R}^2 - \{(0, 0)\}$, then $a, b, c, d, e, f \in \mathbb{R}$ and $a \neq 0, b \neq 0, c \neq 0, d \neq 0, e \neq 0, f \neq 0$.

Since $(a, b) \sim (c, d)$, then there exists $\lambda_1 \in \mathbb{R}$ with $\lambda_1 \neq 0$ such that $(a, b) = (\lambda_1 c, \lambda_1 d)$, so $a = \lambda_1 c$ and $b = \lambda_1 d$.

Since $c \neq 0$ and $d \neq 0$, then we divide to obtain $\lambda_1 = \frac{a}{c} = \frac{b}{d}$.

Since $(c, d) \sim (e, f)$, then there exists $\lambda_2 \in \mathbb{R}$ with $\lambda_2 \neq 0$ such that $(c, d) = (\lambda_2 e, \lambda_2 f)$, so $c = \lambda_2 e$ and $d = \lambda_2 f$.

Since $e \neq 0$ and $f \neq 0$, then we divide to obtain $\lambda_2 = \frac{c}{e} = \frac{d}{f}$.

Let $\lambda = \lambda_1 \cdot \lambda_2$.

Then $\lambda = \frac{a}{c} \cdot \frac{c}{e} = \frac{a}{e} = \frac{b}{d} \cdot \frac{d}{f} = \frac{b}{f}$.

Since a, b, e, f are all nonzero real numbers, then $\lambda = \frac{a}{e} = \frac{b}{f}$ is a nonzero real number.

Since $a = a \cdot 1 = a \cdot \frac{e}{e} = \frac{a}{e} \cdot e = \lambda e$ and $b = b \cdot 1 = b \cdot \frac{f}{f} = \frac{b}{f} \cdot f = \lambda f$, then $(a, b) = (\lambda e, \lambda f)$.

Since λ is a nonzero real number and $(a, b) = (\lambda e, \lambda f)$, then $(a, b) \sim (e, f)$, so $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is an equivalence relation on $\mathbb{R} - \{(0, 0)\}$.

What are the corresponding equivalence classes?

Let $(a, b) \in \mathbb{R}^2 - \{(0, 0)\}$.

Then $a, b \in \mathbb{R}$ and $a \neq 0, b \neq 0$.

The equivalence class of the ordered pair (a, b) is the set

$$\begin{aligned} [(a, b)] &= \{(x, y) : (x, y) \sim (a, b)\} \\ &= \{(x, y) : (\exists \lambda \neq 0)(x, y) = (\lambda a, \lambda b)\} \\ &= \{(x, y) : (\exists \lambda \neq 0)(x = \lambda a, y = \lambda b)\} \\ &= \{(\lambda a, \lambda b) : \lambda \neq 0\}. \end{aligned}$$

This set is a line in $\mathbb{R}^2$ with slope $\frac{b}{a}$ and equation $y = \frac{bx}{a}$ and excludes the origin $(0, 0)$.

Equivalently, this is the same as the linear function defined by $f(x) = \frac{bx}{a}$ and excludes the origin $(0, 0)$.

The collection of all such equivalence classes is a partition of $\mathbb{R}^2 - \{(0, 0)\}$.

The collection of all equivalence classes is the quotient set of $\mathbb{R}^2 - \{(0, 0)\}$ by $\sim$.

The quotient set is the set $\frac{\mathbb{R}^2-\{(0,0)\}}{\sim} = \{[(a,b)] : (a,b) \in \mathbb{R}^2 - \{(0,0)\}\}$.

Therefore the quotient set consists of all lines with slope $\frac{b}{a}$ that exclude the origin for every ordered pair $(a,b) \in \mathbb{R}^2 - \{(0,0)\}$. □

Exercise 69. For all relations R_1 and R_2 defined on set S, if R_1 and R_2 are equivalence relations, then $R_1 \cap R_2$ is an equivalence relation.

Solution. Let S be an arbitrary set.

Let the domain of discourse U be the set of all binary relations defined on S.

We translate the proposition into logic symbols:

$\forall R_1 \forall R_2.[p(R_1, R_2) \rightarrow q(R_1, R_2)]$ with the predicates

$p(R_1, R_2) : R_1$ and R_2 are equivalence relations.

$q(R_1, R_2) : R_1 \cap R_2$ is an equivalence relation.

Based on this form we let R_1 and R_2 be arbitrary relations on S.

The hypothesis is $P(R_1, R_2)$ and conclusion is $Q(R_1, R_2)$. □

Proof. Let R_1 and R_2 be arbitrary relations defined on an arbitrary set S.

To prove $R_1 \cap R_2$ is an equivalence relation means to prove $R_1 \cap R_2$ is reflexive, symmetric, and transitive.

To prove this we assume R_1 and R_2 are equivalence relations.

We first show that $R_1 \cap R_2$ is a relation defined on S.

Observe that $R_1 \cap R_2 = \{(x,y) \in S \times S : (x,y) \in R_1 \wedge (x,y) \in R_2\}$.

Hence $R_1 \cap R_2$ is a subset of $S \times S$, so $R_1 \cap R_2$ is a binary relation on S.

To prove $R_1 \cap R_2$ is reflexive means to prove $(x,x) \in R_1 \cap R_2$ for all $x \in S$.

To prove this we assume a is an arbitrary element of S.

We must prove $(a,a) \in R_1 \cap R_2$; that is, $(a,a) \in R_1$ and $(a,a) \in R_2$.

By assumption $a \in S$, so the ordered pair (a,a) is an element of $S \times S$.

Hence, $(a,a) \in S \times S$.

By hypothesis, we know R_1 and R_2 are reflexive.

Since $(a,a) \in S \times S$ and R_1 is reflexive, then $(a,a) \in R_1$.

Since $(a,a) \in S \times S$ and R_2 is reflexive, then $(a,a) \in R_2$.

Thus, $(a,a) \in R_1$ and $(a,a) \in R_2$, as desired.

To prove $R_1 \cap R_2$ is symmetric means to prove $(x,y) \in R_1 \cap R_2 \rightarrow (y,x) \in R_1 \cap R_2$ for all $x, y \in S$.

To prove this we assume a and b are arbitrary elements of S.

We must prove if $(a,b) \in R_1 \cap R_2$ then $(b,a) \in R_1 \cap R_2$.

To prove $(b,a) \in R_1 \cap R_2$ we must show that $(b,a) \in R_1$ and $(b,a) \in R_2$.

By assumption we know $(a,b) \in S \times S$.

Suppose $(a,b) \in R_1 \cap R_2$.

Then $(a,b) \in R_1$ and $(a,b) \in R_2$.

By hypothesis we know R_1 and R_2 are symmetric.

Since $(a,b) \in R_1$ and R_1 is symmetric, then $(b,a) \in R_1$.

Since $(a,b) \in R_2$ and R_2 is symmetric, then $(b,a) \in R_2$.

Thus we have shown $(b,a) \in R_1$ and $(b,a) \in R_2$, as desired.

To prove $R_1 \cap R_2$ is transitive means to prove $(x,y) \in R_1 \cap R_2 \wedge (y,z) \in R_1 \cap R_2 \rightarrow (x,z) \in R_1 \cap R_2$ for all $x, y, z \in S$.

To prove this we assume a, b, c are arbitrary elements of S.
We must prove $(a, b) \in R_1 \cap R_2 \wedge (b, c) \in R_1 \cap R_2 \rightarrow (a, c) \in R_1 \cap R_2$.
To prove $(a, c) \in R_1 \cap R_2$ we must show that $(a, c) \in R_1$ and $(a, c) \in R_2$.
By assumption we know that $(a, b) \in S \times S$ and $(b, c) \in S \times S$.
Suppose $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$.
Since $(a, b) \in R_1 \cap R_2$ then $(a, b) \in R_1$ and $(a, b) \in R_2$.
Since $(b, c) \in R_1 \cap R_2$ then $(b, c) \in R_1$ and $(b, c) \in R_2$.
By hypothesis, we know R_1 and R_2 are transitive.
Thus, since $(a, b) \in R_1$ and $(b, c) \in R_1$ we conclude $(a, c) \in R_1$.
Since $(a, b) \in R_2$ and $(b, c) \in R_2$ we conclude $(a, c) \in R_2$.
Thus we have shown $(a, c) \in R_1$ and $(a, c) \in R_2$, as desired. □

Exercise 70. Let $f : A \rightarrow B$ be a function.
Let $\cong$ be an equivalence relation on B.
Let $\sim$ be defined on A by $x_1 \sim x_2$ iff $f(x_1) \cong f(x_2)$ for all $x_1, x_2 \in A$.
Then $\sim$ is an equivalence relation on A.

Proof. Let $x \in A$.
Then $f(x) \in B$.
Since $\cong$ is reflexive in B, then $f(x) \cong f(x)$.
Since $x \sim x$ iff $f(x) \cong f(x)$, then $x \sim x$.
Hence, $\sim$ is reflexive.
Let $x_1, x_2 \in A$ such that $x_1 \sim x_2$.
Then $f(x_1) \in B$ and $f(x_2) \in B$ and $f(x_1) \cong f(x_2)$.
Since $\cong$ is symmetric in B, then $f(x_2) \cong f(x_1)$.
Since $x_2 \sim x_1$ iff $f(x_2) \cong f(x_1)$, then $x_2 \sim x_1$.
Thus, $\sim$ is symmetric.
Let $x_1, x_2, x_3 \in A$ such that $x_1 \sim x_2$ and $x_2 \sim x_3$.
Then $f(x_1) \in B$ and $f(x_2) \in B$ and $f(x_2) \in B$ and $f(x_1) \cong f(x_2)$ and $f(x_2) \cong f(x_3)$.
Since $\cong$ is transitive in B, then $f(x_1) \cong f(x_3)$.
Since $x_1 \sim x_3$ iff $f(x_1) \cong f(x_3)$, then $x_1 \sim x_3$.
Thus, $\sim$ is transitive.
Since $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is an equivalence relation on A. □

Exercise 71. Prove $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod 6\} \subseteq \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod 3\}$.

Proof. Let $A = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod 6\}$.

Let $B = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod 3\}$.

We prove $A \subseteq B$.

Suppose $(a, b) \in A$.

Then $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ and $a \equiv b \pmod 6$.

Since $a \equiv b \pmod 6$, then $6|a - b$.

By definition of divisibility there is an integer k for which $a - b = 6k$.

Since $a - b = 6k = 3(2k)$, then $3|a - b$, and consequently $a \equiv b \pmod 3$ by definition of congruence modulo.

So we have $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ and $a \equiv b \pmod 3$, therefore $(a, b) \in B$.

Thus $(a, b) \in A$ implies $(a, b) \in B$, so it follows that $A \subseteq B$. $\square$

Partial Orderings

Example 72. The power set of a set is ordered by inclusion.

Solution. Let S be a set.

Let $\mathscr{P}(S)$ be the power set of S.

To prove the power set of S is ordered by inclusion, we must prove $(\mathscr{P}(S), \subset)$ is a poset.

Thus, we must prove the subset relation on $\mathscr{P}(S)$ is a partial order over $\mathscr{P}(S)$.

Hence, we must prove the subset relation over $\mathscr{P}(S)$ defined by $\{(M, N) \in \mathscr{P}(S) \times \mathscr{P}(S) : M \subset N\}$ is reflexive, antisymmetric, and transitive. $\square$

Proof. Let $\mathscr{P}(S)$ be the power set of a set S.

Let R be the subset relation over $\mathscr{P}(S)$.

Then $R = \{(M, N) \in \mathscr{P}(S) \times \mathscr{P}(S) : M \subset N\}$.

Let $A \in \mathscr{P}(S)$ be arbitrary.

Then $A \subset S$ and $(A, A) \in \mathscr{P}(S) \times \mathscr{P}(S)$.

Since the subset relation is reflexive, then $A \subset A$.

Since $(A, A) \in \mathscr{P}(S) \times \mathscr{P}(S)$ and $A \subset A$, then $(A, A) \in R$.

Therefore, R is reflexive.

Let $A \in \mathscr{P}(S)$ and $B \in \mathscr{P}(S)$ be arbitrary such that $(A, B) \in R$ and $(B, A) \in R$.

Then $A \subset S$ and $B \subset S$ and $A \subset B$ and $B \subset A$.

Since the subset relation is antisymmetric and $A \subset B$ and $B \subset A$, then $A = B$.

Hence, $(A, B) \in R$ and $(B, A) \in R$ implies $A = B$, so R is antisymmetric.

Let $A \in \mathscr{P}(S)$ and $B \in \mathscr{P}(S)$ and $C \in \mathscr{P}(S)$ be arbitrary such that $(A, B) \in R$ and $(B, C) \in R$.

Then $A \subset S$ and $B \subset S$ and $C \subset S$ and $A \subset B$ and $B \subset C$.

Since $A \subset S$ and $C \subset S$, then $A \in \mathscr{P}(S)$ and $C \in \mathscr{P}(S)$, so $(A, C) \in \mathscr{P}(S) \times \mathscr{P}(S)$.

Since the subset relation is transitive and $A \subset B$ and $B \subset C$, then $A \subset C$.

Since $(A, C) \in \mathscr{P}(S) \times \mathscr{P}(S)$ and $A \subset C$, then $(A, C) \in R$.

Therefore, $(A, B) \in R$ and $(B, C) \in R$ implies $(A, C) \in R$, so R is transitive.

Since R is reflexive, antisymmetric, and transitive, then R is a partial ordering over $\mathscr{P}(S)$. $\square$

Example 73. The interval $(1, \infty)$ is unbounded above in $\mathbb{R}$ and bounded below in $\mathbb{R}$.

Proof. Let $I = (1, \infty)$.

We first prove I is bounded below in $\mathbb{R}$.
Let $x \in I$.
Then $x \in \mathbb{R}$ and $1 < x$.
Since $0.5 < 1$, then $0.5 \leq 1$.
Since $1 < x$, then $1 \leq x$.
Since $0.5 \leq 1$ and $1 \leq x$, then by transitivity of $\leq$, $0.5 \leq x$.
Thus, $0.5 \leq x$ for any $x \in I$.
Therefore, 0.5 is a lower bound for I in $\mathbb{R}$, so I is bounded below in $\mathbb{R}$.

To prove I is unbounded above in $\mathbb{R}$, we must prove there is no upper bound for I in $\mathbb{R}$.

Suppose for the sake of contradiction there is an upper bound for I in $\mathbb{R}$.
Let $U \in \mathbb{R}$ be an upper bound for I.
Then $x \leq U$ for every $x \in I$.
To derive a contradiction, we must prove there exists $a \in I$ such that $a > U$.
Let $a = U + 1$.
Since $\mathbb{R}$ is closed under addition and $1, U \in \mathbb{R}$, then $U + 1 \in \mathbb{R}$.
Hence, $a \in \mathbb{R}$.
Since $1 > 0$, then $U + 1 > U$, so $a > U$.
Let $x \in I$.
Then $x \in \mathbb{R}$ and $1 < x$.
Since $x \leq U$ for every $x \in I$ and $x \in I$, then $x \leq U$.
Since $1 < x$ and $x \leq U$, then $1 < U$.
Since $1 < U$ and $U < a$, then $1 < a$.
Since $a \in \mathbb{R}$ and $1 < a$, then $a \in I$.
Thus, there exists $a \in I$ such that $a > U$.
Therefore, there is no upper bound for I in $\mathbb{R}$. $\square$

Exercise 74. Consider the poset $(\mathbb{Z}^+ \cup \{0\}, |)$.

What are the greatest and least elements, if they exist?

Find upper and lower bounds for the following sets: $\{4, 8, 16\}$, $\{4, 6, 10\}$, $\{3, 5, 7\}$.

Find the greatest and least elements of the above sets, if they exist.

Give an example of an unbounded subset of this poset, if any exist.

Describe $m \vee n$ and $m \wedge n$, for nonnegative integers m and n.

Solution. Let $S = \mathbb{Z}^+ \cup \{0\}$ be the set of all nonnegative integers.

Then under the divides relation, $(S, |)$ is a poset.

We've already proved in our relation examples that 0 is the greatest element and 1 is the least element.

Let $A = \{4, 8, 16\}$ and $B = \{4, 6, 10\}$ and $C = \{3, 5, 7\}$.

An element $n \in S$ is an upper bound for A iff $x|n$ for all $x \in A$.

Thus, $n \in S$ is an upper bound for A iff n is a multiple of every element of A.

Thus, $n \in S$ is an upper bound for A iff n is a common multiple of $4, 8, 16$.

Since 32 is a common multiple of 4,8, and 16, then 32 is an upper bound for A.

An element $n \in S$ is a lower bound for A iff $n|x$ for all $x \in A$.

Thus, $n \in S$ is a lower bound for A iff n is a divisor of every element of A.

Thus, $n \in S$ is a lower bound for A iff n is a common divisor of $4, 8, 16$.

Since 2 is a common divisor of 4,8, and 16, then 2 is a lower bound for A.

An element M is the greatest element of A iff $M \in A$ and M is an upper bound for A.

Thus, M is the greatest element of A iff $M \in A$ and M is a common multiple of $4, 8, 16$.

Since $16 \in A$ and 16 is a common multiple of 4,8, and 16, then 16 is the greatest element of A.

An element m is the least element of A iff $m \in A$ and m is a lower bound for A.

Thus, m is the least element of A iff $m \in A$ and m is a common divisor of $4, 8, 16$.

Since $4 \in A$ and 4 is a common multiple of 4,8, and 16, then 4 is the least element of A.

An element $n \in S$ is an upper bound for B iff $x|n$ for all $x \in B$.

Thus, $n \in S$ is an upper bound for B iff n is a multiple of every element of B.

Thus, $n \in S$ is an upper bound for B iff n is a common multiple of $4, 6, 10$.

Since 240 is a common multiple of 4,6, and 10, then 240 is an upper bound for B.

An element $n \in S$ is a lower bound for B iff $n|x$ for all $x \in B$.

Thus, $n \in S$ is a lower bound for B iff n is a divisor of every element of B.

Thus, $n \in S$ is a lower bound for B iff n is a common divisor of $4, 6, 10$.

Since 2 is a common divisor of 4,6, and 10, then 2 is a lower bound for B.

An element M is the greatest element of B iff $M \in B$ and M is an upper bound for B.

Thus, M is the greatest element of B iff $M \in B$ and M is a common multiple of $4, 6, 10$.

Since 4 is not a multiple of 6, then 4 is not a greatest element of B.

Since 6 is not a multiple of 4, then 6 is not a greatest element of B.

Since 10 is not a multiple of 4, then 10 is not a greatest element of B.

Therefore, there is no greatest element of B.

An element m is the least element of B iff $m \in B$ and m is a lower bound for B.

Thus, m is the least element of B iff $m \in B$ and m is a common divisor of $4, 6, 10$.

Since 4 is not a divisor of 6, then 4 is not a least element of B.

Since 6 is not a divisor of 4, then 6 is not a least element of B.

Since 10 is not a divisor of 4, then 10 is not a least element of B.
Therefore, there is no least element of B.
An element $n \in S$ is an upper bound for C iff $x|n$ for all $x \in C$.
Thus, $n \in S$ is an upper bound for C iff n is a multiple of every element of C.
Thus, $n \in S$ is an upper bound for C iff n is a common multiple of $3, 5, 7$.
Since 315 is a common multiple of 3,5, and 7, then 315 is an upper bound for C.
An element $n \in S$ is a lower bound for C iff $n|x$ for all $x \in C$.
Thus, $n \in S$ is a lower bound for C iff n is a divisor of every element of C.
Thus, $n \in S$ is a lower bound for C iff n is a common divisor of $3, 5, 7$.
Since 1 is a common divisor of 3,5, and 7, then 1 is a lower bound for C.
An element M is the greatest element of C iff $M \in C$ and M is an upper bound for C.
Thus, M is the greatest element of C iff $M \in C$ and M is a common multiple of $3, 5, 7$.
Since 3 is not a multiple of 5, then 3 is not a greatest element of C.
Since 5 is not a multiple of 3, then 5 is not a greatest element of C.
Since 7 is not a multiple of 3, then 7 is not a greatest element of C.
Therefore, there is no greatest element of C.
An element m is the least element of C iff $m \in C$ and m is a lower bound for C.
Thus, m is the least element of C iff $m \in C$ and m is a common divisor of $3, 5, 7$.
Since 3 is not a divisor of 5, then 3 is not a least element of C.
Since 5 is not a divisor of 3, then 5 is not a least element of C.
Since 7 is not a divisor of 3, then 7 is not a least element of C.
Therefore, there is no least element of C.
We prove there is no unbounded subset of the set of all nonnegative integers under the divides relation.
Suppose for the sake of contradiction there is an unbounded subset of S.
Let T be an unbounded subset of S under the relation $|$.
Then $T \subset S$ and either T is not bounded above in S or not bounded below in S.
We consider these cases separately.
Case 1: Suppose T is not bounded above in S.
Then T has no upper bound in S.
Either T is empty or not.
Suppose $T \neq \emptyset$.
Then there is an element in T.
Let x be an arbitrary element of T.
Since $x \in T$ and $T \subset S$ and $S \subset \mathbb{Z}$, then $x \in \mathbb{Z}$.
Every integer divides 0, so $x|0$.
Since x is arbitrary, then $x|0$ for all $x \in T$.
Hence, 0 is an upper bound for T in S, so T has an upper bound in S.

Thus, we have T has no upper bound in S and T has an upper bound in S, a contradiction.

Suppose $T = \emptyset$.

Then there is no element in T.

An element $u \in S$ is an upper bound for T iff $x|u$ for all $x \in T$.

Thus, an element $u \in S$ is an upper bound for $\emptyset$ iff $x|u$ for all $x \in \emptyset$.

Since there is no element in T, then the statement there is an element $x \in \emptyset$ such that $x \not| u$ is false.

Hence, the statement for all $x \in \emptyset$, $x|u$ is true.

Therefore, $u \in S$ is an upper bound for $\emptyset$, so $\emptyset$ has an upper bound in S.

Thus, T has an upper bound in S.

Thus, we have T has no upper bound in S and T has an upper bound in S, a contradiction.

In either case, we have a contradiction.

Therefore, T is bounded above in S.

Case 2: Suppose T is not bounded below in S.

Then T has no lower bound in S.

Either T is empty or not.

Suppose $T \neq \emptyset$.

Then there is an element in T.

Let x be an arbitrary element of T.

Since $x \in T$ and $T \subset S$ and $S \subset \mathbb{Z}$, then $x \in \mathbb{Z}$.

Since $1 \in S$ divides every integer, then $1|x$.

Since x is arbitrary, then $1|x$ for all $x \in T$.

Hence, 1 is a lower bound for T in S, so T has a lower bound in S.

Thus, we have T has no lower bound in S and T has a lower bound in S, a contradiction.

Suppose $T = \emptyset$.

Then there is no element in T.

An element $l \in S$ is a lower bound for T iff $l|x$ for all $x \in T$.

Thus, an element $l \in S$ is a lower bound for $\emptyset$ iff $l|x$ for all $x \in \emptyset$.

Since there is no element in T, then the statement there is an element $x \in \emptyset$ such that $l \not| x$ is false.

Hence, the statement for all $x \in \emptyset$, $l|x$ is true.

Therefore, $l \in S$ is a lower bound for $\emptyset$, so $\emptyset$ has a lower bound in S.

Thus, T has a lower bound in S.

Thus, we have T has no lower bound in S and T has a lower bound in S, a contradiction.

In either case, we have a contradiction.

Therefore, T is bounded below in S.

Since T is bounded above in S and bounded below in S, then T is bounded in S.

Therefore, T is not unbounded.

Thus, we have T is unbounded and T is not unbounded, a contradiction.

Therefore, there is no unbounded subset of S.

Let $m, n \in S$.

The join of m and n is $m \vee n = lub\{m, n\}$.

An element $u \in S$ is an upper bound of the set $\{m, n\}$ iff $x|u$ for all $x \in \{m, n\}$.

Thus, an element $u \in S$ is an upper bound for $\{m, n\}$ iff $m|u$ and $n|u$.

Thus, an element $u \in S$ is an upper bound for $\{m, n\}$ iff u is a common multiple of m and n.

Therefore, the least upper bound of $\{m, n\}$ is the least common multiple of m and n.

Hence, the join of m and n is $m \vee n = lub\{m, n\} = lcm(m, n)$.

The meet of m and n is $m \wedge n = glb\{m, n\}$.

An element $l \in S$ is a lower bound of the set $\{m, n\}$ iff $l|x$ for all $x \in \{m, n\}$.

Thus, an element $l \in S$ is a lower bound for $\{m, n\}$ iff $l|m$ and $l|n$.

Thus, an element $l \in S$ is a lower bound for $\{m, n\}$ iff l is a common divisor of m and n.

Therefore, the greatest lower bound of $\{m, n\}$ is the greatest common divisor of m and n.

Hence, the meet of m and n is $m \wedge n = glb\{m, n\} = \gcd(m, n)$. □

Exercise 75. Consider the poset $(\mathbb{Z}^+ \cup \{0\}, \leq)$.

What are the greatest and least elements, if they exist?

Find upper and lower bounds for the following sets: $\{4, 8, 16\}$, $\{4, 6, 10\}$, $\{3, 5, 7\}$.

Find the greatest and least elements of the above sets, if they exist.

Give an example of an unbounded subset of this poset, if any exist.

Describe $m \vee n$ and $m \wedge n$, for nonnegative integers m and n.

Solution. Let $S = \mathbb{Z}^+ \cup \{0\} = \{0, 1, 2, 3, ...\}$ be the set of all nonnegative integers.

Then under the less than or equal to relation, $(S, \leq)$ is a poset.

Clearly, 0 is the least element of S.

We prove there is no greatest element of S.

Suppose there exists a greatest element of S.

Let M be the greatest element of S.

Then $M \in S$ and $x \leq M$ for all $x \in S$.

To derive a contradiction, we show there exists an element of S that is greater than M.

Since $M \in S$, then $M \geq 0$.

Since $M \in S$ and $S \subset \mathbb{Z}$, then $M \in \mathbb{Z}$.

Since M and 1 are integers and $\mathbb{Z}$ is closed under addition, then $M + 1 \in \mathbb{Z}$.

Since $1 > 0$, then $M + 1 > M + 0$, so $M + 1 > M$.

Since $M + 1 > M$ and $M \geq 0$, then $M + 1 > 0$.

Since $M + 1 \in \mathbb{Z}$ and $M + 1 > 0$, then $M \in S$.

Thus, there exists an element of S that is greater than M.

Therefore, there is no greatest element of S.

Let $A = \{4, 8, 16\}$ and $B = \{4, 6, 10\}$ and $C = \{3, 5, 7\}$.

An element $n \in S$ is an upper bound for A iff $x \leq n$ for all $x \in A$.

Since every element of A is less than or equal to 17, then 17 is an upper bound for A.

An element $n \in S$ is a lower bound for A iff $n \leq x$ for all $x \in A$.

Since 1 is less than or equal to every element of A, then 1 is a lower bound for A.

An element M is the greatest element of A iff $M \in A$ and M is an upper bound for A.

Since $16 \in A$ and every element of A is less than or equal to 16, then 16 is the greatest element of A.

An element m is the least element of A iff $m \in A$ and m is a lower bound for A.

Since $4 \in A$ and 4 is less than or equal to every element of A, then 4 is the least element of A.

An element $n \in S$ is an upper bound for B iff $x \leq n$ for all $x \in B$.

Since every element of B is less than or equal to 12, then 12 is an upper bound for B.

An element $n \in S$ is a lower bound for B iff $n \leq x$ for all $x \in B$.

Since 0 is less than or equal to every element of B, then 0 is a lower bound for B.

An element M is the greatest element of B iff $M \in B$ and M is an upper bound for B.

Since $10 \in B$ and every element of B is less than or equal to 1, then 10 is the greatest element of B.

An element m is the least element of B iff $m \in B$ and m is a lower bound for B.

Since $4 \in B$ and 4 is less than or equal to every element of B, then 4 is the least element of B.

An element $n \in S$ is an upper bound for C iff $x \leq n$ for all $x \in C$.

Since every element of C is less than or equal to 11, then 11 is an upper bound for C.

An element $n \in S$ is a lower bound for C iff $n \leq x$ for all $x \in C$.

Since 2 is less than or equal to every element of C, then 2 is a lower bound for C.

An element M is the greatest element of C iff $M \in C$ and M is an upper bound for C.

Since $7 \in C$ and every element of C is less than or equal to 7, then 7 is the greatest element of C.

An element m is the least element of C iff $m \in C$ and m is a lower bound for C.

Since $3 \in C$ and 3 is less than or equal to every element of C, then 3 is the least element of C.

An example of an unbounded subset of the set of all nonnegative integers under $\leq$ is the set of all positive even integers.

Let T be the set of all positive even integers.

Then $T = \{2, 4, 6, 8, ...\} = \{2k : k \in \mathbb{N}\}$ and $T \subset S$.

Suppose there is an upper bound for T in S.

Let M be an upper bound for T in S.
Then $M \in S$ and $x \leq M$ for all $x \in T$.
Since $M \in S$, then there exists $k \in \mathbb{Z}^+$ such that $M = 2k$.
To derive a contradiction, we show there exists an element a of T that is greater than M.
Let $a = M + 2$.
Then $a = 2k + 2 = 2(k + 1)$.
Since $k \in \mathbb{Z}^+$, then $k + 1 \in \mathbb{Z}^+$.
Thus, $a \in T$.
Since $2 > 0$, then $M + 2 > M + 0$, so $a > M$.
Thus, there exists $a \in T$ such that $a > M$.
Therefore, there is no upper bound for T in S.
Hence, T is unbounded above in S.
Let $m, n \in S$.
The join of m and n is $m \vee n = lub\{m, n\}$.
An element $u \in S$ is an upper bound of the set $\{m, n\}$ iff $x \leq u$ for all $x \in \{m, n\}$.
Thus, an element $u \in S$ is an upper bound for $\{m, n\}$ iff $m \leq u$ and $n \leq u$.
Let $M = m \vee n$ be the maximum of m and n.
Then $m \leq M$ and $n \leq M$ and M is either m or n.
Hence, M is an upper bound for $\{m, n\}$ and $M \in \{m, n\}$, so M is the greatest element of $\{m, n\}$.
Therefore, the least upper bound of $\{m, n\}$ is the maximum of m and n.
Hence, the join of m and n is $m \vee n = lub\{m, n\} = \max(m, n)$.
The meet of m and n is $m \wedge n = glb\{m, n\}$.
An element $l \in S$ is a lower bound of the set $\{m, n\}$ iff $l \leq x$ for all $x \in \{m, n\}$.
Thus, an element $l \in S$ is a lower bound for $\{m, n\}$ iff $l \leq m$ and $l \leq n$.
Let $L = m \wedge n$ be the minimum of m and n.
Then $L \leq m$ and $L \leq n$ and L is either m or n.
Hence, L is a lower bound of $\{m, n\}$ and $L \in \{m, n\}$, so L is the least element of $\{m, n\}$.
Therefore, the greatest lower bound of $\{m, n\}$ is the minimum of m and n.
Hence, the meet of m and n is $m \wedge n = glb\{m, n\} = \min(m, n)$. $\square$

Exercise 76. Consider the poset (S, R) where $S = \{1, 2, 3, ..., 9, 10\}$ and R is the union of the following sets of ordered pairs:
$\{(x, x) : x \in S\}$
$\{(1, x) : x \in S\}$
$\{(x, 10) : x \in S\}$
$\{(1, 3), (1, 5), (1, 7), (1, 9), (3, 5), (3, 7), (3, 9), (5, 7), (5, 9), (7, 9)\}$.
a. Compute the lub of the sets $\{1, 3, 5\}$, $\{2, 4, 6\}$.
b. Compute the glb and the least element of $\{2, 4, 6\}$.
c. Compute an upper bound for $\{3, 4, 5, 6, 8\}$.

Solution. Since R is the union of sets of ordered pairs of S, then R is a set of ordered pairs of S, so $R \subset S \times S$. Hence, R is a relation over S.

Observe that R consists of:

$(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),(10,10)$ and
$(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8)(1,9),(1,10)$ and
$(2,10),(3,10),(4,10),(5,10),(6,10),(7,10),(8,10),(9,10)$ and
$(3,5),(3,7),(3,9),(5,7),(5,9),(7,9)$.

Clearly, R is reflexive since the identity relation $\{(x,x) : x \in S\}$ is a subset of R.

Clearly, R is transitive, since if $(1,x) \in R$ and $(x,10) \in R$, then $(1,10) \in R$ for all $x \in S$.

Clearly, R is antisymmetric.

Since R is reflexive, antisymmetric and transitive, then R is a partial order over S, so (S,R) is a poset.

Let $A = \{1,3,5\}$ and $B = \{2,4,6\}$.

An element $U \in S$ is the least upper bound for A iff U is an upper bound for A and U is the least of all upper bounds for A.

Thus, $U \in S$ is $lub(A)$ iff for all $x \in A$, $(x,U) \in R$ and for each upper bound $u \in S$ of A, $(U,u) \in R$.

Thus, $U \in S$ is $lub(A)$ iff $(1,U) \in R \wedge (3,U) \in R \wedge (5,U) \in R$ and for each upper bound $u \in S$ of A, $(U,u) \in R$.

The upper bounds for A are $5,7,9,10$.

The least upper bound U of A satisfies $(U,5) \in R \wedge (U,7) \in R \wedge (U,9) \in R \wedge (U,10) \in R$.

The only upper bound that satisfies this is 5 since $(5,5) \in R \wedge (5,7) \in R \wedge (5,9) \in R \wedge (5,10) \in R$.

Therefore, $lub(A) = 5$.

An element $U \in S$ is the least upper bound for B iff U is an upper bound for B and U is the least of all upper bounds for B.

Thus, $U \in S$ is $lub(B)$ iff for all $x \in B$, $(x,U) \in R$ and for each upper bound $u \in S$ of B, $(U,u) \in R$. Thus, $U \in S$ is $lub(B)$ iff $(2,U) \in R \wedge (4,U) \in R \wedge (6,U) \in R$ and for each upper bound $u \in S$ of B, $(U,u) \in R$.

The only upper bound for B is 10.

The least upper bound U of B satisfies $(U,10) \in R$.

Since $(10,10) \in R$ and 10 is the only upper bound of B, then $lub(B) = 10$.

An element $L \in S$ is the greatest lower bound for B iff L is a lower bound for B and L is the greatest of all lower bounds for B.

Thus, $L \in S$ is $glb(B)$ iff for all $x \in B$, $(L,x) \in R$ and for each lower bound $l \in S$ of B, $(l,L) \in R$.

Thus, $L \in S$ is $glb(B)$ iff $(L,2) \in R \wedge (L,4) \in R \wedge (L,6) \in R$ and for each lower bound $l \in S$ of B, $(l,L) \in R$.

The only element L of S that satisfes $(L,2) \in R \wedge (L,4) \in R \wedge (L,6) \in R$ is 1.

Therefore, the only lower bound for B is 1.

The greatest lower bound L of B satisfies $(1,L) \in R$.

Since $(1,1) \in R$ and 1 is the only lower bound of B, then 1 is the greatest lower bound of B.

Therefore, $glb(B) = 1$.

An element m is the least element of B iff $m \in B$ and m is a lower bound for B.

Thus, an element m is the least element of B iff $m \in B$ and $(m,x) \in R$ for all $x \in B$.

Hence, an element m is the least element of B iff $m \in B$ and $(m,2) \in R \wedge (m,4) \in R \wedge (m,6) \in R$.

Since $2 \in B$ and $(2,2) \in R$ but $(2,4) \notin R$, then 2 is not a least element of B.

Since $4 \in B$ and $(4,2) \notin R$, then 4 is not a least element of B.

Since $6 \in B$ and $(6,2) \notin R$, then 6 is not a least element of B.

Therefore, there is no least element of B.

Let $C = \{3,4,5,6,8\}$.

An element $u \in S$ is an upper bound for C iff $(x,u) \in R$ for all $x \in C$.

Thus, an element $u \in S$ is an upper bound for C iff $(3,u) \in R \wedge (4,u) \in R \wedge (5,u) \in R \wedge (6,u) \in R \wedge (8,u) \in R$.

The only element of S that satisfies this is 10 since $(3,10) \in R \wedge (4,10) \in R \wedge (5,10) \in R \wedge (6,10) \in R \wedge (8,10) \in R$.

Therefore, the only upper bound for C is 10. $\square$

Exercise 77. Let $(S, \leq)$ be a poset.

Let A and B be subsets of S such that $lub(A)$ and $lub(B)$ exist.

If $A \subset B$, then $lub(A) \leq lub(B)$ and $glb(B) \leq glb(A)$.

Solution. Let R be the relation $\leq$ over set S.

Since $lub(A)$ and $lub(B)$ exist, then let $u_A = lub(A)$ and $u_B = lub(B)$.

Suppose $A \subset B$.

To prove $u_A \leq u_B$, we must prove $(u_A, u_B) \in R$.

Let u be an arbitrary upper bound of A in S.

Then $u \in S$ and $(x,u) \in R$ for all $x \in A$ and $(u_A, u) \in R$.

Since $u_A = lub(A)$, then u_A is an upper bound of A and is the least of all upper bounds of A in S.

Since u_A is an upper bound of A, then $(x, u_A) \in R$ for all $x \in A$.

Since $u_B = lub(B)$, then u_B is an upper bound of B and is the least of all upper bounds of B in S.

Since u_B is an upper bound of B, then $(x, u_B) \in R$ for all $x \in B$.

Since u_A is an upper bound of A in S, then the set of all upper bounds of A in S is not empty.

Either $u_A \in B$ or $u_A \notin B$.

We consider these cases separately.

Case 1: Suppose $u_A \in B$.

Since u_B is an upper bound for B, then $(x, u_B) \in R$ for all $x \in B$.

In particular, since $u_A \in B$, then $(u_A, u_B) \in R$.

Case 2: Suppose $u_A \notin B$.

Either $u_A = u_B$ or $u_A \neq u_B$.
If $u_A = u_B$, then $(u_A, u_B) = (u_A, u_A)$.
Since R is reflexive, then $(u_A, u_A) \in R$, so $(u_A, u_B) \in R$.
On the other hand, suppose $u_A \neq u_B$.
By the antisymmetric property of R, if $(u_A, u_B) \in R$ and $(u_B, u_A) \in R$, then $u_A = u_B$.
Thus, if $u_A \neq u_B$, then either $(u_A, u_B) \notin R$ or $(u_B, u_A) \notin R$.
Since $u_A \neq u_B$, then we conclude either $(u_A, u_B) \notin R$ or $(u_B, u_A) \notin R$.
Suppose for the sake of contradiction that $(u_A, u_B) \notin R$.
Then somehow we must invoke the hypothesis that $A \subset B$! ☐

Functions

Exercise 78. Let $f : \mathbb{Q} \mapsto \mathbb{Z}$ be defined by $f(a/b) = a$.
What can we deduce about f?

Solution. We know f is a binary relation since $f \subseteq \mathbb{Q} \times \mathbb{Z}$.
Is f a function?
We observe that $1/2 = 2/4$, but $f(1/2) = 1$ and $f(2/4) = 2$.
Thus, f is not a function, by definition of function. ☐

Exercise 79. Let $f : \mathbb{Q} \to \mathbb{Q}$ be a relation defined by $f(\frac{p}{q}) = \frac{p+1}{p-2}$.
Then f is not a function(map).

Proof. Since division by zero is not defined, then $f(\frac{2}{3})$ is undefined, so f is not a function. ☐

Exercise 80. Let $f : \mathbb{Q} \to \mathbb{Q}$ be a relation defined by $f(\frac{p}{q}) = \frac{3p}{3q}$.
Then f is a function.

Proof. Let $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ such that $\frac{p}{q} = \frac{r}{s}$.
Then $f(\frac{p}{q}) = \frac{3p}{3q} = \frac{p}{q} = \frac{r}{s} = \frac{3r}{3s} = f(\frac{r}{s})$, so $f(\frac{p}{q}) = f(\frac{r}{s})$.
Therefore, f is a function. ☐

Exercise 81. Let $f : \mathbb{Q} \to \mathbb{Q}$ be a relation defined by $f(\frac{p}{q}) = \frac{p+q}{q^2}$.
Then f is not a function.

Proof. Since $f(\frac{1}{2}) = \frac{1+2}{2^2} = \frac{3}{4}$ and $f(\frac{2}{4}) = \frac{2+4}{4^2} = \frac{3}{8}$, then $f(\frac{1}{2}) \neq f(\frac{2}{4})$, even though $\frac{1}{2} = \frac{2}{4}$.
Therefore, f is not a function. ☐

Exercise 82. Let $f : \mathbb{Q} \to \mathbb{Q}$ be a relation defined by $f(\frac{p}{q}) = \frac{3p^2}{7q^2} - \frac{p}{q}$.
Then f is a function.

Proof. Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ such that $\frac{a}{b} = \frac{c}{d}$.
Then

$$
\begin{aligned}
f(\frac{a}{b}) &= \frac{3a^2}{7b^2} - \frac{a}{b} \\
&= \frac{3}{7}(\frac{a}{b})^2 - \frac{a}{b} \\
&= \frac{3}{7}(\frac{c}{d})^2 - \frac{c}{d} \\
&= \frac{3c^2}{7d^2} - \frac{c}{d} \\
&= f(\frac{c}{d}).
\end{aligned}
$$

Therefore, f is a function. $\square$

Exercise 83. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = e^x$.
Then f is injective, but not surjective.

Proof. Let $a, b \in \mathbb{R}$ such that $f(a) = f(b)$.
Then $e^a = f(a) = f(b) = e^b$, so $a = b$.
Therefore, f is injective.

Since $e^x > 0$ for all $x \in \mathbb{R}$, then $e^x \neq 0$ for all $x \in \mathbb{R}$, so $f(x) \neq 0$ for all $x \in \mathbb{R}$.
Since 0 is a real number and $f(x) \neq 0$ for all $x \in \mathbb{R}$, then f is not surjective.

The range of f is the set $\{f(x) : x \in \mathbb{R}\} = \{e^x : x \in \mathbb{R}\} = (0, \infty)$.
Therefore, the range of f is the interval $(0, \infty)$. $\square$

Exercise 84. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(n) = n^2 + 3$.
Then f is not injective and not surjective.

Proof. Observe that $f(-1) = (-1)^2 + 3 = 4 = 1^2 + 3 = f(1)$, so $f(-1) = f(1)$.
Since $-1 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$ and $-1 \neq 1$ and $f(-1) = f(1)$, then f is not injective. $\square$

Proof. We next prove f is not surjective.
Suppose there is an integer n such that $f(n) = 5$.
Then $5 = f(n) = n^2 + 3$, so $2 = n^2$.
But, there is no integer whose square is 2, so we conclude there is no integer n such that $f(n) = 5$.
Hence, $f(n) \neq 5$ for all $n \in \mathbb{Z}$.
Since $5 \in \mathbb{Z}$ and $f(n) \neq 5$ for all $n \in \mathbb{Z}$, then f is not surjective.

The range of f is the set $\{f(n) : n \in \mathbb{Z}\} = \{n^2+3 : n \in \mathbb{Z}\} = \{3, 4, 7, 12, 19, 28, ...\}$. $\square$

Exercise 85. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \sin x$.
Then f is not injective and not surjective.

Proof. Since $f(\pi) = \sin \pi = 0 = \sin 0 = f(0)$, then $f(\pi) = f(0)$.
Since $\pi \in \mathbb{R}$ and $0 \in \mathbb{R}$ and $\pi \neq 0$ and $f(\pi) = f(0)$, then f is not injective.

The range of f is the set $\{f(x) : x \in \mathbb{R}\} = \{\sin x : x \in \mathbb{R}\} = [-1, 1]$.
Therefore, the range of f is the interval $[-1, 1]$ and f is not surjective. $\square$

Exercise 86. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $f(n) = n^2$.
Then f is not injective and not surjective.

Proof. Since $1 \in \mathbb{Z}$ and $-1 \in \mathbb{Z}$ and $1 \neq -1$ and $f(1) = 1^2 = 1 = (-1)^2 = f(-1)$, then f is not injective.

Since $n^2 \geq 0$ for all $n \in \mathbb{Z}$, then $n^2 > -1$ for all $n \in \mathbb{Z}$, so $n^2 \neq -1$ for all $n \in \mathbb{Z}$.
Hence, $f(n) \neq -1$ for all $n \in \mathbb{Z}$.
Since $-1 \in \mathbb{Z}$ and $f(n) \neq -1$ for all $n \in \mathbb{Z}$, then f is not surjective. $\square$

Exercise 87. Let $f : \mathbb{Z} \to \mathbb{Q}$ be a relation defined by $f(n) = \frac{n}{1}$.
Then f is a function and f is injective, but f is not surjective.

Proof. We first prove the relation f is actually a function.

Let $a, b \in \mathbb{Z}$ such that $a = b$.
Then $f(a) = \frac{a}{1} = a = b = \frac{b}{1} = f(b)$, so $f(a) = f(b)$.
Therefore, f is a function.

We next prove f is injective.
Let $a, b \in \mathbb{Z}$ such that $f(a) = f(b)$.
Then $a = \frac{a}{1} = f(a) = f(b) = \frac{b}{1} = b$, so $a = b$.
Therefore, f is injective.

We next prove f is not surjective.
Let $b = \frac{1}{2} \in \mathbb{Q}$.
Let $a \in \mathbb{Z}$ be arbitrary.
Then $f(a) = \frac{a}{1} = a \neq \frac{1}{2} = b$, so $f(a) \neq b$.
Therefore, f is not surjective. $\square$

Exercise 88. Let $g : \mathbb{Q} \to \mathbb{Z}$ be defined by $g(\frac{p}{q}) = p$ for $\frac{p}{q}$ expressed in lowest terms with a positive denominator.
Then the function g is surjective, but g is not injective.

Proof. Let $b \in \mathbb{Z}$ be arbitrary.
Since $\frac{b}{1} \in \mathbb{Q}$ and $g(\frac{b}{1}) = b$, then g is surjective.

Observe that $\frac{1}{2} \in \mathbb{Q}$ and $\frac{1}{3} \in \mathbb{Q}$ and $\frac{1}{2} \neq \frac{1}{3}$, but $g(\frac{1}{2}) = 1 = g(\frac{1}{3})$.
Therefore, g is not injective. $\square$

Exercise 89. Let $f : \mathbb{R} \to \mathbb{R}$ be a map defined by $f(x) = x^3$.
Then f is injective.

Proof. Let $x, y \in \mathbb{R}$ such that $x \neq y$.
To prove f is injective, we must prove $x^3 \neq y^3$.
Since $x \neq y$, then either $x < y$ or $x > y$.
Without loss of generality, assume $x < y$.
Then either $0 < x < y$ or $0 = x < y$ or $x < 0 < y$ or $x < 0 = y$ or $x < y < 0$.
We consider these cases separately.
Case 1: Suppose $0 < x < y$.
Then $0 < x^2 < y^2$, so $0 < x^3 < y^3$.
Hence, $x^3 < y^3$.
Case 2: Suppose $0 = x < y$.
Since $y > 0$, then $y^3 > 0 = x^3$, so $x^3 < y^3$.
Case 3: Suppose $x < 0 < y$.
Then $x < 0$ and $y > 0$, so $x^3 < 0$ and $y^3 > 0$.
Thus, $x^3 < 0 < y^3$, so $x^3 < y^3$.
Case 4: Suppose $x < 0 = y$.
Since $x < 0$, then $x^3 < 0 = y^3$, so $x^3 < y^3$.
Case 5: Suppose $x < y < 0$.
Then $-x > -y > 0$, so $x^2 > y^2 > 0$.
Hence, $-x^3 > -y^3 > 0$, so $-x^3 > -y^3$.
Thus, $x^3 < y^3$.
In all cases, $x^3 < y^3$, so $x^3 \neq y^3$, as desired. $\square$

Exercise 90. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^4 + 5x^2$.
Then f is injective.

Proof. Let $a, b \in [0, \infty)$ such that $f(a) = f(b)$.
Then $a \geq 0$ and $b \geq 0$ and $a^4 + 5a^2 = b^4 + 5b^2$.
Since $a^4 + 5a^2 = b^4 + 5b^2$, then $a^4 - b^4 + 5a^2 - 5b^2 = 0$, so $(a^2 - b^2)(a^2 + b^2) + 5(a^2 - b^2) = 0$.
Hence, $(a^2 - b^2)(a^2 + b^2 + 5) = 0$, so either $a^2 - b^2 = 0$ or $a^2 + b^2 + 5 = 0$.
Since $a \geq 0$ and $b \geq 0$, then $a^2 \geq 0$ and $b^2 \geq 0$, so $a^2 + b^2 \geq 0$.
Thus, $a^2 + b^2 + 5 \geq 5 > 0$, so $a^2 + b^2 + 5 > 0$.
Hence, $a^2 + b^2 + 5 \neq 0$, so $a^2 - b^2 = 0$.
Therefore, $a^2 = b^2$, so $|a| = |b|$.
Since $a \geq 0$ and $b \geq 0$, then $a = b$, as desired. $\square$

Exercise 91. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - x$.
Then f is not injective.

Solution. Since $0 \neq 1$ and $f(0) = 0^3 - 0 = 0 = 1^3 - 1 = f(1)$, then f is not injective. $\square$

Exercise 92. Let $f : A \to B$ and $g : B \to C$ be functions.
If $g \circ f$ is onto, then g is onto.

Proof. Suppose $g \circ f$ is onto.
To prove g is onto, let $c \in C$.
We must prove there exists $b \in B$ such that $g(b) = c$.
Since $g \circ f$ is onto, then there exists $a \in A$ such that $(g \circ f)(a) = c$.
Since f is a function, then there exists $b \in B$ such that $f(a) = b$.
Thus,

$$\begin{aligned} c &= (g \circ f)(a) \\ &= g(f(a)) \\ &= g(b). \end{aligned}$$

Therefore, $g(b) = c$, so g is onto. $\square$

Exercise 93. Let $f : A \to B$ and $g : B \to C$ be functions.
If $g \circ f$ is one to one, then f is one to one.

Proof. Suppose $g \circ f$ is one to one.
To prove f is one to one, let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$.
Then $g(f(a_1)) = g(f(a_2))$, so $(g \circ f)(a_1) = (g \circ f)(a_2)$.
Since $g \circ f$ is one to one, then this implies $a_1 = a_2$.
Therefore, f is one to one. $\square$

Exercise 94. Let $f : A \to B$ and $g : B \to C$ be functions.
If $g \circ f$ is one to one and f is onto, then g is one to one.

Proof. Suppose $g \circ f$ is one to one and f is onto.
To prove g is one to one, let $b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$.
We must prove $b_1 = b_2$.
Since f is onto, then there exist $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$.
Thus,

$$\begin{aligned} (g \circ f)(a_1) &= g(f(a_1)) \\ &= g(b_1) \\ &= g(b_2) \\ &= g(f(a_2)) \\ &= (g \circ f)(a_2). \end{aligned}$$

Since $g \circ f$ is one to one, then this implies $a_1 = a_2$.
Hence,

$$\begin{aligned} b_1 &= f(a_1) \\ &= f(a_2) \\ &= b_2. \end{aligned}$$

Therefore, $b_1 = b_2$, so g is one to one, as desired. $\square$

Exercise 95. Let $f : A \to B$ and $g : B \to C$ be functions.
If $g \circ f$ is onto and g is one to one, then f is onto.

Proof. Suppose $g \circ f$ is onto and g is one to one.
To prove f is onto, let $b \in B$.
We must prove there exists $a \in A$ such that $f(a) = b$.
Since g is a function, then there is a unique $c \in C$ such that $g(b) = c$.
Since $g \circ f$ is onto, then there exists at least one $a \in A$ such that $(g \circ f)(a) = c$.
Thus,

$$\begin{aligned} g(b) &= c \\ &= (g \circ f)(a) \\ &= g(f(a)). \end{aligned}$$

Hence, $g(b) = g(f(a))$.
Since g is one to one, then this implies $b = f(a)$.
Thus, there exists $a \in A$ such that $f(a) = b$.
Therefore, f is onto. $\square$

Exercise 96. Devise a bijection from $S \times T$ to $T \times S$.

Solution. Observe that $S \times T = \{(s,t) : s \in S, t \in T\}$ and $T \times S = \{(t,s) : t \in T, s \in S\}$.
Let $\phi : S \times T \mapsto T \times S$ be a function defined by $\phi(x,y) = (y,x)$.
We can prove that ϕ is both 1-1 and onto.

We prove ϕ is injective.
Let $(a,b),(c,d) \in S \times T$.
Suppose $\phi(a,b) = \phi(c,d)$.
Then $(b,a) = (d,c)$.
Hence $b = d$ and $a = c$ so $(a,b) = (c,d)$.
Therefore $\phi(a,b) = \phi(c,d) \to (a,b) = (c,d)$ so ϕ is injective.

We prove ϕ is surjective.
Let $(a,b) \in T \times S$.
Then $a \in T$ and $b \in S$ so $(b,a) \in S \times T$.
Observe that $\phi(b,a) = (a,b)$.
Hence ϕ is surjective.
Since ϕ is injective and surjective then ϕ is bijective.
Therefore ϕ is a one to one correspondence from $S \times T$ to $T \times S$. $\square$

Exercise 97. Let $A = \{a,b,c\}$.
Let $f : A \to \mathbb{Z}$ be the function defined by $f(a) = 1$ and $f(b) = 5$ and $f(c) = 5$.
1. The image of f is the set $\{1,5\}$.

2. The pre-image of 5 is the set $\{b, c\}$.
3. The function f is not one to one.

Proof. The image of f is the set $\{f(x) \in \mathbb{Z} : x \in A\} = \{1, 5\}$.

The pre-image of 5 is the set $\{x \in A : f(x) = 5\} = \{b, c\}$.

Since $b \neq c$ and $f(b) = 5 = f(c)$, then f is not one to one. $\square$

Exercise 98. Let $f : S \to T$ be a surjective function.

Let $\{P_a\}_{a \in I}$ be a partition of T.

Then $\{f^{-1}(P_a) : a \in I\}$ is a partition of S.

Proof. Let $T' = \{P_a\}_{a \in I}$ for some index set I.

Then T' is a partition of T.

Thus, T' is a collection of nonempty subsets of T such that $T = \cup(P_a)_{a \in I}$ and if $P_a \neq P_b$ for $a, b \in I$, then $P_a \cap P_b = \emptyset$.

Since T' is a partition of T, then there exists a subset of T in T'.

Hence, there exists $a \in I$ such that $P_a \in T'$.

Since $P_a \in T'$, then $P_a \subset T$ and $P_a \neq \emptyset$.

Since P_a is not empty, then there exists $y \in P_a$.

Since $P_a \subset T$, then $y \in T$.

Since f is surjective, then there exists $x \in S$ such that $f(x) = y$.

Hence, $x \in S$ and $f(x) \in P_a$, so $x \in f^{-1}(P_a)$.

Therefore, $f^{-1}(P_a) \neq \emptyset$.

Let $S' = \{f^{-1}(P_a) : a \in I\}$.

Then S' is a collection of nonempty sets $f^{-1}(P_a)$ for some $a \in I$.

We prove S' is a partition of S.

We first prove $\cup f^{-1}(P_a) = S$.

Let $x \in \cup f^{-1}(P_a)$.

Then there exists $a \in I$ such that $x \in f^{-1}(P_a)$.

Thus, $x \in S$.

Hence, $x \in \cup f^{-1}(P_a)$ implies $x \in S$, so $\cup f^{-1}(P_a) \subset S$.

Let $y \in S$.

Then $f(y) \in T$.

Since T' is a partition of T, then every element of T is contained in some set in T'.

In particular, $f(y)$ is contained in some set in T'.

Hence, there exists $a \in I$ such that $f(y) \in P_a$ and $P_a \in T'$.

Since $y \in S$ and $f(y) \in P_a$, then $y \in f^{-1}(P_a)$.

Thus, there exists $a \in I$ such that $y \in f^{-1}(P_a)$, so $y \in \cup f^{-1}(P_a)$.

Therefore, $y \in S$ implies $y \in \cup f^{-1}(P_a)$, so $S \subset \cup f^{-1}(P_a)$.

Since $\cup f^{-1}(P_a) \subset S$ and $S \subset \cup f^{-1}(P_a)$, then $S = \cup f^{-1}(P_a)$.

Let $a, b \in I$ such that $a \neq b$.

Since T' is a partition of T and $P_a, P_b \in T'$, then $P_a \neq P_b$.

Hence, $P_a \cap P_b = \emptyset$.

Since $P_a \neq P_b$, then $f^{-1}P_a \neq f^{-1}P_b$.

Observe that

$$\begin{aligned} f^{-1}(P_a) \cap f^{-1}(P_b) &= f^{-1}(P_a \cap P_b) \\ &= f^{-1}(\emptyset) \\ &= \emptyset. \end{aligned}$$

Therefore, if $f^{-1}P_a$ and $f^{-1}P_b$ are distinct, then $f^{-1}P_a$ and $f^{-1}P_b$ are disjoint.
Hence, S' is a partition of S. □

Exercise 99. If $f : A \to B$ is a bijective function, then $f^{-1} : B \to A$ is a bijective function.

Proof. Let $f : A \to B$ be a bijective function.
Since f is bijective, then the inverse function $f^{-1} : B \to A$ exists.
Hence, $f \circ f^{-1} = f^{-1} \circ f = id$.

We prove f^{-1} is injective.
Let $a, b \in B$ such that $f^{-1}(a) = f^{-1}(b)$.
Since $f \circ f^{-1} = id$, then $(f \circ f^{-1})(x) = x$ for all $x \in B$.
In particular, $(f \circ f^{-1})(a) = a$.
Thus,

$$\begin{aligned} a &= (f \circ f^{-1})(a) \\ &= f(f^{-1}(a)) \\ &= f(f^{-1}(b)) \\ &= (f \circ f^{-1})(b) \\ &= id(b) \\ &= b. \end{aligned}$$

Hence, $f^{-1}(a) = f^{-1}(b)$ implies $a = b$, so f^{-1} is injective.

We prove f^{-1} is surjective.
Let $a \in A$.
Since f is a function, then there exists $b \in B$ such that $f(a) = b$.
Observe that

$$\begin{aligned} f^{-1}(b) &= f^{-1}(f(a)) \\ &= (f^{-1} \circ f)(a) \\ &= id(a) \\ &= a. \end{aligned}$$

Therefore, f^{-1} is surjective.
Hence, f^{-1} is bijective. □

Exercise 100. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 + x$.
Then f is injective.

Proof. Let $a, b \in \mathbb{R}$ such that $f(a) = f(b)$.

Then $a^3 + a = b^3 + b$, so $a^3 - b^3 + a - b = 0$.

Hence, $(a-b)(a^2 + ab + b^2) + a - b = 0$, so $(a-b)(a^2 + ab + b^2 + 1) = 0$.

Thus, either $a - b = 0$ or $a^2 + ab + b^2 + 1 = 0$.

Suppose $a^2 + ab + b^2 + 1 = 0$.

Then $a^2 + ab + b^2 = -1$.

Since $a, b \in \mathbb{R}$, then either $a \neq 0$ or $a = 0$ and either $b \neq 0$ or $b = 0$.

Thus, either $a \neq 0$ and $b \neq 0$ or $a \neq 0$ and $b = 0$ or $a = 0$ and $b \neq 0$ or $a = 0$ and $b = 0$.

Since $a \neq 0$ and $b \neq 0$ iff $a > 0$ or $a < 0$ and $b > 0$ or $b < 0$, then either $a > 0$ and $b > 0$ or $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$ or $a < 0$ and $b < 0$.

We consider these cases separately.

Case 1: Suppose $a > 0$ and $b > 0$.

Then $a^2 > 0$ and $b^2 > 0$ and $ab > 0$, so $a^2 + ab + b^2 > 0$.

Hence, $-1 > 0$, a contradiction.

Case 2: Suppose $a > 0$ and $b < 0$.

Since $a > 0$, then $a^2 > 0$, so $a^2 + 1 > 0$.

Hence, $f(a) = a^3 + a = a(a^2 + 1) > 0$.

Since $b < 0$, then $b^2 > 0$, so $b^2 + 1 > 0$.

Thus, $f(b) = b^3 + b = b(b^2 + 1) < 0$, so $f(b) < 0 < f(a)$.

Hence, $f(b) < f(a)$, so $f(b) \neq f(a)$.

Thus, we have $f(a) \neq f(b)$ and $f(a) = f(b)$, a contradiction.

Case 3: Suppose $a < 0$ and $b > 0$.

Since $a < 0$, then $a^2 > 0$, so $a^2 + 1 > 0$.

Hence, $f(a) = a^3 + a = a(a^2 + 1) < 0$.

Since $b > 0$, then $b^2 > 0$, so $b^2 + 1 > 0$.

Thus, $f(b) = b^3 + b = b(b^2 + 1) > 0$, so $f(a) < 0 < f(b)$.

Hence, $f(a) < f(b)$, so $f(a) \neq f(b)$.

Thus, we have $f(a) \neq f(b)$ and $f(a) = f(b)$, a contradiction.

Case 4: Suppose $a < 0$ and $b < 0$.

Then $a^2 > 0$ and $b^2 > 0$ and $ab > 0$, so $a^2 + ab + b^2 > 0$.

Hence, $-1 > 0$, a contradiction.

Case 5: Suppose $a \neq 0$ and $b = 0$.

Then $-1 = a^2 + ab + b^2 = a^2 + a0 + 0^2 = a^2 + 0 + 0 = a^2 > 0$, so $-1 > 0$, a contradiction.

Case 6: Suppose $a = 0$ and $b \neq 0$.

Then $-1 = a^2 + ab + b^2 = 0^2 + 0b + b^2 = 0 + 0 + b^2 = b^2 > 0$, so $-1 > 0$, a contradiction.

Case 7: Suppose $a = 0$ and $b = 0$.

Then $-1 = a^2 + ab + b^2 = 0^2 + 00 + 0^2 = 0 + 0 + 0 = 0$, so $-1 = 0$, a contradiction.

Since a contradiction is reached in all cases, then $a^2 + ab + b^2 + 1 \neq 0$.

Therefore, $a - b = 0$, so $a = b$, as desired. $\square$

Exercise 101. Let $f : A \to B$ and $g : B \to C$ be maps.

If $g \circ f$ is injective and f is surjective, then g is injective.

Proof. Let $g \circ f : A \to C$ be the map such that $g \circ f$ is injective and f is surjective.

To prove g is injective, let $b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$.

Since f is surjective and $b_1 \in B$, then there exists $a_1 \in A$ such that $f(a_1) = b_1$.

Since f is surjective and $b_2 \in B$, then there exists $a_2 \in A$ such that $f(a_2) = b_2$.

Observe that

$$\begin{aligned}(g \circ f)(a_1) &= g(f(a_1)) \\ &= g(b_1) \\ &= g(b_2) \\ &= g(f(a_2)) \\ &= (g \circ f)(a_2).\end{aligned}$$

Since $g \circ f$ is injective and $(g \circ f)(a_1) = (g \circ f)(a_2)$, then $a_1 = a_2$.

Since f is a function, then this implies $f(a_1) = f(a_2)$.

Thus, $b_1 = f(a_1) = f(a_2) = b_2$, so $b_1 = b_2$.

Therefore, g is injective. ☐

Exercise 102. Let $f : A \to B$ and $g : B \to C$ be maps

If $g \circ f$ is surjective and g is injective, then f is surjective.

Proof. Let $g \circ f : A \to C$ be the map such that $g \circ f$ is surjective and g is injective.

To prove f is surjective, let $b \in B$ be arbitrary.

Since $g \circ f$ is surjective and $g(b) \in C$, then there exists $a \in A$ such that $(g \circ f)(a) = g(b)$.

Thus, $g(b) = (g \circ f)(a) = g(f(a))$.

Since g is injective and $g(f(a)) = g(b)$, then $f(a) = b$.

Therefore, there exists $a \in A$ such that $f(a) = b$, so f is surjective. ☐

Exercise 103. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by $f(n) = 2n - 1$ for each $n \in \mathbb{N}$.

Analyze f.

Proof. The domain of f is the set $\mathbb{N}$.

The range of f is $f(\mathbb{N}) = \{2n-1 : n \in \mathbb{N}\}$, the set of all odd natural numbers.

We prove f is not surjective.

Suppose there exists $a \in \mathbb{N}$ such that $f(a) = 2$.

Then $a \in \mathbb{N}$ and $2a - 1 = 2$, so $2a = 3$.

This implies 3 is even which contradicts the fact that 3 is odd.

Therefore, there is no $a \in \mathbb{N}$ such that $f(a) = 2$, so $f(a) \neq 2$ for each $a \in \mathbb{N}$.

Since $2 \in \mathbb{N}$ and $f(a) \neq 2$ for each $a \in \mathbb{N}$, then f is not surjective.

We prove f is injective.
Let $a, b \in \mathbb{N}$ such that $f(a) = f(b)$.
Then $2a - 1 = 2b - 1$, so $2a = 2b$.
Hence, $a = b$, so f is injective.

Since f is injective, then the inverse relation f^{-1} is a function and $domf^{-1} = rngf = \{2n - 1 : n \in \mathbb{N}\}$.
Since f^{-1} is the inverse of f, then $f(x) = y$ iff $f^{-1}(y) = x$.
Hence, $2x - 1 = y$ iff $f^{-1}(y) = x$, so $x = \frac{y+1}{2}$ iff $f^{-1}(y) = x$.
Let $(x, y) \in f$.
Then $f(x) = y$, so $f^{-1}(y) = x$.
Thus, $f^{-1}(y) = x = \frac{y+1}{2}$.
Therefore, $f^{-1}(n) = \frac{n+1}{2}$ for each odd natural number n. $\square$

Exercise 104. Define a function $f : \mathbb{N} \to \mathbb{N}$ that is surjective but not injective.

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

.

We prove f is surjective.
Let $b \in \mathbb{N}$.
Then either b is even or b is odd.
If b is even, then $f(b) = \frac{b}{2}$.
If b is odd, then $f(b) = \frac{b+1}{2}$.
Therefore, f is surjective.

We prove f is not injective.
Observe that $f(1) = \frac{1+1}{2} = 1 = \frac{2}{2} = f(2)$.
Since $1 \in \mathbb{N}$ and $2 \in \mathbb{N}$ and $1 \neq 2$ and $f(1) = f(2)$, then f is not injective. $\square$

Exercise 105. Let f be a function defined on $\mathbb{R}$ by $f(x) = \frac{x+1}{x-1}$.
Analyze f.
That is, compute the domain of f, the range of f and inverse of f and $f \circ f^{-1}$ and $f^{-1} \circ f$.

Solution. Since division by zero is not allowed, then the domain of f is the set $domf = \mathbb{R} - \{1\}$ and the range of f is the set $rngf = \mathbb{R} - \{1\}$.

We prove f is injective.

Let $a, b \in domf$ such that $f(a) = f(b)$.

Then $a, b \in \mathbb{R}$ and $a \neq 1$ and $b \neq 1$ and $\frac{a+1}{a-1} = \frac{b+1}{b-1}$.

Thus, $(a+1)(b-1) = (a-1)(b+1)$, so $ab - a + b - 1 = ab + a - b - 1$.

Hence, $-a + b = a - b$, so $2b = 2a$.

Therefore, $b = a$, so $a = b$.

Consequently, f is injective, so the inverse relation f^{-1} is a function.

The domain of f^{-1} is the set $domf^{-1} = rngf = \mathbb{R} - \{1\}$ and the range of f^{-1} is the set $rngf^{-1} = domf = \mathbb{R} - \{1\}$.

We determine the formula for f^{-1}.

Since f and f^{-1} are inverses, then $f(x) = y$ iff $f^{-1}(y) = x$.

Let $x \in domf$ and $y = f(x)$.

Then $x \in \mathbb{R}$ and $x \neq 1$ and $y \in \mathbb{R}$ and $y \neq 1$ and $y = f(x) = \frac{x+1}{x-1}$, so $y(x-1) = x + 1$.

Hence, $yx - y = x + 1$, so $yx - x = y + 1$.

Thus, $x(y-1) = y + 1$, so $x = \frac{y+1}{y-1}$.

Since $f(x) = y$ and $f(x) = y$ iff $f^{-1}(y) = x$, then $f^{-1}(y) = x = \frac{y+1}{y-1}$.

Therefore, $f^{-1}(x) = \frac{x+1}{x-1}$.

Let $x \in dom(f \circ f^{-1})$.

Then $(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(\frac{x+1}{x-1}) = x$.

Therefore, $f \circ f^{-1}$ is the identity function $I(x) = x$ on $\mathbb{R} - \{1\}$.

Similarly, $f^{-1} \circ f$ is the identity function $I(x) = x$ on $\mathbb{R} - \{1\}$. $\square$

Proposition 106. *When is function composition commutative?*

Solution. We need to determine necessary and sufficient conditions to guarantee $g \circ f = f \circ g$.

Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions.

Suppose $g \circ f = f \circ g$.

Then $g \circ f : A \mapsto C$ and $f \circ g$ are functions.

By definition of equal functions we have domain$(g \circ f)$ = domain $(f \circ g)$ and codomain$(g \circ f)$ = codomain $(f \circ g)$ and $(g \circ f)(x) = (f \circ g)(x)$ for each $x \in A$.

Since domain$(g \circ f)$ = domain $(f \circ g)$ then domain of $f \circ g$ is A.

Since codomain$(g \circ f)$ = codomain $(f \circ g)$ then codomain of $f \circ g$ is C.

By definition of function composition for $f \circ g$, the domain of f is the codomain of g.

Thus, $A = C$.

Hence, the codomain of g is A and the codomain of f is A.

By definition of function composition for $g \circ f$, the domain of g is the codomain of f.

Thus, $B = A$.

Since $A = B = C$ then function f is $f : A \mapsto A$ and function g is $g : A \mapsto A$.

Hence, function $g \circ f$ is $g \circ f : A \mapsto A$ and function $f \circ g$ is $f \circ g : A \mapsto A$.

Assume f and g are inverse functions.

Then $(g \circ f)(x) = (f \circ g)(x) = I(x) = x$ for all $x \in A$ by definition of inverse function.

Thus, $g \circ f = f \circ g$.

Moreover, every invertible function is bijective, so f and g are bijections.

Since $f : A \mapsto A$ and $g : A \mapsto A$ are bijective functions, then f and g are permutation maps on A, by definition of permutation map. Hence, if f and g are permutation maps such that $g = f^{-1}$ then $g \circ f = f \circ g$. $\square$

Exercise 107. Let $f : \mathbb{Z} \mapsto \mathbb{Q}$ be defined by $f(n) = n/1$.

What can we deduce about f? Is f a function? If so, is f one to one or onto?

Solution. We know f is a binary relation from $\mathbb{Z}$ to $\mathbb{Q}$ since $f \subseteq \mathbb{Z} \times \mathbb{Q}$.

For each $n \in \mathbb{Z}$, $f(n)$ exists, so each element in the domain has at least one image.

Let $a \in \mathbb{Z}$ such that $f(a) = b_1 \wedge f(a) = b_2$ with $b_1, b_2 \in \mathbb{Q}$.

Then $a/1 = b_1$ and $a/1 = b_2$, so $b_1 = b_2$.

Hence, each element in the domain has at most one image.

Since each element in the domain has at least one image and each element in the domain has at most one image, then each element in the domain has exactly one image.

Thus, f is a function, by definition of function.

Let $a_1, a_2 \in \mathbb{Z}$ such that $f(a_1) = f(a_2)$.

Then $a_1/1 = a_2/1$, so $a_1 = a_2$.

Hence $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

Therefore, f is one to one(injective).

There is no integer k such that $f(k) = 1/2$, so f is not onto $\mathbb{Q}$.

Thus, f is not surjective. $\square$

Exercise 108. Let f be the function defined on $\mathbb{R}$ by $f(x) = \frac{x}{x+2}$.

Analyze f.

Solution. Since $x \in \mathbb{R}$, then $f(x) \in \mathbb{R}$ iff $\frac{x}{x+2} \in \mathbb{R}$ iff $x + 2 \neq 0$ iff $x \neq -2$.

Since f is a function, then $f(x) \in \mathbb{R}$ iff $x \neq -2$.

Thus, if $x \neq -2$, then $f(x) \in \mathbb{R}$, so for every $x \in \mathbb{R}$ with $x \neq -2$, $f(x) \in \mathbb{R}$.

Therefore, the domain of f is the set $dom f = \mathbb{R} - \{-2\}$.

We prove the range of f is the set $\mathbb{R} - \{1\}$.

Let $y \in rng f$.

Then $y = f(x) = \frac{x}{x+2}$ and $x \in \mathbb{R} - \{-2\}$, so $x \in \mathbb{R}$ and $x \neq -2$.

Since $x \neq -2$, then $x + 2 \neq 0$, so $\frac{x}{x+2} = y \in \mathbb{R}$.

Suppose $y = 1$.

Then $1 = \frac{x}{x+2}$.

Since $x + 2 \neq 0$, then $x + 2 = x$, so $2 = 0$, a contradiction.

Therefore, $y \neq 1$.

Since $y \in \mathbb{R}$ and $y \neq 1$, then $y \in \mathbb{R} - \{1\}$.

Hence, if $y \in rng f$, then $y \in \mathbb{R} - \{1\}$, so $rng f \subset \mathbb{R} - \{1\}$.

Suppose $t \in \mathbb{R} - \{1\}$.
Then $t \in \mathbb{R}$ and $t \neq 1$.
Since $t \neq 1$, then $t - 1 \neq 0$, so $\frac{-2t}{t-1} \in \mathbb{R}$.
Let $x = \frac{-2t}{t-1}$.
Then $x \in \mathbb{R}$.
Since $t - 1 \neq 0$, then $x(t-1) = -2t$, so $xt - x = -2t$.
Hence, $xt + 2t = x$, so $t(x+2) = x$.
Suppose $x + 2 = 0$.
Then $x = -2$ and $x = t \cdot 0 = 0$, so $x = -2$ and $x = 0$, a contradiction.
Thus, $x + 2 \neq 0$.
Since $t(x+2) = x$ and $x + 2 \neq 0$, then $t = \frac{x}{x+2}$.
Since $x + 2 \neq 0$, then $x \neq -2$.
Since $x \in \mathbb{R}$ and $x \neq -2$, then $x \in \mathbb{R} - \{-2\}$.
Therefore, there exists $x \in \mathbb{R} - \{-2\}$ such that $t = \frac{x}{x+2}$, so $t \in rngf$.
Hence, if $t \in \mathbb{R} - \{1\}$, then $t \in rngf$, so $\mathbb{R} - \{1\} \subset rngf$.
Since $rngf \subset \mathbb{R} - \{1\}$ and $\mathbb{R} - \{1\} \subset rngf$, then $rngf = \mathbb{R} - \{1\}$.
Therefore, $rngf = \mathbb{R} - \{1\}$.

We prove f is injective.
Let $a, b \in \mathbb{R} - \{-2\}$ such that $f(a) = f(b)$.
Then $a \in \mathbb{R}$ and $a \neq -2$ and $b \in \mathbb{R}$ and $b \neq -2$ and $\frac{a}{a+2} = \frac{b}{b+2}$.
Since $a \neq -2$, then $a + 2 \neq 0$, so $a = \frac{(a+2)b}{b+2}$.
Since $b \neq -2$, then $b + 2 \neq 0$, so $a(b+2) = (a+2)b$.
Hence, $ab + 2a = ab + 2b$, so $2a = 2b$.
Therefore, $a = b$, so f is injective.

Since f is injective, then the inverse relation f^{-1} is a function and $domf^{-1} = rngf = \mathbb{R} - \{1\}$.
Since f^{-1} is the inverse of f, then $f(x) = y$ iff $f^{-1}(y) = x$.
Hence, $\frac{x}{x+2} = y$ iff $f^{-1}(y) = x$.
Let $x \in domf$.
Then $x \in \mathbb{R}$ and $x \neq -2$ and $f(x) \in rngf$.
Let $y = f(x)$.
Then $y \in rngf$, so $y = \frac{x}{x+2}$ and $y \neq 1$.
Since $x \neq -2$, then $x + 2 \neq 0$.
Since $y = \frac{x}{x+2}$, then $(x+2)y = x$, so $xy + 2y = x$.
Hence, $2y = x - xy = x(1-y)$.
Since $y \neq 1$, then $1 \neq y$, so $1 - y \neq 0$.
Thus, $\frac{2y}{1-y} = x$.
Since $\frac{x}{x+2} = y$, then $f^{-1}(y) = x = \frac{2y}{1-y}$.
Thus, $f^{-1}(y) = \frac{2y}{1-y}$ and $y \neq 1$.
Therefore, $f^{-1}(x) = \frac{2x}{1-x}$ for each $x \neq 1$. $\square$

Exercise 109. Let π be a permutation defined on the set $\{1, 2, 3\}$ by

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

.

What is the inverse permutation π^{-1}?

Solution. The inverse permutation π^{-1} is given by

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

.

$\square$

Exercise 110. If $f : A \to B$ is an injective map, then $f^{-1} : rngf \to A$ is an injective map.

Proof. Suppose $f : A \to B$ is an injective map.
Then f is an injective function.
Hence, the inverse relation f^{-1} is a function.
To prove $f^{-1} : rngf \to A$ is a map, we must prove $domf^{-1} = rngf$ and $rngf^{-1} \subset A$.

We first prove $domf^{-1} = rngf$.
Either $f = \emptyset$ or $f \neq \emptyset$.
Case 1: Suppose $f = \emptyset$.
Then f is the empty relation, so $domf = \emptyset = rngf$.
Thus, f^{-1} is empty, so $f^{-1} = \emptyset$.
Hence, $domf^{-1} = \emptyset = rngf$.
Case 2: Suppose $f \neq \emptyset$.
Then f is a nonempty relation, so $domf^{-1} = rngf$.
Therefore, in either case, $domf^{-1} = rngf$.

We next prove $rngf^{-1} \subset A$.
Suppose $b \in rngf^{-1}$.
Then there exists $a \in domf^{-1}$ such that $(a, b) \in f^{-1}$.
Since f^{-1} is the inverse of f, then $(b, a) \in f$, so $b \in domf$.
Since $domf = A$, then $b \in A$.
Hence, $rngf^{-1} \subset A$.

Since f^{-1} is a function and $domf^{-1} = rngf$ and $rngf^{-1} \subset A$, then $f^{-1} : rngf \to A$ is a map.

To prove f^{-1} is injective, let $b_1, b_2 \in dom f^{-1}$ such that $f^{-1}(b_1) = f^{-1}(b_2)$.

Since $b_1, b_2 \in dom f^{-1}$ and $dom f^{-1} = rng f$, then $b_1, b_2 \in rng f$, so there exist $a_1, a_2 \in dom f$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$.

Since f^{-1} is the inverse of f, then $f^{-1}(b_1) = a_1$ and $f^{-1}(b_2) = a_2$.

Hence, $a_1 = f^{-1}(b_1) = f^{-1}(b_2) = a_2$, so $a_1 = a_2$.

Since f is a function and $a_1 = a_2$ and $f(a_1) = b_1$ and $f(a_2) = b_2$, then $b_1 = b_2$.

Therefore, f^{-1} is injective. ☐

Exercise 111. Let A be a set and $X \subset A$.

Then the inclusion map $i : X \to A$ is injective.

Proof. Let $a, b \in X$ such that $i(a) = i(b)$.

Since $i(x) = x$ for all $x \in X$ and $a, b \in X$, then $i(a) = a$ and $i(b) = b$.

Therefore, $a = b$, so i is injective. ☐

Exercise 112. Restriction of an injective map is injective.

Let $f : A \to B$ be a map and $S \subset A$.

If f is injective, then the restriction $f|_S : S \to B$ is injective.

Proof. Suppose f is injective.

Let $a, b \in S$ such that $f|_S(a) = f|_S(b)$.

Since $a, b \in S$, then $f|_S(a) = f(a)$ and $f|_S(b) = f(b)$.

Thus, $f(a) = f|_S(a) = f|_S(b) = f(b)$.

Since f is injective and $f(a) = f(b)$, then $a = b$, so $f|_S$ is injective. ☐

Exercise 113. Provide an example of two functions f and g such that $f \circ g = g \circ f$.

Solution. Let $f = \{(1,3),(4,7),(9,8)\}$ and $g = \{(1,9),(3,8),(4,1),(7,3)\}$.

Then f and g are functions and $f \circ g = \{(1,8),(4,3)\} = g \circ f$. ☐

Exercise 114. Composition of linear functions is a linear function

Let $M, B, N, C \in \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a map defined by $f(x) = Mx + B$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a map defined by $g(x) = Nx + C$.

I. Then $g \circ f$ is a linear map with slope MN and $f \circ g$ is a linear map with slope MN.

II. If $B(N-1) = C(M-1)$, then $f \circ g = g \circ f$.

III. Provide an example of two linear functions f and g such that $f \circ g = g \circ f$.

Proof. We prove I.

Since $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are maps, then $g \circ f : \mathbb{R} \to \mathbb{R}$ is a map and $(g \circ f)(x) = g(f(x))$ for all $x \in \mathbb{R}$ and $f \circ g : \mathbb{R} \to \mathbb{R}$ is a map and $(f \circ g)(x) = f(g(x))$ for all $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$.

Then $(g \circ f)(x) = g(f(x)) = g(Mx+B) = N(Mx+B)+C = NMx+NB+C = MNx + (NB + C)$.

Hence, $(g \circ f)(x) = MNx + (NB + C)$, so $g \circ f$ is a linear map with slope MN.

Let $x \in \mathbb{R}$.

Then $(f \circ g)(x) = f(g(x)) = f(Nx+C) = M(Nx+C)+B = MNx+MC+B = MNx + (MC+B)$.

Hence, $(f \circ g)(x) = MNx + (MC+B)$, so $f \circ g$ is a linear map with slope MN. $\square$

Proof. We prove II.

Suppose $B(N-1) = C(M-1)$.

Then $BN - B = CM - C$, so $BN + C = CM + B$.

Thus, $NB + C = MC + B$.

The domain of $g \circ f$ is $\mathbb{R}$ which is the same as the domain of $f \circ g$ and the codomain of $g \circ f$ is $\mathbb{R}$ which is the codomain of $f \circ g$.

Let $x \in \mathbb{R}$.

Then

$$\begin{aligned} (f \circ g)(x) &= MNx + (MC+B) \\ &= MNx + (NB+C) \\ &= (g \circ f)(x). \end{aligned}$$

Thus, $(f \circ g)(x) = (g \circ f)(x)$ for all $x \in \mathbb{R}$, so $f \circ g = g \circ f$. $\square$

Solution. III.

Let $f : \mathbb{R} \to \mathbb{R}$ by given by $f(x) = 3x + 5$.

Let $g : \mathbb{R} \to \mathbb{R}$ by given by $g(x) = 7x + 15$.

Let $x \in \mathbb{R}$.

Then $(g \circ f)(x) = g(f(x)) = g(3x+5) = 7(3x+5) + 15 = 21x + 35 + 15 = 21x+50$ and $(f \circ g)(x) = f(g(x)) = f(7x+15) = 3(7x+15)+5 = 21x+45+5 = 21x + 50$.

Thus, $(g \circ f)(x) = 21x + 50 = (f \circ g)(x)$. $\square$

Exercise 115. Let f and g be functions such that $f \neq \emptyset$ and $g \neq \emptyset$.

If $rngf \cap domg = \emptyset$, then $g \circ f = \emptyset$.

Proof. Suppose $rngf \cap domg = \emptyset$.

Since f and g are functions, then $g \circ f$ is a function.

We prove $g \circ f = \emptyset$ by contradiction.

Suppose $g \circ f \neq \emptyset$.

Then there exists an ordered pair $(a, b) \in g \circ f$.

Since $f \neq \emptyset$ and $g \neq \emptyset$, then there exists c such that $(a, c) \in f$ and $(c, b) \in g$.

Since $(a, c) \in f$, then $c \in rngf$.

Since $(c, b) \in g$, then $c \in domg$.

Thus, $c \in rngf$ and $c \in domg$, so $c \in rngf \cap domg$.

Hence, $rngf \cap domg \neq \emptyset$.

Thus, we have $rngf \cap domg = \emptyset$ and $rngf \cap domg \neq \emptyset$, a contradiction.

Therefore, $g \circ f = \emptyset$. $\square$

Exercise 116. Let f and g be functions such that $rngf \subset domg$.
Let $E \subset domf$.
Then $(g \circ f)(E) = g(f(E))$.

Proof. We prove $g(f(E)) \subset (g \circ f)(E)$.
Let $b \in g(f(E))$.
Then $b = g(a)$ for some $a \in f(E)$.
Since $a \in f(E)$, then $a = f(x)$ for some $x \in E$.
Thus, $b = g(a) = g(f(x)) = (g \circ f)(x)$ for some $x \in E$, so $b \in (g \circ f)(E)$.
Hence, $g(f(E)) \subset (g \circ f)(E)$.

We prove $(g \circ f)(E) \subset g(f(E))$.
Let $x \in (g \circ f)(E)$.
Then $x = (g \circ f)(y)$ for some $y \in E$.
Since $y \in E$ and $E \subset domf$, then $y \in domf$, so $f(y) \in f(E)$.
Since $E \subset domf$, then $f(E) \subset rngf$, so $f(y) \in rngf$.
Since $rngf \subset domg$, then $f(y) \in domg$.
Thus, $g(f(y)) \in g(f(E))$, so $(g \circ f)(y) \in g(f(E))$.
Hence, $x \in g(f(E))$, so $(g \circ f)(E) \subset g(f(E))$.
Since $g(f(E)) \subset (g \circ f)(E)$ and $(g \circ f)(E) \subset g(f(E))$, then $(g \circ f)(E) = g(f(E))$. ∎

Exercise 117. Let $f : A \to B$ be a map.
Let I_A be the identity map on A and I_B be the identity map on B.
1. If f is injective, then $f^{-1} \circ f = I_A$.
2. Let $X \subset A$.
Let i be the inclusion map of X into A.
Let $f|_X$ be the restriction of f to X.
Then $f \circ i_X = f|_X$.

Proof. We prove 1.
Suppose f is injective.
Then the inverse relation f^{-1} is a function.
Since f is a function, then $f^{-1} \circ f$ is a function and $domf^{-1} \circ f = \{x \in domf : f(x) \in domf^{-1}\}$ and $(f^{-1} \circ f)(x) = f^{-1}(f(x))$ for all $x \in domf^{-1} \circ f$.
We prove $domf^{-1} \circ f = A$.
Since $domf = A$, then $domf^{-1} \circ f = \{x \in A : f(x) \in domf^{-1}\}$, so $domf^{-1} \circ f \subset A$.
Let $a \in A$.
Then $a \in domf$, so $f(a) \in rngf$.
Since $rngf = domf^{-1}$, then $f(a) \in domf^{-1}$.
Since $a \in domf$ and $f(a) \in domf^{-1}$, then $a \in domf^{-1} \circ f$, so $A \subset domf^{-1} \circ f$.
Thus, $domf^{-1} \circ f \subset A$ and $A \subset domf^{-1} \circ f$, so $domf^{-1} \circ f = A$.
Hence, $(f^{-1} \circ f)(x) = f^{-1}(f(x))$ for all $x \in A$.
Let $x \in A$.
Let $b = (f^{-1} \circ f)(x) = f^{-1}(f(x))$.

Since f^{-1} is the inverse of f, then $f^{-1}(f(x)) = b$ iff $f(b) = f(x)$.
Thus, $f(b) = f(x)$.
Since f is injective, then $b = x$.
Hence, $I_A(x) = x = b = (f^{-1} \circ f)(x)$, so $I_A(x) = (f^{-1} \circ f)(x)$ for all $x \in A$.
Therefore, $I_A = f^{-1} \circ f$. □

Proof. We prove 2.
Since i is the inclusion map of X into A, then $i : X \to A$ is a map defined by $i(x) = x$ for all $x \in X$.
Since $f|_X$ is the restriction of f to X, then $f|_X : X \to B$ is a map defined by $f|_X(x) = f(x)$ for all $x \in X$.
Since $i : X \to A$ is a map and $f : A \to B$ is a map, then $f \circ i : X \to B$ is a map and $(f \circ i)(x) = f(i(x))$ for all $x \in X$.
We prove the maps $f \circ i$ and $f|_X$ are equal.
Observe that $dom f \circ i = X = dom f|_X$ and the codomain of $f \circ i$ is B which is the codomain of $f|_X$.
Let $x \in X$.
Then $(f \circ i)(x) = f(i(x)) = f(x) = f|_X(x)$, so $(f \circ i)(x) = f|_X(x)$ for all $x \in X$.
Therefore, $f \circ i = f|_X$. □

Exercise 118. i. Give an example of maps $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is injective, but g is not injective.
ii. Give an example of maps $f : A \to B$ and $g : B \to C$ such that f is injective, but $g \circ f$ is not injective.

Solution. Let $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 8, 9\}$ and $C = \{5, 6, 7, 10\}$.
i. Here is an example such that $g \circ f$ is injective, but g is not injective.
Let $f : A \to B$ be a map given by $f = \{(1, 3), (2, 5), (3, 4), (7, 8)\}$.
Let $g : B \to C$ be a map given by $g = \{(3, 6), (4, 7), (5, 10), (8, 5), (9, 6)\}$.
Since $g(3) = 6$ and $g(9) = 6$, then g is not injective.
Observe that $g \circ f = \{(1, 6), (2, 10), (3, 7), (7, 5)\}$ is injective.
ii. Here is an example such that f is injective, but $g \circ f$ is not injective.
Let $f : A \to B$ be a map given by $f = \{(1, 3), (2, 5), (3, 4), (7, 8)\}$.
Then f is injective.
Let $g : B \to C$ be a map given by $g = \{(3, 6), (4, 7), (5, 10), (8, 6), (9, 2)\}$.
Then $g \circ f = \{(1, 6), (2, 10), (3, 7), (7, 6)\}$ is not injective since $(g \circ f)(1) = 6$ and $(g \circ f)(7) = 6$. □

Exercise 119. Let $f : A \to B$ and $g : C \to D$ be maps.
If $rng f \subset C$, then $dom(g \circ f) = A$ and $rng(g \circ f) \subset D$.

Proof. Suppose $rng f \subset C$.
Since $f : A \to B$ and $g : C \to D$ are maps, then f and g are functions, so $g \circ f$ is a function and $dom g \circ f = \{x \in dom f : f(x) \in dom g\}$ and $(g \circ f)(x) = g(f(x))$ for all $x \in dom g \circ f$.
Since $dom f = A$ and $dom g = C$, then $dom g \circ f = \{x \in A : f(x) \in C\}$, so $dom g \circ f \subset A$.

Let $a \in A$.

Then $a \in domf$, so $f(a) \in rngf$.

Since $rngf \subset C$, then $f(a) \in C$.

Since $a \in A$ and $f(a) \in C$, then $a \in domg \circ f$, so $A \subset domg \circ f$.

Since $domg \circ f \subset A$ and $A \subset domg \circ f$, then $dom(g \circ f) = A$, so $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

Let $y \in rng(g \circ f)$.

Then there exists $x \in domg \circ f$ such that $(g \circ f)(x) = y$.

Thus, there exists $x \in A$ such that $g(f(x)) = y$.

Since $x \in A$, then $f(x) \in rngf$.

Since $rngf \subset C$, then $f(x) \in C$.

Since $g : C \to D$ is a map, then $g(f(x)) \in D$, so $y \in D$.

Therefore, $rng(g \circ f) \subset D$, as desired. $\square$

Exercise 120. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be maps.

Let $f \vee g$ be a map from $\mathbb{R}$ into $\mathbb{R}$ defined by $(f \vee g)(x) = \max\{f(x), g(x)\}$ for all $x \in \mathbb{R}$.

Let $f \wedge g$ be a map from $\mathbb{R}$ into $\mathbb{R}$ defined by $(f \wedge g)(x) = \min\{f(x), g(x)\}$ for all $x \in \mathbb{R}$.

Provide an example to show that f and g can be one to one, yet $f \vee g$ is not one to one.

Provide an example to show that f and g can be one to one, yet $f \wedge g$ is not one to one.

Solution. Let $f(x) = e^x$ and $g(x) = e^{-x}$ for all $x \in \mathbb{R}$.

Then f and g are one to one functions.

Since $e > 1$, then $1 > \frac{1}{e}$, so $e > 1 > \frac{1}{e}$.

Thus, $e > \frac{1}{e}$, so $\frac{1}{e} < e$.

Since $f(-1) = e^{-1} = \frac{1}{e} < e = e^1 = e^{-(-1)} = g(-1)$, then $f(-1) < g(-1)$, so $(f \vee g)(-1) = \max\{f(-1), g(-1)\} = g(-1) = e$.

Since $f(1) = e > \frac{1}{e} = e^{-1} = g(1)$, then $f(1) > g(1)$, so $(f \vee g)(1) = \max\{f(1), g(1)\} = f(1) = e$.

Thus, $(f \vee g)(-1) = e = (f \vee g)(1)$, so $f \vee g$ is not one to one.

Since $f(-1) = e^{-1} = \frac{1}{e} < e = e^1 = e^{-(-1)} = g(-1)$, then $f(-1) < g(-1)$, so $(f \wedge g)(-1) = \min\{f(-1), g(-1)\} = f(-1) = \frac{1}{e}$.

Since $f(1) = e > \frac{1}{e} = e^{-1} = g(1)$, then $f(1) > g(1)$, so $(f \wedge g)(1) = \min\{f(1), g(1)\} = g(1) = \frac{1}{e}$.

Thus, $(f \wedge g)(-1) = \frac{1}{e} = (f \wedge g)(1)$, so $f \wedge g$ is not one to one. $\square$

Exercise 121. Let $f : A \to B$ and $g : C \to D$ be maps.

a. If $A \cap C = \emptyset$, then $f \cup g : A \cup C \to B \cup D$ is a map.

b. If $A \cap C = B \cap D = \emptyset$ and f and g are injective, then $f \cup g$ is injective.

c. If $A \cap C = \emptyset$, then $(f \cup g)|_A = f$ and $(f \cup g)|_C = g$.

Proof. We prove a.

Suppose $A \cap C = \emptyset$.

To prove $f \cup g : A \cup C \to B \cup D$ is a map, we must prove $f \cup g$ is a function and $dom(f \cup g) = A \cup C$ and $rng(f \cup g) \subset B \cup D$.

Since $f : A \to B$ and $g : C \to D$ are maps, then f and g are functions, so f and g are relations.

Thus, $f \cup g$ is a relation.

To prove $f \cup g$ is a function, let $(a, b) \in f \cup g$ and $(a, b') \in f \cup g$.

Then either $(a, b) \in f$ or $(a, b) \in g$, and either $(a, b') \in f$ or $(a, b') \in g$.

Hence, either $(a, b) \in f$ and $(a, b') \in f$ or $(a, b) \in f$ and $(a, b') \in g$ or $(a, b) \in g$ and $(a, b') \in f$ or $(a, b) \in g$ and $(a, b') \in g$.

Suppose $(a, b) \in f$ and $(a, b') \in g$ or $(a, b) \in g$ and $(a, b') \in f$.

Then $a \in domf$ and $a \in domg$, so $a \in A$ and $a \in C$.

Hence, $a \in A \cap C$, so $A \cap C \neq \emptyset$.

This contradicts the assumption $A \cap C = \emptyset$.

Thus, it cannot be the case that $(a, b) \in f$ and $(a, b') \in g$ or $(a, b) \in g$ and $(a, b') \in f$.

Hence, either $(a, b) \in f$ and $(a, b') \in f$ or $(a, b) \in g$ and $(a, b') \in g$.

We consider these cases separately.

Case 1: Suppose $(a, b) \in f$ and $(a, b') \in f$.

Since f is a function, then $b = b'$.

Case 2: Suppose $(a, b) \in g$ and $(a, b') \in g$.

Since g is a function, then $b = b'$.

Thus, in all cases, $b = b'$, so $f \cup g$ is a function, as desired.

We prove $dom(f \cup g) = A \cup C$.

Let $x \in dom(f \cup g)$.

Then there exists y such that $(x, y) \in f \cup g$, so either $(x, y) \in f$ or $(x, y) \in g$.

Hence, either $x \in domf$ or $x \in domg$, so $x \in domf \cup domg$.

Thus, $x \in A \cup C$, so $dom(f \cup g) \subset A \cup C$.

Let $y \in A \cup C$.

Then either $y \in A$ or $y \in C$, so either $y \in domf$ or $y \in domg$.

We consider these cases separately.

Case 1: Suppose $y \in domf$.

Then there exists z_1 such that $(y, z_1) \in f$.

Since $f \subset f \cup g$, then $(y, z_1) \in f \cup g$.

Thus, there exists z_1 such that $(y, z_1) \in f \cup g$.

Case 2: Suppose $y \in domg$.

Then there exists z_2 such that $(y, z_2) \in g$.

Since $g \subset f \cup g$, then $(y, z_2) \in f \cup g$.

Thus, there exists z_2 such that $(y, z_2) \in f \cup g$.

Hence, in all cases, there exists z such that $(y, z) \in f \cup g$, so $y \in dom(f \cup g)$.

Hence, $A \cup C \subset dom(f \cup g)$.

Since $dom(f \cup g) \subset A \cup C$ and $A \cup C \subset dom(f \cup g)$, then $dom(f \cup g) = A \cup C$, as desired.

We prove $rng(f \cup g) \subset B \cup D$.

Let $b \in rng(f \cup g)$.

Then there exists a such that $(a, b) \in f \cup g$, so either $(a, b) \in f$ or $(a, b) \in g$.

We consider these cases separately.
Case 1: Suppose $(a, b) \in f$.
Since $f : A \to B$ is a map, then $b \in B$.
Case 2: Suppose $(a, b) \in g$.
Since $g : C \to D$ is a map, then $b \in D$.
Thus, either $b \in B$ or $b \in D$, so $b \in B \cup D$.
Therefore, $rng(f \cup g) \subset B \cup D$, as desired. □

Proof. We prove b.
Suppose $A \cap C = B \cap D = \emptyset$ and f and g are injective.
We must prove $f \cup g$ is injective.
Since $A \cap C = \emptyset$, then $f \cup g : A \cup C \to B \cup D$ is a map.
To prove $f \cup g$ is injective, let $a, b \in A \cup C$ such that $(f \cup g)(a) = (f \cup g)(b)$.
We must prove $a = b$.
Let $y = (f \cup g)(a)$.
Then $(a, y) \in f \cup g$, so either $(a, y) \in f$ or $(a, y) \in g$.
Since $a \in A \cup C$, then either $a \in A$ or $a \in C$.
We consider these cases separately.
Case 1: Suppose $a \in A$.
Since $A \cap C = \emptyset$, then $a \notin C$, so $a \notin domg$.
Hence, $(a, y) \notin g$.
Since either $(a, y) \in f$ or $(a, y) \in g$, then this implies $(a, y) \in f$.
Thus, $f(a) = y = (f \cup g)(a) = (f \cup g)(b)$, so $y = (f \cup g)(b)$.
Hence, $(b, y) \in f \cup g$, so either $(b, y) \in f$ or $(b, y) \in g$.
Since $b \in A \cup C$, then either $b \in A$ or $b \in C$.
Case 1a: Suppose $b \in C$.
Since $A \cap C = \emptyset$, then $b \notin A$, so $b \notin domf$.
Thus, $(b, y) \notin f$.
Since either $(b, y) \in f$ or $(b, y) \in g$, then this implies $(b, y) \in g$.
Since $(a, y) \in f$ and $f : A \to B$ is a map, then $y \in B$.
Since $(b, y) \in g$ and $g : C \to D$ is a map, then $y \in D$.
Thus, $y \in B$ and $y \in D$, so $y \in B \cap D$.
Hence, $B \cap D \neq \emptyset$.
But, this contradicts the hypothesis $B \cap D = \emptyset$.
Therefore, it is not possible that $a \in A$ and $b \in C$.
Case 1b: Suppose $b \in A$.
Since $A \cap C = \emptyset$, then $b \notin C$, so $b \notin domg$.
Thus, $(b, y) \notin g$.
Since either $(b, y) \in f$ or $(b, y) \in g$, then this implies $(b, y) \in f$.
Hence, $f(b) = y = f(a)$.
Since $f(a) = f(b)$ and f is injective, then $a = b$.
Case 2: Suppose $a \in C$.
Since $A \cap C = \emptyset$, then $a \notin A$, so $a \notin domf$.
Hence, $(a, y) \notin f$.
Since either $(a, y) \in f$ or $(a, y) \in g$, then this implies $(a, y) \in g$.
Thus, $g(a) = y = (f \cup g)(a) = (f \cup g)(b)$, so $y = (f \cup g)(b)$.

Hence, $(b, y) \in f \cup g$, so either $(b, y) \in f$ or $(b, y) \in g$.
Since $b \in A \cup C$, then either $b \in A$ or $b \in C$.
Case 2a: Suppose $b \in A$.
Since $A \cap C = \emptyset$, then $b \notin C$, so $b \notin domg$.
Thus, $(b, y) \notin g$.
Since either $(b, y) \in f$ or $(b, y) \in g$, then this implies $(b, y) \in f$.
Since $(a, y) \in g$ and $g : C \to D$ is a map, then $y \in D$.
Since $(b, y) \in f$ and $f : A \to B$ is a map, then $y \in B$.
Thus, $y \in B$ and $y \in D$, so $y \in B \cap D$.
Hence, $B \cap D \neq \emptyset$.
But, this contradicts the hypothesis $B \cap D = \emptyset$.
Therefore, it is not possible that $a \in C$ and $b \in A$.
Case 2b: Suppose $b \in C$.
Since $A \cap C = \emptyset$, then $b \notin A$, so $b \notin domf$.
Thus, $(b, y) \notin f$.
Since either $(b, y) \in f$ or $(b, y) \in g$, then this implies $(b, y) \in g$.
Hence, $g(b) = y = g(a)$.
Since $g(a) = g(b)$ and g is injective, then $a = b$.
Thus, we conclude $a = b$, so $f \cup g$ is injective, as desired. □

Proof. We prove c.
Suppose $A \cap C = \emptyset$.
Then $f \cup g : A \cup C \to B \cup D$ is a map.
Since $f \cup g : A \cup C \to B \cup D$ is a map and $A \subset A \cup C$, then the restriction $(f \cup g)|_A : A \to B \cup D$ is a map, so $(f \cup g)|_A$ is a function.

We prove $(f \cup g)|_A = f$.
Since the domain of $(f \cup g)|_A$ is A and the domain of f is A, then the functions $(f \cup g)|_A$ and f have the same domain.
Let $x \in A$.
Let $y = (f \cup g)|_A(x)$.
Since $x \in A$, then $(f \cup g)|_A(x) = (f \cup g)(x)$, so $y = (f \cup g)(x)$.
Hence, $(x, y) \in f \cup g$, so either $(x, y) \in f$ or $(x, y) \in g$.
Since $x \in A$ and $A \cap C = \emptyset$, then $x \notin C$, so $x \notin domg$.
Thus, $(x, y) \notin g$, so $(x, y) \in f$.
Since f is a function, then $f(x) = y$.
Hence, $(f \cup g)|_A(x) = (f \cup g)(x) = y = f(x)$, so $(f \cup g)|_A = f$.

Since $f \cup g : A \cup C \to B \cup D$ is a map and $C \subset A \cup C$, then the restriction $(f \cup g)|_C : C \to B \cup D$ is a map, so $(f \cup g)|_C$ is a function.

We prove $(f \cup g)|_C = g$.
Since the domain of $(f \cup g)|_C$ is C and the domain of g is C, then the functions $(f \cup g)|_C$ and g have the same domain.
Let $x \in C$.

Let $y = (f \cup g)|_C(x)$.
Since $x \in C$, then $(f \cup g)|_C(x) = (f \cup g)(x)$, so $y = (f \cup g)(x)$.
Hence, $(x, y) \in f \cup g$, so either $(x, y) \in f$ or $(x, y) \in g$.
Since $x \in C$ and $A \cap C = \emptyset$, then $x \notin A$, so $x \notin dom f$.
Thus, $(x, y) \notin f$, so $(x, y) \in g$.
Since g is a function, then $g(x) = y$.
Hence, $(f \cup g)|_C(x) = (f \cup g)(x) = y = g(x)$, so $(f \cup g)|_C = g$. $\square$

Exercise 122. A map $g : C \to B$ is an extension of a map $f : A \to B$ iff $f \subset g$.
a. If a map $g : C \to B$ is an extension of a map $f : A \to B$, then $A \subset C$.
b. If $f : A \to B$ and $g : C \to D$ are maps such that $A \cap C = \emptyset$, then $f \cup g : A \cup C \to B \cup D$ is an extension of both f and g.
c. If a map $g : C \to B$ is an extension of a map $f : A \to B$, then $g|_A = f$.
d. Find an extension of the map $f : \mathbb{R} - \{-5\} \to \mathbb{R}$ given by $f(x) = \frac{x^2-25}{x+5}$ whose domain is $\mathbb{R}$ and that is continuous on $\mathbb{R}$.
e. Let $f : \mathbb{R} \to \mathbb{C}$ be a map defined by $f(x) = e^x$.
Let $z = x + yi$ for $x, y \in \mathbb{R}$.
Let $g : \mathbb{C} \to \mathbb{C}$ be defined by $g(z) = e^x(\cos y + i \sin y)$ for all $z \in C$.
Then g is an extension of f.

Proof. We prove a.
Suppose a map $g : C \to B$ is an extension of a map $f : A \to B$.
Then $f \subset g$.
Let $a \in A$.
Since $A = dom f$, then $a \in dom f$, so $(a, f(a)) \in f$.
Since $f \subset g$, then $(a, f(a)) \in g$, so $a \in dom g$.
Since $dom g = C$, then $a \in C$, so $A \subset C$, as desired. $\square$

Proof. We prove b.
Suppose $f : A \to B$ and $g : C \to D$ are maps such that $A \cap C = \emptyset$.
Then, by a previous exercise, $f \cup g : A \cup C \to B \cup D$ is a map.
Since $f : A \to B$ is a map, then f is a function, so f is a relation.
Hence, f is a set.
Since $g : C \to D$ is a map, then g is a function, so g is a relation.
Hence, g is a set.
Since $f \subset f \cup g$, then $f \cup g$ is an extension of f.
Since $g \subset f \cup g$, then $f \cup g$ is an extension of g. $\square$

Proof. We prove c.
Suppose a map $g : C \to B$ is an extension of a map $f : A \to B$.
Then $f \subset g$ and $A \subset C$.
Since $f : A \to B$ is a map, then f is a function.
Since $g : C \to B$ is a map, then g is a function.
Since $g : C \to B$ is a map and $A \subset C$, then the restriction $g|_A : A \to B$ is a map.
Observe that *dom* $\mathrm{g}|_A = A = dom f$.
Let $a \in A$ be arbitrary.

Then $(g|_A)(a) = g(a)$.
Since $a \in A$ and f is a function, then $(a, f(a)) \in f$.
Since $f \subset g$, then $(a, f(a)) \in g$.
Since g is a function, then $g(a) = f(a)$.
Thus, $(g|_A)(a) = g(a) = f(a)$, so $(g|_A)(a) = f(a)$ for all $a \in A$.
Therefore, $g|_A = f$, as desired. $\square$

Proof. We solve d.
Let $g : \mathbb{R} \to \mathbb{R}$ be a map defined by $g(x) = x - 5$.
Then g is a function and $domg = \mathbb{R}$.
Since g is a polynomial function, then g is continuous, so g is continuous on $\mathbb{R}$.
We prove g is an extension of f.
Let $x \in domf$ be arbitrary.
Then $(x, f(x)) \in f$.
Since $x \in domf$ and $domf = \mathbb{R} - \{-5\}$, then $x \in \mathbb{R} - \{-5\}$, so $x \in \mathbb{R}$ and $x \neq -5$.
Since $x + 5 = 0$ iff $x = -5$ and $x \neq -5$, then $x + 5 \neq 0$.
Thus, $g(x) = x - 5 = (x - 5) \cdot \frac{x+5}{x+5} = \frac{x^2-25}{x+5} = f(x)$, so $g(x) = f(x)$.
Since g is a function, then $(x, g(x)) \in g$, so $(x, f(x)) \in g$.
Hence, $f \subset g$.
Therefore, g is an extension of f. $\square$

Proof. We prove e.
Let $r \in domf$.
Then $r \in \mathbb{R}$ and $f(r) = e^r$.
Since f is a function, then $(r, f(r)) \in f$.
Since $r \in \mathbb{R}$, then $r = r + 0 = r + i \cdot 0$, so $g(r) = e^r(\cos 0 + i \sin 0) = e^r(1 + 0) = e^r \cdot 1 = e^r = f(r)$.
Thus, $g(r) = f(r)$, so $r \in domg$.
Since g is a function, then $(r, g(r)) \in g$, so $(r, f(r)) \in g$.
Thus, $(r, f(r)) \in f$ implies $(r, f(r)) \in g$, so $f \subset g$.
Therefore, g is an extension of f. $\square$

Exercise 123. a. Give an example of maps $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is surjective, but f is not surjective.
b. Give an example of maps $f : A \to B$ and $g : B \to C$ such that g is surjective, but $g \circ f$ is not surjective.
c. Give an example of maps $f : A \to B$ and $g : B \to C$ such that f is surjective, but $g \circ f$ is not surjective.

Solution. Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8, 9\}$ and $C = \{10, 11, 12, 13\}$.
a. Let $f = \{(1, 5), (2, 6), (3, 7), (4, 8)\}$ and $g = \{(5, 10), (6, 11), (7, 12), (8, 13), (9, 10)\}$.
Then f and g are functions.
Since f is a function and $domf = A$ and $rngf = \{5, 6, 7, 8\} \subset B$, then $f : A \to B$ is a map.

Since g is a function and $domg = B$ and $rngg = C \subset C$, then $g : B \to C$ is a map.

Since $f : A \to B$ and $g : B \to C$ are maps, then $g \circ f : A \to C$ is a map and $g \circ f = \{(1, 10), (2, 11), (3, 12), (4, 13)\}$.

Since $rng(g \circ f) = C$, then $g \circ f$ is surjective.

Since $9 \in B$, but $9 \notin rngf$, then $rngf \neq B$, so f is not surjective.

We know that if $g \circ f$ is surjective, then g is surjective.

Since $g \circ f$ is surjective, then g is surjective.

Since $rngg = C$, then g is surjective, as predicted by theory.

b. Let $f = \{(1, 5), (2, 6), (3, 7), (4, 8)\}$ and $g = \{(5, 10), (6, 11), (7, 12), (8, 12), (9, 13)\}$.

Then f and g are functions.

Since f is a function and $domf = A$ and $rngf = \{5, 6, 7, 8\} \subset B$, then $f : A \to B$ is a map.

Since g is a function and $domg = B$ and $rngg = C \subset C$, then $g : B \to C$ is a map.

Since $f : A \to B$ and $g : B \to C$ are maps, then $g \circ f : A \to C$ is a map and $g \circ f = \{(1, 10), (2, 11), (3, 12), (4, 12)\}$.

Since $rngg = C$, then g is surjective.

Observe that $rng(g \circ f) = \{10, 11, 12\}$.

Since $13 \in C$, but $13 \notin rng(g \circ f)$, then $rng(g \circ f) \neq C$, so $g \circ f$ is not surjective.

We know that if f and g are surjective, then $g \circ f$ is surjective, so if $g \circ f$ is not surjective, then either f is not surjective or g is not surjective.

Since $g \circ f$ is not surjective, then either f is not surjective or g is not surjective.

Since g is surjective, then this implies f is not surjective.

Since $9 \in B$, but $9 \notin rngf$, then $rngg \neq B$, so f is not surjective, as predicted by theory.

Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$ and $C = \{10, 11, 12, 13\}$.

c. Let $f = \{(1, 5), (2, 6), (3, 7), (4, 8)\}$ and $g = \{(5, 10), (6, 11), (7, 12), (8, 12)\}$.

Then f and g are functions.

Since f is a function and $domf = A$ and $rngf = B \subset B$, then $f : A \to B$ is a map.

Since g is a function and $domg = B$ and $rngg = \{10, 11, 12\} \subset C$, then $g : B \to C$ is a map.

Since $f : A \to B$ and $g : B \to C$ are maps, then $g \circ f : A \to C$ is a map and $g \circ f = \{(1, 10), (2, 11), (3, 12), (4, 12)\}$.

Since $rngf = B$, then f is surjective.

Observe that $rng(g \circ f) = \{10, 11, 12\}$.

Since $13 \in C$, but $13 \notin rng(g \circ f)$, then $rng(g \circ f) \neq C$, so $g \circ f$ is not surjective.

We know that if f and g are surjective, then $g \circ f$ is surjective, so if $g \circ f$ is not surjective, then either f is not surjective or g is not surjective.

Since $g \circ f$ is not surjective, then either f is not surjective or g is not surjective.

Since f is surjective, then this implies g is not surjective.

Since $13 \in C$, but $13 \notin rngg$, then $rngg \neq C$, so g is not surjective, as predicted by theory. $\square$

Exercise 124. Let $f : A \to B$ and $g : B \to C$ be maps such that f is surjective.

Then $g \circ f : A \to C$ is a surjection iff g is surjective.

Proof. Since $f : A \to B$ and $g : B \to C$ are maps, then $g \circ f : A \to C$ is a map.

We must prove $g \circ f$ is surjective iff g is surjective.

Suppose $g \circ f$ is surjective.

Then g is surjective.

Conversely, suppose g is surjective.

Since f is surjective and g is surjective, then the composition $g \circ f$ is surjective. $\square$

Exercise 125. Let $f : A \to B$ and $g : B \to C$ be maps such that g is injective.

Then $g \circ f : A \to C$ is an injection iff f is injective.

Proof. Since $f : A \to B$ and $g : B \to C$ are maps, then $g \circ f : A \to C$ is a map.

We must prove $g \circ f$ is injective iff f is injective.

Suppose $g \circ f$ is injective.

Then f is injective.

Conversely, suppose f is injective.

Since f is injective and g is injective, then the composition $g \circ f$ is injective. $\square$

Exercise 126. Give an example of maps $f : A \to B$ and $g : B \to C$ such that:

1. $g \circ f : A \to C$ is a bijection, but f is not onto and g is not one to one.
2. f is bijective, but $g \circ f$ is not bijective.
3. g is bijective, but $g \circ f$ is not bijective.

Solution. 1. We give an example such that $g \circ f$ is bijective and f is not onto and g is not one to one.

Let $A = \{1, 2, 3, 4\}$ and $B = \{10, 20, 30, 40, 50\}$ and $C = \{3, 5, 7, 9\}$.

Let $f = \{(1, 10), (2, 20), (3, 30), (4, 40)\}$.

Let $g = \{(10, 3), (20, 5), (30, 7), (40, 9), (50, 9)\}$.

Since f is a function and $domf = A$ and $rngf = \{10, 20, 30, 40\} \subset B$, then $f : A \to B$ is a map.

Since $50 \in B$, but $50 \notin rngf$, then $rngf \neq B$, so f is not onto B.

Since g is a function and $domg = B$ and $rngg = C \subset C$, then $g : B \to C$ is a map.

Since $g(40) = 9 = g(50)$, but $40 \neq 50$, then g is not one to one.

Since $f : A \to B$ is a map and $g : B \to C$ is a map, then the composition $g \circ f : A \to C$ is a map and $g \circ f = \{(1,3),(2,5),(3,7),(4,9)\}$.

Clearly, $g \circ f$ is one to one and onto, so $g \circ f$ is bijective. $\square$

Solution. 2. We give an example such that f is bijective and $g \circ f$ is not bijective:

Let $A = \{1,2,3,4\}$ and $B = \{10,20,30,40\}$ and $C = \{3,5,7,9\}$.

Let $f = \{(1,10),(2,20),(3,30),(4,40)\}$.

Let $g = \{(10,3),(20,5),(30,7),(40,7)\}$.

Since f is a function and $domf = A$ and $rngf = B \subset B$, then $f : A \to B$ is a map.

Clearly, f is one to one.

Since $rngf = B$, then f is onto, so f is bijective.

Since g is a function and $domg = B$ and $rngg = \{3,5,7\} \subset C$, then $g : B \to C$ is a map.

Since $f : A \to B$ is a map and $g : B \to C$ is a map, then the composition $g \circ f : A \to C$ is a map and $g \circ f = \{(1,3),(2,5),(3,7),(4,7)\}$.

Since $(g \circ f)(3) = 7 = (g \circ f)(4)$, but $3 \neq 4$, then $g \circ f$ is not one to one, so $g \circ f$ is not bijective. $\square$

Solution. 3. We give an example such that g is bijective and $g \circ f$ is not bijective:

Let $A = \{1,2,3,4\}$ and $B = \{10,20,30,40\}$ and $C = \{3,5,7,9\}$.

Let $f = \{(1,10),(2,20),(3,30),(4,30)\}$.

Let $g = \{(10,3),(20,5),(30,7),(40,9)\}$.

Since f is a function and $domf = A$ and $rngf = \{10,20,30\} \subset B$, then $f : A \to B$ is a map.

Since g is a function and $domg = B$ and $rngg = C \subset C$, then $g : B \to C$ is a map.

Clearly, g is one to one.

Since $rngg = C$, then g is onto.

Hence, g is bijective.

Since $f : A \to B$ is a map and $g : B \to C$ is a map, then the composition $g \circ f : A \to C$ is a map and $g \circ f = \{(1,3),(2,5),(3,7),(4,7)\}$.

Since $(g \circ f)(3) = 7 = (g \circ f)(4)$, but $3 \neq 4$, then $g \circ f$ is not one to one, so $g \circ f$ is not bijective. $\square$

Exercise 127. Let $f : X \to Y$ and $g : Y \to Z$ be maps.

If $g \circ f$ is injective, then f is injective.

Prove this statement using the left cancellation property of injective maps.

Proof. Suppose $g \circ f$ is injective.

To prove f is injective using the left cancellation property of injective maps, let W be a set and let $h : W \to X$ and $k : W \to X$ be maps such that $f \circ h = f \circ k$.

We must prove $h = k$.

Since $g \circ f$ is injective, then by the left cancellation property of injective maps, if $h : W \to X$ and $k : W \to X$ are maps such that $(g \circ f) \circ h = (g \circ f) \circ k$, then $h = k$.

Since f, g, h are functions, then $(g \circ f) \circ h = g \circ (f \circ h) = g \circ (f \circ k) = (g \circ f) \circ k$.

Since $h : W \to X$ and $k : W \to X$ are maps and $(g \circ f) \circ h = (g \circ f) \circ k$, then we conclude $h = k$, as desired. $\square$

Exercise 128. Let $f : X \to Y$ and $g : Y \to Z$ be maps.

If $g \circ f$ is surjective, then g is surjective.

Prove this statement using the right cancellation property of surjective maps.

Proof. Suppose $g \circ f$ is surjective.

To prove g is surjective using the right cancellation property of surjective maps, let W be a set and let $h : Z \to W$ and $k : Z \to W$ be maps such that $h \circ g = k \circ g$.

We must prove $h = k$.

Since $g \circ f$ is surjective, then by the right cancellation property of surjective maps, if $h : Z \to W$ and $k : Z \to W$ are maps such that $h \circ (g \circ f) = k \circ (g \circ f)$, then $h = k$.

Since f, g, h are functions, then $h \circ (g \circ f) = (h \circ g) \circ f = (k \circ g) \circ f = k \circ (g \circ f)$.

Since $h : Z \to W$ and $k : Z \to W$ are maps and $h \circ (g \circ f) = k \circ (g \circ f)$, then we conclude $h = k$, as desired. $\square$

Exercise 129. Let $f : X \to Y$ and $g : Y \to Z$ be maps.

If f and g are injective, then $g \circ f$ is injective.

Prove this statement using the left cancellation property of injective maps.

Proof. Suppose f and g are injective.

Since $f : X \to Y$ and $g : Y \to Z$ are maps, then $g \circ f : X \to Z$ is a map.

To prove $g \circ f$ is injective using the left cancellation property of injective maps, let W be a set and let $h : W \to X$ and $k : W \to X$ be maps such that $(g \circ f) \circ h = (g \circ f) \circ k$.

We must prove $h = k$.

Since g is injective, then by the left cancellation property of injective maps, if $f \circ h : W \to Y$ and $f \circ k : W \to Y$ are maps such that $g \circ (f \circ h) = g \circ (f \circ k)$, then $f \circ h = f \circ k$.

Since $h : W \to X$ and $f : X \to Y$ are maps, then $f \circ h : W \to Y$ is a map.

Since $k : W \to X$ and $f : X \to Y$ are maps, then $f \circ k : W \to Y$ is a map.

Since f, g, h are functions, then $g \circ (f \circ h) = (g \circ f) \circ h = (g \circ f) \circ k = g \circ (f \circ k)$.

Since $f \circ h : W \to Y$ is a map and $f \circ k : W \to Y$ is a map and $g \circ (f \circ h) = g \circ (f \circ k)$, then we conclude $f \circ h = f \circ k$.

Since f is injective, then by the left cancellation property of injective maps, if $h : W \to X$ and $k : W \to X$ are maps such that $f \circ h = f \circ k$, then $h = k$.

Since $h : W \to X$ and $k : W \to X$ are maps and $f \circ h = f \circ k$, then we conclude $h = k$, as desired. $\square$

Exercise 130. Give an example of a map $f : X \to Y$ and subsets A, B of X such that $A \subset B$, but $f(A) = f(B)$.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be the map defined by $f(x) = \sin x$.

Let $A = [0, \frac{\pi}{2}]$ and $B = [0, \pi]$.

Then $A \subset B$.

Observe that

$$\begin{aligned} f(A) &= \{f(x) : x \in A\} \\ &= \{\sin x : x \in [0, \frac{\pi}{2}]\} \\ &= \{\sin x : 0 \leq x \leq \frac{\pi}{2}\} \\ &= [0, 1] \\ &= \{\sin x : 0 \leq x \leq \pi\} \\ &= \{f(x) : x \in [0, \pi]\} \\ &= \{f(x) : x \in B\} \\ &= f(B). \end{aligned}$$

$\square$

Exercise 131. Let $f : X \to Y$ be a map.

Let A and B be subsets of X.

If $f(A) = f(B)$ and f is one to one, then $A = B$.

Proof. Suppose $f(A) = f(B)$ and f is one to one.

We first prove $A \subset B$.

Let $a \in A$.

Then $f(a) \in f(A)$.

Since $f(A) = f(B)$, then $f(a) \in f(B)$.

Hence, there exists $b \in B$ such that $f(a) = f(b)$.

Since f is one to one, then $a = b$.

Since $b \in B$, then $a \in B$.

Thus, $A \subset B$.

We next prove $B \subset A$.

Let $b \in B$.

Then $f(b) \in f(B)$.

Since $f(B) = f(A)$, then $f(b) \in f(A)$.

Hence, there exists $a \in A$ such that $f(b) = f(a)$.

Since f is one to one, then $b = a$.

Since $a \in A$, then $b \in A$.

Thus, $B \subset A$.

Since $A \subset B$ and $B \subset A$, then $A = B$, as desired. $\square$

Exercise 132. Let $f : A \to B$ be a map.
Let $Y \subset B$.
Then $f(f^{-1}(Y)) = Y$ iff $Y \subset rngf$.

Proof. We must prove $f(f^{-1}(Y)) = Y$ iff $Y \subset rngf$.

We first prove if $f(f^{-1}(Y)) = Y$, then $Y \subset rngf$.
Suppose $f(f^{-1}(Y)) = Y$.
To prove $Y \subset rngf$, let $b \in Y$.
Since $Y = f(f^{-1}(Y))$, then $y \in f(f^{-1}(Y))$.
Hence, $y = f(x)$ for some $x \in f^{-1}(Y)$.
Since $x \in f^{-1}(Y)$, then $x \in A$ and $f(x) \in Y$.
Thus, there exists $x \in A$ such that $f(x) = y$, so $y \in rngf$, as desired.

Conversely, we prove if $Y \subset rngf$, then $f(f^{-1}(Y)) = Y$.
Suppose $Y \subset rngf$.
We first prove $Y \subset f(f^{-1}(Y))$.
Let $y \in Y$.
Since $Y \subset rngf$, then $y \in rngf$, so there exists $x \in A$ such that $f(x) = y$.
Thus, there exists $x \in A$ such that $f(x) \in Y$, so $x \in f^{-1}(Y)$.
Hence, there exists $x \in f^{-1}(Y)$ such that $y = f(x)$, so $y \in f(f^{-1}(Y))$.
Therefore, $Y \subset f(f^{-1}(Y))$.
Since $f(f^{-1}(Y)) \subset Y$ and $Y \subset f(f^{-1}(Y))$, then $f(f^{-1}(Y)) = Y$, as desired.
$\square$

Exercise 133. Let $f : A \to B$ be a map.
Let $X \subset A$.
Then $f^{-1}(f(X)) = X$ iff the restriction of f to the subset $f^{-1}(f(X))$ of A is one to one.

Proof. We must prove $f^{-1}(f(X)) = X$ iff the restriction of f to the subset $f^{-1}(f(X))$ of A is one to one.

We first prove if $f^{-1}(f(X)) = X$, then the restriction of f to the subset $f^{-1}(f(X))$ of A is one to one.
Suppose $f^{-1}(f(X)) = X$.
Then f is one to one.
Let $f|_X : X \to B$ be the restriction of f to X defined by $f|_X(x) = f(x)$ for all $x \in X$.
To prove $f|_X$ is one to one, let $a, b \in X$ such that $f|_X(a) = f|_X(b)$.
Then $f(a) = f(b)$.
Since f is one to one, then $a = b$.
Therefore, $f|_X$ is one to one, as desired.

Conversely, we prove if the restriction of f to the subset $f^{-1}(f(X))$ of A is one to one, then $f^{-1}(f(X)) = X$.

Suppose the restriction of f to the subset $f^{-1}(f(X))$ of A is one to one.

Let $g : f^{-1}(f(X)) \to B$ be the restriction of f to $f^{-1}(f(X))$ defined by $g(x) = f(x)$ for all $x \in f^{-1}(f(X))$.

Then g is one to one.

We first prove $f^{-1}(f(X)) \subset X$.

Let $a \in f^{-1}(f(X))$.

Then $a \in A$ and $f(a) \in f(X)$.

Since $f(a) \in f(X)$, then $f(a) = f(b)$ for some $b \in X$.

Since $b \in X$ and $X \subset A$, then $b \in A$.

Since $f(b) = f(a)$ and $f(a) \in f(X)$, then $f(b) \in f(X)$.

Thus, $b \in A$ and $f(b) \in f(X)$, so $b \in f^{-1}(f(X))$.

Hence, $g(a) = f(a) = f(b) = g(b)$.

Since g is one to one and $g(a) = g(b)$, then $a = b$.

Since $b \in X$, then $a \in X$.

Therefore, $f^{-1}(f(X)) \subset X$.

Since $f^{-1}(f(X)) \subset X$ and $X \subset f^{-1}(f(X))$, then $f^{-1}(f(X)) = X$, as desired. $\square$

Exercise 134. Let $f : A \to B$ be a map.

Let $X \subset A$ and $Y \subset B$. Then

1. $f^{-1}(B - Y) = A - f^{-1}(Y)$.
2. $f(X) \subset Y$ iff $X \subset f^{-1}(Y)$.
3. If f is bijective, then $f(X) = Y$ iff $f^{-1}(Y) = X$.

Proof. We prove 1.

We must prove $f^{-1}(B - Y) = A - f^{-1}(Y)$.

We first prove $f^{-1}(B - Y) \subset A - f^{-1}(Y)$.

Let $x \in f^{-1}(B - Y)$.

Then $x \in A$ and $f(x) \in B - Y$.

Since $f(x) \in B - Y$, then $f(x) \in B$ and $f(x) \notin Y$.

Since $x \in A$ and $f(x) \notin Y$, then $x \notin f^{-1}(Y)$.

Since $x \in A$ and $x \notin f^{-1}(Y)$, then $x \in A - f^{-1}(Y)$.

Thus, $f^{-1}(B - Y) \subset A - f^{-1}(Y)$.

We next prove $A - f^{-1}(Y) \subset f^{-1}(B - Y)$.

Let $y \in A - f^{-1}(Y)$.

Then $y \in A$ and $y \notin f^{-1}(Y)$, so $f(y) \notin Y$.

Since $y \in A$ and $f : A \to B$ is a map, then $f(y) \in B$.

Since $f(y) \in B$ and $f(y) \notin Y$, then $f(y) \in B - Y$.

Hence, $y \in A$ and $f(y) \in B - Y$, so $y \in f^{-1}(B - Y)$.

Thus, $A - f^{-1}(Y) \subset f^{-1}(B - Y)$.

Since $f^{-1}(B-Y) \subset A - f^{-1}(Y)$ and $A - f^{-1}(Y) \subset f^{-1}(B-Y)$, then $f^{-1}(B-Y) = A - f^{-1}(Y)$. $\square$

Proof. We prove 2.
We must prove $f(X) \subset Y$ iff $X \subset f^{-1}(Y)$.

We first prove if $f(X) \subset Y$, then $X \subset f^{-1}(Y)$.
Suppose $f(X) \subset Y$.
To prove $X \subset f^{-1}(Y)$, let $x \in X$.
Then $f(x) \in f(X)$.
Since $f(X) \subset Y$, then $f(x) \in Y$.
Since $x \in X$ and $f(x) \in Y$, then $x \in f^{-1}(Y)$.
Thus, $X \subset f^{-1}(Y)$.

Conversely, we prove if $X \subset f^{-1}(Y)$, then $f(X) \subset Y$.
Suppose $X \subset f^{-1}(Y)$.
To prove $f(X) \subset Y$, let $y \in f(X)$.
Then there exists $x \in X$ such that $y = f(x)$.
Since $x \in X$ and $X \subset f^{-1}(Y)$, then $x \in f^{-1}(Y)$, so $f(x) \in Y$.
Thus, $y \in Y$, so $f(X) \subset Y$. $\square$

Proof. We prove 3.
We prove if f is bijective, then $f(X) = Y$ iff $f^{-1}(Y) = X$.
Suppose f is bijective.
Then f is injective and surjective.
We must prove $f(X) = Y$ iff $f^{-1}(Y) = X$.

We first prove if $f(X) = Y$, then $f^{-1}(Y) = X$.
Suppose $f(X) = Y$.
Then $f(X) \subset Y$, so $X \subset f^{-1}(Y)$.
Let $x \in f^{-1}(Y)$.
Then $x \in A$ and $f(x) \in Y$.
Since $Y = f(X)$, then $f(x) \in f(X)$.
Hence, there exists $a \in X$ such that $f(x) = f(a)$.
Since f is injective, then $x = a$.
Thus, $x \in X$, so $f^{-1}(Y) \subset X$.
Since $f^{-1}(Y) \subset X$ and $X \subset f^{-1}(Y)$, then $f^{-1}(Y) = X$, as desired.

Conversely, we prove if $f^{-1}(Y) = X$, then $f(X) = Y$.
Suppose $f^{-1}(Y) = X$.
Then $X \subset f^{-1}(Y)$, so $f(X) \subset Y$.
Let $y \in Y$.
Since $Y \subset B$, then $y \in B$.
Since f is surjective, then there exists $x \in A$ such that $f(x) = y$, so $f(x) \in Y$.
Since $x \in A$ and $f(x) \in Y$, then $x \in f^{-1}(Y)$.
Since $f^{-1}(Y) = X$, then $x \in X$, so $f(x) \in f(X)$.

Hence, $y \in f(X)$, so $Y \subset f(X)$.
Since $f(X) \subset Y$ and $Y \subset f(X)$, then $f(X) = Y$, as desired. $\square$

Exercise 135. image of a difference

Let $f : X \to Y$ be a map with $A, B \subset X$. Then
1. $f(A) - f(B) \subset f(A - B)$.
2. If f is injective, then $f(A - B) = f(A) - f(B)$.

Proof. We prove 1.
We prove $f(A) - f(B) \subset f(A - B)$.
Let $y \in f(A) - f(B)$.
Then $y \in f(A)$ and $y \notin f(B)$.
Since $y \in f(A)$, then $y = f(a)$ for some $a \in A$.
Since $y \in f(B)$ iff $a \in B$ and $f(a) = y$, then $y \notin f(B)$ iff either $a \notin B$ or $f(a) \neq y$.
Since $y \notin f(B)$, then either $a \notin B$ or $f(a) \neq y$.
Since $f(a) = y$, then we conclude $a \notin B$.
Since $a \in A$ and $a \notin B$, then $a \in A - B$.
Thus, there exists $a \in A - B$ such that $f(a) = y$, so $y \in f(A - B)$.
Therefore, $f(A) - f(B) \subset f(A - B)$. $\square$

Proof. We prove 2.
We prove if f is injective, then $f(A - B) = f(A) - f(B)$.
Suppose f is injective.

We first prove $f(A - B) \subset f(A) - f(B)$.
Let $y \in f(A - B)$.
Then there exists $a \in A - B$ such that $f(a) = y$.
Since $a \in A - B$, then $a \in A$ and $a \notin B$.
Since $y = f(a)$ and $a \in A$, then $y \in f(A)$.
Suppose for the sake of contradiction $y \in f(B)$.
Then there exists $b \in B$ such that $f(b) = y$.
Thus, $f(a) = y = f(b)$.
Since f is injective and $f(a) = f(b)$, then $a = b$.
Since $b \in B$ and $b = a$, then $a \in B$.
Thus, we have $a \in B$ and $a \notin B$, a contradiction.
Hence, $y \notin f(B)$.
Since $y \in f(A)$ and $y \notin f(B)$, then $y \in f(A) - f(B)$.
Therefore, $f(A - B) \subset f(A) - f(B)$.

Since $f(A-B) \subset f(A)-f(B)$ and $f(A)-f(B) \subset f(A-B)$, then $f(A-B) = f(A) - f(B)$. $\square$

Exercise 136. inverse image of a difference equals difference of inverse images

Let $f : X \to Y$ be a map with $C, D \subset Y$.
Then $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.

Proof. We must prove $f^{-1}(C-D) = f^{-1}(C) - f^{-1}(D)$.

We first prove $f^{-1}(C-D) \subset f^{-1}(C) - f^{-1}(D)$.
Let $x \in f^{-1}(C-D)$.
Then $x \in X$ and $f(x) \in (C-D)$.
Since $f(x) \in (C-D)$, then $f(x) \in C$ and $f(x) \notin D$.
Since $x \in X$ and $f(x) \in C$, then $x \in f^{-1}(C)$.
Since $x \in X$ and $f(x) \notin D$, then $x \notin f^{-1}(D)$.
Thus, $x \in f^{-1}(C)$ and $x \notin f^{-1}(D)$, so $x \in f^{-1}(C) - f^{-1}(D)$.
Hence, $f^{-1}(C-D) \subset f^{-1}(C) - f^{-1}(D)$.

We next prove $f^{-1}(C) - f^{-1}(D) \subset f^{-1}(C-D)$.
Let $a \in f^{-1}(C) - f^{-1}(D)$.
Then $a \in f^{-1}(C)$ and $a \notin f^{-1}(D)$.
Since $a \in f^{-1}(C)$, then $a \in X$ and $f(a) \in C$.
Since $a \in X$ and $a \notin f^{-1}(D)$, then $f(a) \notin D$.
Thus, $f(a) \in C$ and $f(a) \notin D$, so $f(a) \in (C-D)$.
Since $a \in X$ and $f(a) \in (C-D)$, then $a \in f^{-1}(C-D)$.
Hence, $f^{-1}(C) - f^{-1}(D) \subset f^{-1}(C-D)$.

Since $f^{-1}(C-D) \subset f^{-1}(C) - f^{-1}(D)$ and $f^{-1}(C) - f^{-1}(D) \subset f^{-1}(C-D)$, then $f^{-1}(C-D) = f^{-1}(C) - f^{-1}(D)$. $\square$

Exercise 137. Let $f : A \to B$ and $g : B \to C$ be maps.
Let $S \subset C$.
Then $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$.

Proof. Since $f : A \to B$ and $g : B \to C$ are maps, then $g \circ f : A \to C$ is a map.
Let $x \in (g \circ f)^{-1}(S)$.
Then $x \in A$ and $(g \circ f)(x) \in S$.
Since $x \in A$ and $f : A \to B$ is a map, then $f(x) \in B$.
Since $(g \circ f)(x) = g(f(x))$ and $(g \circ f)(x) \in S$, then $g(f(x)) \in S$.
Since $f(x) \in B$ and $g(f(x)) \in S$, then $f(x) \in g^{-1}(S)$.
Since $x \in A$ and $f(x) \in g^{-1}(S)$, then $x \in f^{-1}(g^{-1}(S))$.
Therefore, $(g \circ f)^{-1}(S) \subset f^{-1}(g^{-1}(S))$.

Let $y \in f^{-1}(g^{-1}(S))$.
Then $y \in A$ and $f(y) \in g^{-1}(S)$.
Since $f(y) \in g^{-1}(S)$, then $f(y) \in B$ and $g(f(y)) \in S$.
Since $g(f(y)) = (g \circ f)(y)$ and $g(f(y)) \in S$, then $(g \circ f)(y) \in S$.
Since $y \in A$ and $(g \circ f)(y) \in S$, then $y \in (g \circ f)^{-1}(S)$.
Therefore, $f^{-1}(g^{-1}(S)) \subset (g \circ f)^{-1}(S)$.

Since $(g \circ f)^{-1}(S) \subset f^{-1}(g^{-1}(S))$ and $f^{-1}(g^{-1}(S)) \subset (g \circ f)^{-1}(S)$, then $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$, as desired. $\square$

Proposition 138. *Let $f : X \mapsto Y$ be a function with $A_1, A_2 \subseteq X$ and $B_1, B_2 \subseteq Y$.*

Then $f^{-1}(Y - B_1) = X - f^{-1}(B_1)$.

Solution. We can use the definition of pre-image under a function.

We know for any $B \subseteq Y$, the pre-image of B under f is the set $f^{-1}(B) = \{a \in X : f(a) \in B\} \subseteq X$.

Thus,

$f^{-1}(B_1) = \{a \in X : f(a) \in B_1\}$

$f^{-1}(Y - B_1) = \{a \in X : f(a) \in Y - B_1\}$

We use the definition of set equality to prove this. $\square$

Proof. We know that $f^{-1}(B_1) = \{a \in X : f(a) \in B_1\}$ and $f^{-1}(Y - B_1) = \{a \in X : f(a) \in Y - B_1\}$.

Observe that

$$\begin{aligned}
f^{-1}(Y - B_1) &= \{a \in X : f(a) \in Y - B_1\} \\
&= \{a \in X : f(a) \in Y \wedge f(a) \notin B_1\} \\
&= \{a \in X : T \wedge f(a) \notin B_1\} \\
&= \{a \in X : f(a) \notin B_1\} \\
&= \{a \in X : a \notin f^{-1}(B_1)\} \\
&= \{a : a \in X \wedge a \notin f^{-1}(B_1)\} \\
&= X - f^{-1}(B_1)
\end{aligned}$$

$\square$