Set Theory

Jason Sass

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Sets

Proposition 1. *Every set is a subset of itself.*

Proof. Let S be an arbitrary set.
To prove $S \subset S$, let x be an arbitrary object.
We must prove $x \in S$ implies $x \in S$.
The statement $x \in S \to x \in S$ has the form $P \to P$, where P is the predicate $x \in S$.
Since $P \to P$ is a tautology, then $x \in S \to x \in S$ is true.
Therefore, $x \in S$ implies $x \in S$. $\square$

Proposition 2. ***necessary and sufficient conditions for set equality***
Let A and B be sets.
Then $A = B$ iff $A \subset B$ and $B \subset A$.

Proof. Observe that

$$\begin{aligned} \forall x.(x \in A \leftrightarrow x \in B) &\Leftrightarrow \forall x.[(x \in A \to x \in B) \wedge (x \in B \to x \in A)] \\ &\Leftrightarrow \forall x.(x \in A \to x \in B) \wedge \forall x.(x \in B \to x \in A) \\ &\Leftrightarrow A \subset B \wedge B \subset A \end{aligned}$$

Therefore $A = B$ iff $A \subset B \wedge B \subset A$. $\square$

Proposition 3. *Every set equals itself.*

Proof. Let S be an arbitrary set.
We must prove $S = S$.
Let x be an arbitrary object.
To prove $S = S$, we must prove $x \in S \leftrightarrow x \in S$.
The biconditional $x \in S \leftrightarrow x \in S$ has the form $p \leftrightarrow p$, which is a tautology.
Hence, $x \in S \leftrightarrow x \in S$ is always true. $\square$

Lemma 4. *There is at most one empty set.*

Proof. Let A and B be arbitrary sets such that A is empty and B is empty.

To prove there is at most one empty set, we must prove $A = B$.

Let x be an arbitrary element.

To prove $A = B$, we must prove $x \in A \Leftrightarrow x \in B$.

Since A is empty, then there is no element in A, so $x \notin A$.

Since B is empty, then there is no element in B, so $x \notin B$.

Hence, $x \notin A$ and $x \notin B$ implies the biconditional $x \in A \leftrightarrow x \in B$ is true.

Therefore, $x \in A \Leftrightarrow x \in B$. □

Proposition 5. *The empty set is unique.*

Proof. Let S be a set that contains no elements.

Then there is no element in S, so S is empty.

Therefore, there is at least one empty set.

Since there is at least one empty set and there is at most one empty set, then there is exactly one empty set.

Therefore, the empty set is unique. □

Proposition 6. *The empty set is a subset of every set.*

Proof. Let S be an arbitrary set.

To prove $\emptyset \subset S$, let x be an arbitrary object.

We must prove $x \in \emptyset$ implies $x \in S$.

Since $\emptyset$ is empty, then there is no element in $\emptyset$, so $x \notin \emptyset$.

Therefore, the conditional $x \in \emptyset$ implies $x \in S$ is vacuously true. □

Theorem 7. *The subset relation is a partial order.*

Proof. Since every set is a subset of itself, then $S \subset S$ for every set S.

Therefore, the subset relation is reflexive. □

Proof. Let A, B be arbitrary sets such that $A \subset B$ and $B \subset A$.

Since $A \subset B$ and $B \subset A$ iff $A = B$, then $A = B$.

Therefore, the subset relation is antisymmetric. □

Proof. Let A, B, C be arbitrary sets such that $A \subset B$ and $B \subset C$.

Either $A = \emptyset$ or $A \neq \emptyset$.

We consider these cases separately.

Case 1: Suppose $A = \emptyset$.

Since $\emptyset \subset B$ and $B \subset C$ and $\emptyset \subset C$, then $\emptyset \subset B$ and $B \subset C$ implies $\emptyset \subset C$, so $A \subset B$ and $B \subset C$ implies $A \subset C$.

Case 2: Suppose $A \neq \emptyset$.

Then there is some element in A.

Let $x \in A$.

Since $x \in A$ and $A \subset B$, then $x \in B$.

Since $x \in B$ and $B \subset C$, then $x \in C$.

Therefore, $x \in C$ for every $x \in A$, so $A \subset C$.

Thus, $A \subset B$ and $B \subset C$ implies $A \subset C$.

Since $A \subset B$ and $B \subset C$ implies $A \subset C$ in all cases, then the subset relation is transitive.

Since the subset relation is reflexive, antisymmetric, and transitive, then the subset relation is a partial order. □

Theorem 8. *The set equality relation is an equivalence relation.*

Proof. Since every set equals itself, then $S = S$ for every set S, so set equality is reflexive. □

Proof. Let A, B be arbitrary sets such that $A = B$.
Since $A = B$ iff $A \subset B$ and $B \subset A$, then $A \subset B$ and $B \subset A$.
Thus, $B \subset A$ and $A \subset B$, so $B = A$.
Therefore $A = B$ implies $B = A$, so set equality is symmetric. □

Proof. Let A, B, C be arbitrary sets such that $A = B$ and $B = C$.
Then $A \subset B$ and $B \subset A$ and $B \subset C$ and $C \subset B$.
Since $A \subset B$ and $B \subset C$, then $A \subset C$.
Since $C \subset B$ and $B \subset A$, then $C \subset A$.
Since $A \subset C$ and $C \subset A$ iff $A = C$, then $A = C$.
Hence, $A = B$ and $B = C$ implies $A = C$, so set equality is transitive.

Since set equality is reflexive, symmetric, and transitive, then set equality is an equivalence relation. □

Algebraic Properties of Sets

Proposition 9. ***Idempotent law for set union***
Let A be a set.
Then $A \cup A = A$.

Proof. Observe that $x \in A \cup A \Leftrightarrow x \in A \vee x \in A \Leftrightarrow x \in A$.
Therefore, $A \cup A = A$. □

Proposition 10. ***Idempotent law for set intersection***
Let A be a set.
Then $A \cap A = A$.

Proof. Observe that $x \in A \cap A \Leftrightarrow x \in A \wedge x \in A \Leftrightarrow x \in A$.
Therefore, $A \cap A = A$. □

Proposition 11. ***empty set is the identity for set union***
Let A be a set.
Then $A \cup \emptyset = A$.

Proof. Let $x \in A \cup \emptyset$.
Then either $x \in A$ or $x \in \emptyset$.
Since $\emptyset$ contains no elements, then $x \notin \emptyset$.
Hence, $x \in A$, so $A \cup \emptyset \subset A$.

Let $y \in A$.
Then either $y \in A$ or $y \in \emptyset$, so $y \in A \cup \emptyset$.
Hence, $A \subset A \cup \emptyset$.
Since $A \cup \emptyset \subset A$ and $A \subset A \cup \emptyset$, then $A \cup \emptyset = A$. □

Proposition 12. ***universal set is the identity for set intersection***
Let U be a set and $A \subset U$.
Then $A \cap U = A$.

Proof. Let $x \in A \cap U$.
Then $x \in A$ and $x \in U$, so $x \in A$.
Therefore, $A \cap U \subset A$.

Let $y \in A$.
Since $A \subset U$, then $y \in U$.
Since $y \in A$ and $y \in U$, then $y \in A \cap U$, so $A \subset A \cap U$.
Since $A \cap U \subset A$ and $A \subset A \cap U$, then $A \cap U = A$. □

Proposition 13. ***Involution law for sets***
Let A be a set.
Then $\overline{\overline{A}} = A$.

Proof. To prove $\overline{\overline{A}} = A$, we must prove $x \in \overline{\overline{A}} \Leftrightarrow x \in A$.
Observe that

$$\begin{aligned} x \in \overline{\overline{A}} &\Leftrightarrow x \notin \overline{A} \\ &\Leftrightarrow \neg(x \in \overline{A}) \\ &\Leftrightarrow \neg(x \notin A) \\ &\Leftrightarrow x \in A. \end{aligned}$$

□

Proposition 14. ***Domination law for set intersection***
Let A be a set.
Then $A \cap \emptyset = \emptyset$.

Proof. Suppose $A \cap \emptyset \neq \emptyset$.
Then there exists $x \in A \cap \emptyset$, so $x \in A$ and $x \in \emptyset$.
Since $\emptyset$ is empty, then $x \notin \emptyset$.
Thus we have $x \in \emptyset$ and $x \notin \emptyset$, a contradiction.
Therefore, $A \cap \emptyset = \emptyset$. □

Proposition 15. *Let $A \subset U$.*
Then $A \cup U = U$.

Proof. Let $x \in A \cup U$.
Then either $x \in A$ or $x \in U$.
We consider these cases separately.
Case 1: Suppose $x \in A$.
Since $A \subset U$, then $x \in U$.
Case 2: Suppose $x \in U$.
The $x \in U$.
Therefore, in all cases, $x \in U$.
Since $x \in A \cup U$ implies $x \in U$, then $A \cup U \subset U$.

Let $y \in U$.
Then either $y \in A$ or $y \in U$, so $y \in A \cup U$.
Hence, $U \subset A \cup U$.
Since $A \cup U \subset U$ and $U \subset A \cup U$, then $A \cup U = U$. □

Proposition 16. ***intersection of a set and its complement is empty***
Let A be a set.
Then $A \cap \overline{A} = \emptyset$.

Proof. Suppose $A \cap \overline{A} \neq \emptyset$.
Then there exists $x \in A \cap \overline{A}$, so $x \in A$ and $x \in \overline{A}$.
Since $x \in \overline{A}$, then $x \notin A$.
Thus, we have $x \in A$ and $x \notin A$, a contradiction.
Therefore, $A \cap \overline{A} = \emptyset$. □

Proposition 17. $\overline{U} = \emptyset$.

Proof. Suppose $\overline{U} \neq \emptyset$.
Then there exists $x \in \overline{U}$.
By definition of $\overline{U}$, $x \in U$ and $x \notin U$.
Thus, we have a contradiction $x \in U$ and $x \notin U$, so $\overline{U} = \emptyset$. □

Proposition 18. *Let U be a set.*
Then $\overline{\emptyset} = U$.

Proof. Since any set is a subset of the universal set U, then, in particular, $\overline{\emptyset}$ is a subset of U, so $\overline{\emptyset} \subset U$.

Let $x \in U$.
Since $\emptyset$ is empty, then $x \notin \emptyset$, so $x \in \overline{\emptyset}$.
Therefore, $U \subset \overline{\emptyset}$.
Since $\overline{\emptyset} \subset U$ and $U \subset \overline{\emptyset}$, then $\overline{\emptyset} = U$. □

Proposition 19. *Let U, A be sets.*
Then $A \cup \overline{A} = U$.

Proof. Observe that

$$\begin{aligned} x \in A \cup \overline{A} &\Leftrightarrow x \in U \wedge (x \in A \vee x \in \overline{A}) \\ &\Leftrightarrow x \in U \wedge (x \in A \vee x \notin A) \\ &\Leftrightarrow x \in U. \end{aligned}$$

Therefore, $A \cup \overline{A} = U$. □

Proposition 20. ***Commutative law of set union***
Let A, B be sets.
Then $A \cup B = B \cup A$.

Proof. Observe that

$$\begin{aligned} x \in A \cup B &\Leftrightarrow x \in A \vee x \in B \\ &\Leftrightarrow x \in B \vee x \in A \\ &\Leftrightarrow x \in B \cup A. \end{aligned}$$

Therefore, $A \cup B = B \cup A$. □

Proposition 21. ***Commutative law of set intersection***
Let A, B be sets.
Then $A \cap B = B \cap A$.

Proof. Observe that

$$\begin{aligned} x \in A \cap B &\Leftrightarrow x \in A \wedge x \in B \\ &\Leftrightarrow x \in B \wedge x \in A \\ &\Leftrightarrow x \in B \cap A. \end{aligned}$$

Therefore, $A \cap B = B \cap A$. □

Proposition 22. ***Associative law of set union***
Let A, B, C be sets.
Then $(A \cup B) \cup C = A \cup (B \cup C)$.

Proof. Observe that

$$\begin{aligned} x \in (A \cup B) \cup C &\Leftrightarrow (x \in A \cup B) \vee (x \in C) \\ &\Leftrightarrow (x \in A \vee x \in B) \vee (x \in C) \\ &\Leftrightarrow x \in A \vee (x \in B \vee x \in C) \\ &\Leftrightarrow x \in A \vee (x \in B \cup C) \\ &\Leftrightarrow x \in A \cup (B \cup C). \end{aligned}$$

Therefore, $(A \cup B) \cup C = A \cup (B \cup C)$. □

Proposition 23. ***Associative law of set intersection***

Let A, B, C be sets.

Then $(A \cap B) \cap C = A \cap (B \cap C)$.

Proof. Observe that

$$\begin{aligned} x \in (A \cap B) \cap C &\Leftrightarrow (x \in A \cap B) \wedge (x \in C) \\ &\Leftrightarrow (x \in A \wedge x \in B) \wedge (x \in C) \\ &\Leftrightarrow x \in A \wedge (x \in B \wedge x \in C) \\ &\Leftrightarrow x \in A \wedge (x \in B \cap C) \\ &\Leftrightarrow x \in A \cap (B \cap C). \end{aligned}$$

Therefore, $(A \cap B) \cap C = A \cap (B \cap C)$. □

Proposition 24. ***Distributive law of set intersection over union***

Let A, B, C be sets.

Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. Observe that

$$\begin{aligned} x \in A \cap (B \cup C) &\Leftrightarrow x \in A \wedge (x \in B \cup C) \\ &\Leftrightarrow x \in A \wedge (x \in B \vee x \in C) \\ &\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\ &\Leftrightarrow (x \in A \cap B) \vee (x \in A \cap C) \\ &\Leftrightarrow x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

Proposition 25. ***Distributive law of set union over intersection***

Let A, B, C be sets.

Then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. Observe that

$$\begin{aligned} x \in A \cup (B \cap C) &\Leftrightarrow (x \in A) \vee (x \in B \cap C) \\ &\Leftrightarrow (x \in A) \vee (x \in B \wedge x \in C) \\ &\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \\ &\Leftrightarrow (x \in A \cup B) \wedge (x \in A \cup C) \\ &\Leftrightarrow x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. □

Theorem 26. ***DeMorgan law for absolute complements***

Let A, B be sets.

Then $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof. Observe that

$$\begin{aligned} x \in \overline{A \cup B} &\Leftrightarrow x \notin A \cup B \\ &\Leftrightarrow \neg(x \in A \cup B) \\ &\Leftrightarrow \neg(x \in A \vee x \in B) \\ &\Leftrightarrow x \notin A \wedge x \notin B \\ &\Leftrightarrow x \in \overline{A} \wedge x \in \overline{B} \\ &\Leftrightarrow x \in \overline{A} \cap \overline{B}. \end{aligned}$$

Therefore, $\overline{A \cup B} = \overline{A} \cap \overline{B}$. $\square$

Theorem 27. ***DeMorgan law for absolute complements***
Let A, B be sets.
Then $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Proof. Observe that

$$\begin{aligned} x \in \overline{A \cap B} &\Leftrightarrow x \notin A \cap B \\ &\Leftrightarrow \neg(x \in A \cap B) \\ &\Leftrightarrow \neg(x \in A \wedge x \in B) \\ &\Leftrightarrow x \notin A \vee x \notin B \\ &\Leftrightarrow x \in \overline{A} \vee x \in \overline{B} \\ &\Leftrightarrow x \in \overline{A} \cup \overline{B}. \end{aligned}$$

Therefore, $\overline{A \cap B} = \overline{A} \cup \overline{B}$. $\square$

Proposition 28. ***Absorption law***
Let A, B be sets.
Then $A \cup (A \cap B) = A$.

Proof. Let $x \in A \cup (A \cap B)$.
Observe that

$$\begin{aligned} x \in A \cup (A \cap B) &\Leftrightarrow x \in A \vee (x \in A \cap B) \\ &\Leftrightarrow x \in A \vee (x \in A \wedge x \in B) \\ &\Leftrightarrow (x \in A \vee x \in A) \wedge (x \in A \vee x \in B) \\ &\Leftrightarrow x \in A \wedge (x \in A \vee x \in B) \\ &\Rightarrow x \in A. \end{aligned}$$

Since $x \in A \cup (A \cap B)$ and $x \in A \cup (A \cap B)$ implies $x \in A$, then $x \in A$.
Therefore, $A \cup (A \cap B) \subset A$.

Let $y \in A$.
Observe that

$$\begin{aligned} y \in A &\Rightarrow y \in A \vee (x \in A \cap B) \\ &\Leftrightarrow y \in A \cup (A \cap B). \end{aligned}$$

Since $y \in A$ and $y \in A$ implies $y \in A \cup (A \cap B)$, then $y \in A \cup (A \cap B)$.
Therefore, $A \subset A \cup (A \cap B)$.
Since $A \cup (A \cap B) \subset A$ and $A \subset A \cup (A \cap B)$, then $A \cup (A \cap B) = A$. □

Proposition 29. *Let A and B be sets.*
Then $A \subset B$ iff $A \cup B = B$.

Proof. We prove if $A \cup B = B$, then $A \subset B$.
Suppose $A \cup B = B$.
Let $x \in A$.
Then either $x \in A$ or $x \in B$, so $x \in A \cup B$.
Since $A \cup B = B$, then $x \in B$.
Therefore, $A \subset B$, as desired. □

Proof. Conversely, we prove if $A \subset B$, then $A \cup B = B$.
Suppose $A \subset B$.
To prove $A \cup B = B$, we prove $A \cup B \subset B$ and $B \subset A \cup B$.

Let $x \in A \cup B$.
Then either $x \in A$ or $x \in B$.
We consider these cases separately.
Case 1: Suppose $x \in A$.
Since $A \subset B$, then $x \in B$.
Case 2: Suppose $x \in B$.
Then $x \in B$.
In either case, we conclude $x \in B$.
Therefore, $A \cup B \subset B$, as desired.

Let $y \in B$.
Then either $y \in B$ or $y \in A$, so either $y \in A$ or $y \in B$.
Hence, $y \in A \cup B$.
Therefore, $B \subset A \cup B$, as desired.

Since $A \cup B \subset B$ and $B \subset A \cup B$, then $A \cup B = B$. □

Proposition 30. *Let A and B be sets.*
Then $A \subset B$ iff $A \cap B = A$.

Proof. We prove if $A \subset B$ then $A \cap B = A$.
Suppose $A \subset B$.

Let $x \in A \cap B$.
Then $x \in A$ and $x \in B$, so $x \in A$.
Therefore, $A \cap B \subset A$.

Let $y \in A$.
Since $A \subset B$, then $y \in B$.
Since $y \in A$ and $y \in B$, then $y \in A \cap B$.
Therefore, $A \subset A \cap B$.

Since $A \cap B \subset A$ and $A \subset A \cap B$, then $A \cap B = A$, as desired.

Conversely, we prove if $A \cap B = A$ then $A \subset B$.
Suppose $A \cap B = A$.
Let $a \in A$.
Since $A = A \cap B$, then $a \in A \cap B$.
Hence, $a \in B$.
Therefore, $A \subset B$, as desired. $\square$

Proposition 31. *Let A and B be sets.*
Then $(A \cap B) \subset A \subset (A \cup B)$.

Proof. We prove $A \cap B \subset A$. Observe that

$$\begin{aligned} x \in A \cap B &\Leftrightarrow x \in A \wedge x \in B \\ &\Rightarrow x \in A. \end{aligned}$$

Therefore, $x \in A \cap B$ implies $x \in A$, so $A \cap B \subset A$.

We prove $A \subset A \cup B$.
Observe that

$$\begin{aligned} x \in A &\Rightarrow x \in A \vee x \in B \\ &\Leftrightarrow x \in A \cup B. \end{aligned}$$

Therefore, $x \in A$ implies $x \in A \cup B$, so $A \subset A \cup B$.

Since $A \cap B \subset A$ and $A \subset A \cup B$, then $(A \cap B) \subset A \subset (A \cup B)$, as desired. $\square$

Corollary 32. *Let A and B be sets.*
Then $A \cap B \subset A \cup B$.

Proof. Since the subset relation is transitive and $A \cap B \subset A$ and $A \subset A \cup B$, then $A \cap B \subset A \cup B$. $\square$

Set difference

Proposition 33. *Let A and B be sets.*

Then

1. $A - B = A \cap \overline{B} = A - (A \cap B)$.
2. $A - A = \emptyset$.
3. $A - \emptyset = A$.
4. $A \subset B$ *iff* $A - B = \emptyset$.

Proof. We prove 1.

First, we prove $A - B = A \cap \overline{B}$.

Observe that

$$\begin{aligned} x \in A - B &\Leftrightarrow x \in A \wedge x \notin B \\ &\Leftrightarrow x \in A \wedge x \in \overline{B} \\ &\Leftrightarrow x \in A \cap \overline{B}. \end{aligned}$$

Therefore, $x \in A - B$ iff $x \in A \cap \overline{B}$, so $A - B = A \cap \overline{B}$.

We prove $A - B = A - A \cap B$.

Observe that

$$\begin{aligned} A - A \cap B &= A \cap \overline{A \cap B} \\ &= A \cap (\overline{A} \cup \overline{B}) \\ &= (A \cap \overline{A}) \cup (A \cap \overline{B}) \\ &= \emptyset \cup (A \cap \overline{B}) \\ &= (A \cap \overline{B}) \cup \emptyset \\ &= A \cap \overline{B} \\ &= A - B. \end{aligned}$$

□

Proof. We prove 2.

Observe that $A - A = A \cap \overline{A} = \emptyset$. □

Proof. We prove 3.

Let U be a universal set.

Then

$$\begin{aligned} A - \emptyset &= A \cap \overline{\emptyset} \\ &= A \cap U \\ &= A. \end{aligned}$$

□

Proof. We prove 4.

We first prove if $A \subset B$, then $A - B = \emptyset$.
Suppose $A \subset B$.
To prove $A - B = \emptyset$, we use proof by contradiction.
Suppose $A - B \neq \emptyset$.
Then there exists an element in $A - B$.
Let x be an element of $A - B$.
Then $x \in A$ and $x \notin B$.
Since $x \in A$ and $A \subset B$, then $x \in B$.
Thus, we have $x \in B$ and $x \notin B$, a contradiction.
Therefore, $A - B = \emptyset$.

Conversely, we prove if $A - B = \emptyset$, then $A \subset B$.
We use proof by contrapositive.
Suppose $A \not\subset B$.
We must prove $A - B \neq \emptyset$.
Since $A \not\subset B$, then there exists an element of A that is not in B.
Let x be some element of A that is not in B.
Then $x \in A$ and $x \notin B$.
Hence, $x \in A - B$, so $A - B$ is not empty.
Therefore, $A - B \neq \emptyset$, as desired. $\square$

Proposition 34. $A \cup B$ ***is a union of 3 disjoint sets.***
Let A and B be sets. Then
$A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$.

Proof. Let U be the universal set for A and B.
Observe that

$$\begin{aligned}
(A - B) \cup (A \cap B) \cup (B - A) &= (A \cap \overline{B}) \cup (A \cap B) \cup (B \cap \overline{A}) \\
&= [A \cap (\overline{B} \cup B)] \cup (B \cap \overline{A}) \\
&= (A \cap U) \cup (B \cap \overline{A}) \\
&= A \cup (B \cap \overline{A}) \\
&= (A \cup B) \cap (A \cup \overline{A}) \\
&= (A \cup B) \cap U \\
&= A \cup B.
\end{aligned}$$

$\square$

Symmetric difference

Proposition 35. ***symmetric difference is logic exclusive OR***
Let A and B be sets. Then
$(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

Proof. Observe that

$$\begin{aligned}
(A \cup B) - (A \cap B) &= (A \cup B) \cap \overline{A \cap B} \\
&= (A \cup B) \cap (\overline{A} \cup \overline{B}) \\
&= [(A \cup B) \cap \overline{A}] \cup [(A \cup B) \cap \overline{B}] \\
&= [\overline{A} \cap (A \cup B)] \cup [\overline{B} \cap (A \cup B)] \\
&= [(\overline{A} \cap A) \cup (\overline{A} \cap B)] \cup [(\overline{B} \cap A) \cup (\overline{B} \cap B)] \\
&= [(A \cap \overline{A}) \cup (B \cap \overline{A})] \cup [(A \cap \overline{B}) \cup (B \cap \overline{B})] \\
&= [\emptyset \cup (B - A)] \cup [(A - B) \cup \emptyset] \\
&= [(B - A) \cup \emptyset] \cup (A - B) \\
&= (B - A) \cup (A - B) \\
&= (A - B) \cup (B - A).
\end{aligned}$$

□

Proposition 36. ***Properties of symmetric difference***

1. $A \triangle B = B \triangle A$ for all sets A, B. (commutative)
2. $(A \triangle B) \triangle C = A \triangle (B \triangle C)$ for all sets A, B, C.(associative)
3. $A \triangle A = \emptyset$ for every set A.
4. $A \triangle \emptyset = A$ for every set A.

Proof. We prove 1.
Let A, B be sets.
Observe that

$$\begin{aligned}
A \triangle B &= (A \cup B) - (A \cap B) \\
&= (B \cup A) - (A \cap B) \\
&= (B \cup A) - (B \cap A) \\
&= B \triangle A.
\end{aligned}$$

□

Proof. We prove 2.
Let A, B, C be sets.

Observe that

$$
\begin{aligned}
(A \triangle B) \triangle C &= ((A \triangle B) - C) \cup (C - (A \triangle B)) \\
&= ((A \triangle B) \cap \overline{C}) \cup (C \cap \overline{A \triangle B}) \\
&= (\overline{C} \cap (A \triangle B)) \cup (C \cap \overline{(A \cup B) - (A \cap B)}) \\
&= (\overline{C} \cap ((A - B) \cup (B - A))) \cup (C \cap \overline{(A \cup B) \cap \overline{(A \cap B)}}) \\
&= (\overline{C} \cap (A - B)) \cup (\overline{C} \cap (B - A)) \cup (C \cap (\overline{A \cup B} \cup (A \cap B))) \\
&= (\overline{C} \cap A \cap \overline{B}) \cup (\overline{C} \cap B \cap \overline{A}) \cup (C \cap \overline{A \cup B}) \cup (C \cap A \cap B) \\
&= (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (C \cap \overline{A} \cap \overline{B}) \cup (A \cap B \cap C) \\
&= (A \cap \overline{B} \cap \overline{C}) \cup (A \cap B \cap C) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap C \cap \overline{B}) \\
&= (A \cap \overline{B \cup C}) \cup (A \cap B \cap C) \cup (\overline{A} \cap (B - C)) \cup (\overline{A} \cap (C - B)) \\
&= (A \cap (\overline{B \cup C} \cup (B \cap C))) \cup (\overline{A} \cap ((B - C) \cup (C - B))) \\
&= (A \cap (\overline{B \cup C} \cup (B \cap C))) \cup (\overline{A} \cap (B \triangle C)) \\
&= (A \cap \overline{(B \cup C) \cap \overline{B \cap C}}) \cup ((B \triangle C) \cap \overline{A}) \\
&= (A \cap \overline{(B \cup C) - (B \cap C)}) \cup ((B \triangle C) - A) \\
&= (A \cap \overline{B \triangle C}) \cup ((B \triangle C) - A) \\
&= (A - (B \triangle C)) \cup ((B \triangle C) - A) \\
&= A \triangle (B \triangle C).
\end{aligned}
$$

□

Proof. We prove 3.
Let A be a set.
Observe that

$$
\begin{aligned}
A \triangle A &= (A \cup A) - (A \cap A) \\
&= A - A \\
&= \emptyset.
\end{aligned}
$$

□

Proof. We prove 4.
Let A be a set.
Observe that

$$
\begin{aligned}
A \triangle \emptyset &= (A \cup \emptyset) - (A \cap \emptyset) \\
&= A - \emptyset \\
&= A.
\end{aligned}
$$

□

Cartesian Product

Proposition 37. *Let (a, b) and (c, d) be ordered pairs.*
Then $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

Proof. We prove if $(a, b) = (c, d)$, then $a = c$ and $b = d$.
Suppose $(a, b) = (c, d)$.
Then $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$.
Either $a = b$ or $a \neq b$.
We consider these cases separately.
Case 1: Suppose $a = b$.
Then

$$\begin{aligned} \{\{c\}, \{c, d\}\} &= \{\{a\}, \{a, b\}\} \\ &= \{\{a\}, \{a, a\}\} \\ &= \{\{a\}, \{a\}\} \\ &= \{\{a\}\}. \end{aligned}$$

Therefore, $\{c\} = \{c, d\}$, so $c = d$.
Observe that

$$\begin{aligned} \{\{a\}\} &= \{\{c\}, \{c, d\}\} \\ &= \{\{c\}, \{c, c\}\} \\ &= \{\{c\}, \{c\}\} \\ &= \{\{c\}\}. \end{aligned}$$

Therefore, $\{a\} = \{c\}$, so $a = c$, as desired.
Observe that

$$\begin{aligned} \{\{b\}\} &= \{\{a\}\} \\ &= \{\{c\}, \{c, d\}\} \\ &= \{\{d\}, \{d, d\}\} \\ &= \{\{d\}, \{d\}\} \\ &= \{\{d\}\}. \end{aligned}$$

Therefore, $\{b\} = \{d\}$, so $b = d$, as desired.
Case 2: Suppose $a \neq b$.
Either $\{a, b\} = \{c\}$ or $\{a, b\} = \{c, d\}$.
Since $a \neq b$, then the set $\{a, b\}$ contains exactly two elements.
Since $\{c\}$ contains exactly one element, then $\{a, b\} \neq \{c\}$.
Therefore, $\{a, b\} = \{c, d\}$.
Either $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$.
Since $\{a, b\}$ contains exactly two elements and $\{a, b\} = \{c, d\}$, then the set $\{c, d\}$ contains exactly two elements.
Hence, $\{a\} \neq \{c, d\}$.

Therefore, $\{a\} = \{c\}$, so $a = c$, as desired.
Since $\{c, d\} = \{a, b\} = \{c, b\}$, then $b = d$, as desired.
Conversely, we prove if $a = c$ and $b = d$, then $(a, b) = (c, d)$.
Suppose $a = c$ and $b = d$.
Then

$$\begin{aligned} (a, b) &= \{\{a\}, \{a, b\}\} \\ &= \{\{c\}, \{c, b\}\} \\ &= \{\{c\}, \{c, d\}\} \\ &= (c, d). \end{aligned}$$

Therefore, $(a, b) = (c, d)$, as desired. □

Proposition 38. *Let A, B be finite sets.*
Then $|A \times B| = |A||B|$.

Proof. TODO □

Proposition 39. ***Domination law for cartesian product***
$A \times \emptyset = \emptyset \times A = \emptyset$ for every set A.

Proof. Let A be a set.
Suppose for the sake of contradiction that $A \times \emptyset \neq \emptyset$.
Then the set $A \times \emptyset$ is not empty, so there exists some element x of $A \times \emptyset$.
Thus there exist $a \in A$ and $b \in \emptyset$ such that $x = (a, b)$.
Since $\emptyset$ is empty, then $b \notin \emptyset$.
Thus, $b \in \emptyset$ and $b \notin \emptyset$, a contradiction.
Hence, $A \times \emptyset$ must be empty, so $A \times \emptyset = \emptyset$.

Similarly, suppose for the sake of contradiction that $\emptyset \times A \neq \emptyset$.
Then the set $\emptyset \times A$ is not empty, so there exists an element y of $\emptyset \times A$.
Thus, there exist $c \in \emptyset$ and $d \in A$ such that $y = (c, d)$.
Since $\emptyset$ is empty, then $c \notin \emptyset$.
Thus, $c \in \emptyset$ and $c \notin \emptyset$, a contradiction.
Hence, $\emptyset \times A$ must be empty, so $\emptyset \times A = \emptyset$.

Therefore, $A \times \emptyset = \emptyset = \emptyset \times A$. □

Proposition 40. ***Distributive properties of cartesian product***
Let A, B, C be sets. Then
1. *$A \times (B \cup C) = (A \times B) \cup (A \times C)$. (left distributive $\times$ over $\cup$)*
2. *$(B \cup C) \times A = (B \times A) \cup (C \times A)$. (right distributive $\times$ over $\cup$)*
3. *$A \times (B \cap C) = (A \times B) \cap (A \times C)$. (left distributive $\times$ over $\cap$)*
4. *$(B \cap C) \times A = (B \times A) \cap (C \times A)$. (right distributive $\times$ over $\cap$)*
5. *$(B - C) \times A = (B \times A) - (C \times A)$. (right distributive $\times$ over $-$)*

Proof. We prove 1.

Let (x, y) be arbitrary.

Then

$$\begin{aligned}
(x,y) \in A \times (B \cup C) &\Leftrightarrow (x \in A) \wedge (y \in B \cup C) \\
&\Leftrightarrow (x \in A) \wedge (y \in B \vee y \in C) \\
&\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \\
&\Leftrightarrow (x,y) \in (A \times B) \vee (x,y) \in (A \times C) \\
&\Leftrightarrow (x,y) \in (A \times B) \cup (A \times C).
\end{aligned}$$

□

Proof. We prove 2.

Let (x, y) be arbitrary.

Then

$$\begin{aligned}
(x,y) \in (B \cup C) \times A &\Leftrightarrow (x \in B \cup C) \wedge (y \in A) \\
&\Leftrightarrow (x \in B \vee x \in C) \wedge (y \in A) \\
&\Leftrightarrow (x \in B \wedge y \in A) \vee (x \in C \wedge y \in A) \\
&\Leftrightarrow (x,y) \in (B \times A) \vee (x,y) \in C \times A \\
&\Leftrightarrow (x,y) \in (B \times A) \cup (C \times A).
\end{aligned}$$

□

Proof. We prove 3.

Let (x, y) be arbitrary.

Then

$$\begin{aligned}
(x,y) \in A \times (B \cap C) &\Leftrightarrow (x \in A) \wedge (y \in B \cap C) \\
&\Leftrightarrow (x \in A) \wedge (y \in B \wedge y \in C) \\
&\Leftrightarrow (x \in A \wedge x \in A) \wedge (y \in B \wedge y \in C) \\
&\Leftrightarrow x \in A \wedge (x \in A \wedge y \in B) \wedge y \in C \\
&\Leftrightarrow x \in A \wedge (y \in B \wedge x \in A) \wedge y \in C \\
&\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C) \\
&\Leftrightarrow (x,y) \in (A \times B) \wedge (x,y) \in (A \times C) \\
&\Leftrightarrow (x,y) \in (A \times B) \cap (A \times C).
\end{aligned}$$

□

Proof. We prove 4.

Let (x, y) be arbitrary.

Then

$$
\begin{aligned}
(x,y) \in (B \cap C) \times A &\Leftrightarrow x \in (B \cap C) \wedge (y \in A) \\
&\Leftrightarrow (x \in B \wedge x \in C) \wedge (y \in A) \\
&\Leftrightarrow (x \in B \wedge x \in C) \wedge (y \in A \wedge y \in A) \\
&\Leftrightarrow x \in B \wedge (x \in C \wedge y \in A) \wedge y \in A \\
&\Leftrightarrow x \in B \wedge (y \in A \wedge x \in C) \wedge y \in A \\
&\Leftrightarrow (x \in B \wedge y \in A) \wedge (x \in C \wedge y \in A) \\
&\Leftrightarrow (x,y) \in (B \times A) \wedge (x,y) \in (C \times A) \\
&\Leftrightarrow (x,y) \in (B \times A) \cap (C \times A).
\end{aligned}
$$

□

Proof. We prove 5.

To prove $(B - C) \times A \subset (B \times A) - (C \times A)$, let $s \in (B - C) \times A$.

To prove $s \in (B \times A) - (C \times A)$, we must prove $s \in B \times A$ and $s \notin C \times A$.

Since $s \in (B - C) \times A$, then there exist $x \in B - C$ and $y \in A$ such that $s = (x, y)$.

Since $x \in B - C$, then $x \in B$.

Since $x \in B$ and $y \in A$, then $s = (x, y) \in B \times A$.

We will be done if we can show $s \notin C \times A$.

Suppose for the sake of contradiction that $s \in C \times A$.

Then $x \in C$ and $y \in A$, since $s = (x, y)$.

Since $x \in B - C$, then $x \notin C$.

Thus, we have $x \in C$ and $x \notin C$, a contradiction.

Therefore, $s \notin C \times A$.

Since $s \in B \times A$ and $s \notin C \times A$, then $s \in (B \times A) - (C \times A)$, so $(B - C) \times A \subset (B \times A) - (C \times A)$, as desired.

To prove $(B \times A) - (C \times A) \subset (B - C) \times A$, let $t \in (B \times A) - (C \times A)$.

To prove $t \in (B - C) \times A$, we must prove there exist $x \in B - C$ and $y \in A$ such that $t = (x, y)$.

Now $t \in (B \times A) - (C \times A)$ implies $t \in B \times A$, so there exist $x \in B$ and $y \in A$ such that $t = (x, y)$.

We will be done if we can show $x \notin C$.

Suppose for the sake of contradiction that $x \in C$.

Since $x \in C$ and $y \in A$, then $t = (x, y) \in C \times A$.

Since $t \in (B \times A) - (C \times A)$, then $t \notin C \times A$.

Thus, we have $t \in C \times A$ and $t \notin C \times A$, a contradiction.

Therefore, $x \notin C$, so $x \in B - C$.

Since $x \in B - C$ and $y \in A$, then $t = (x, y) \in (B - C) \times A$, so $(B \times A) - (C \times A) \subset (B - C) \times A$, as desired.

Since $(B - C) \times A \subset (B \times A) - (C \times A)$ and $(B \times A) - (C \times A) \subset (B - C) \times A$, then $(B - C) \times A = (B \times A) - (C \times A)$, as desired. □

Proposition 41. *Let A, B be sets.*

If $A \times B = \emptyset$, then either $A = \emptyset$ or $B = \emptyset$.

Solution. The hypothesis is: $A \times B = \emptyset$

The conclusion is: either $A = \emptyset$ or $B = \emptyset$

Let $P : A \times B = \emptyset$.

Let $Q : A = \emptyset$.

Let $R : B = \emptyset$.

Then we must prove $P \to (Q \vee R)$.

Direct proof won't help here because we have a disjunction.

So, let's try indirect proof.

Since $P \to (Q \vee R) \Leftrightarrow (P \wedge \neg Q) \to R$, we could prove by negating one of Q and R and proving the other.

However, a cleaner approach would be to use proof by contrapositive.

Therefore, we assume the negation of the conclusion.

Hence, we assume $\neg(Q \vee R) \Leftrightarrow \neg Q \wedge \neg R$.

Thus, we assume $A \neq \emptyset$ and $B \neq \emptyset$.

We must prove the negation of the hypothesis.

Thus, we must prove $A \times B \neq \emptyset$. □

Proof. We use proof by contrapositive.

Suppose $A \neq \emptyset$ and $B \neq \emptyset$.

We must prove $A \times B \neq \emptyset$.

Since A and B are not empty, let a and b be some elements of A and B, respectively.

Then $a \in A$ and $b \in B$.

Hence, $(a, b) \in A \times B$, so $A \times B$ is not empty.

Therefore, $A \times B \neq \emptyset$, as desired. □

Proposition 42. *Let A, B be sets.*

If $A \times B = B \times A$, then either $A = \emptyset$ or $B = \emptyset$ or $A = B$.

Solution. The hypothesis is: $A \times B = B \times A$

The conclusion is: either $A = \emptyset$ or $B = \emptyset$ or $A = B$

Let $P : A \times B = B \times A$

Let $Q : A = \emptyset$

Let $R : B = \emptyset$.

Let $S : A = B$.

Then we must prove $P \to (Q \vee R \vee S)$.

We know that $P \to (Q \vee R \vee S) \Leftrightarrow (P \wedge \neg Q \wedge \neg R) \to S$.

Thus, we may prove indirectly by proving $A \times B = B \times A$ and $A \neq \emptyset$ and $B \neq \emptyset$ imply $A = B$. □

Proof. Suppose $A \times B = B \times A$ and $A \neq \emptyset$ and $B \neq \emptyset$.

Let $x \in A$.

Since $B \neq \emptyset$, then there exists $b \in B$.

Since $x \in A$ and $b \in B$, then $(x, b) \in A \times B$.

Since $A \times B = B \times A$, then $(x, b) \in B \times A$, so $x \in B$.
Therefore, $A \subset B$.

Let $y \in B$.
Since $A \neq \emptyset$, then there exists $a \in A$.
Since $y \in B$ and $a \in A$, then $(y, a) \in B \times A$.
Since $B \times A = A \times B$, then $(y, a) \in A \times B$, so $y \in A$.
Therefore, $B \subset A$.
Since $A \subset B$ and $B \subset A$, then $A = B$, as desired. □

Proposition 43. ***left cancellation law***
Let A, B, C be sets.
If $A \times B = A \times C$ and $A \neq \emptyset$, then $B = C$.

Proof. Suppose $A \times B = A \times C$ and $A \neq \emptyset$.
Since $A \neq \emptyset$, then A is not empty, so let a be some element of A.
Let $b \in B$.
Since $a \in A$ and $b \in B$, then $(a, b) \in A \times B$.
Since $A \times B = A \times C$, then $(a, b) \in A \times C$.
Hence, $b \in C$.
Therefore, if $b \in B$, then $b \in C$, so $B \subset C$.
Let $c \in C$.
Since $a \in A$ and $c \in C$, then $(a, c) \in A \times C$.
Since $A \times C = A \times B$, then $(a, c) \in A \times B$.
Hence, $c \in B$.
Therefore, if $c \in C$, then $c \in B$, so $C \subset B$.
Since $B \subset C$ and $C \subset B$, then $B = C$, as desired. □

Proposition 44. ***right cancellation law***
Let A, B, C be sets.
If $B \times A = C \times A$ and $A \neq \emptyset$, then $B = C$.

Proof. Suppose $B \times A = C \times A$ and $A \neq \emptyset$.
Since $A \neq \emptyset$, then A is not empty, so let a be some element of A.
Let $b \in B$.
Since $b \in B$ and $a \in A$, then $(b, a) \in B \times A$.
Since $B \times A = C \times A$, then $(b, a) \in C \times A$.
Hence, $b \in C$.
Thus, $b \in B$ implies $b \in C$, so $B \subset C$.
Let $c \in C$.
Since $c \in C$ and $a \in A$, then $(c, a) \in C \times A$.
Since $C \times A = B \times A$, then $(c, a) \in B \times A$.
Hence, $c \in B$.
Thus, $c \in C$ implies $c \in B$, so $C \subset B$.
Since $B \subset C$ and $C \subset B$, then $B = C$. □

Indexed Sets

Proposition 45. ***DeMorgan law for relative complements***

For all sets A, B, C,

1. $A - (B \cup C) = (A - B) \cap (A - C)$. (Complement of union = intersection of complements).

2. $A - (B \cap C) = (A - B) \cup (A - C)$. (Complement of intersection = union of complements).

Proof. We prove 1.

Observe that

$$\begin{aligned}
x \in A - (B \cup C) &\Leftrightarrow x \in A \wedge x \notin B \cup C \\
&\Leftrightarrow x \in A \wedge \neg(x \in B \cup C) \\
&\Leftrightarrow x \in A \wedge \neg(x \in B \vee x \in C) \\
&\Leftrightarrow x \in A \wedge (x \notin B \wedge x \notin C) \\
&\Leftrightarrow (x \in A \wedge x \in A) \wedge (x \notin B \wedge x \notin C) \\
&\Leftrightarrow x \in A \wedge (x \in A \wedge x \notin B) \wedge x \notin C \\
&\Leftrightarrow x \in A \wedge (x \notin B \wedge x \in A) \wedge x \notin C \\
&\Leftrightarrow (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C) \\
&\Leftrightarrow (x \in A - B) \wedge (x \in A - C) \\
&\Leftrightarrow x \in (A - B) \cap (A - C).
\end{aligned}$$

Hence, $x \in A - (B \cup C)$ iff $x \in (A - B) \cap (A - C)$.

Therefore, $A - (B \cup C) = (A - B) \cap (A - C)$. □

Proof. We prove 2.

Observe that

$$\begin{aligned}
x \in A - (B \cap C) &\Leftrightarrow x \in A \wedge x \notin B \cap C \\
&\Leftrightarrow x \in A \wedge \neg(x \in B \cap C) \\
&\Leftrightarrow x \in A \wedge \neg(x \in B \wedge x \in C) \\
&\Leftrightarrow x \in A \wedge (x \notin B \vee x \notin C) \\
&\Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) \\
&\Leftrightarrow (x \in A - B) \vee (x \in A - C) \\
&\Leftrightarrow x \in (A - B) \cup (A - C).
\end{aligned}$$

Hence, $x \in A - (B \cap C)$ iff $x \in (A - B) \cup (A - C)$.

Therefore, $A - (B \cap C) = (A - B) \cup (A - C)$. □

Theorem 46. ***Generalized DeMorgan law***

Let S be a set.

Let $\{A_i : i \in I\}$ be a family of sets indexed by a set I.

Then

1. $S - \cup_{i\in I} A_i = \cap_{i\in I}(S - A_i)$. (Complement of union = intersection of complements)

2. $S - \cap_{i\in I} A_i = \cup_{i\in I}(S - A_i)$. (Complement of intersection = union of complements)

Proof. We prove 1.

Observe that

$$\begin{aligned}
x \in (S - \cup_{i\in I} A_i) &\Leftrightarrow x \in S \wedge x \notin \cup_{i\in I} A_i \\
&\Leftrightarrow x \in S \wedge \neg(x \in \cup_{i\in I} A_i) \\
&\Leftrightarrow x \in S \wedge \neg(\exists i \in I)(x \in A_i) \\
&\Leftrightarrow x \in S \wedge (\forall i \in I)(x \notin A_i) \\
&\Leftrightarrow (\forall i \in I)(x \in S) \wedge (\forall i \in I)(x \notin A_i) \\
&\Leftrightarrow (\forall i \in I)(x \in S \wedge x \notin A_i) \\
&\Leftrightarrow (\forall i \in I)(x \in (S - A_i)) \\
&\Leftrightarrow x \in \cap_{i\in I}(S - A_i).
\end{aligned}$$

□

Proof. We prove 2.

Observe that

$$\begin{aligned}
x \in (S - \cap_{i\in I} A_i) &\Leftrightarrow x \in S \wedge x \notin \cap_{i\in I} A_i \\
&\Leftrightarrow x \in S \wedge \neg(x \in \cap_{i\in I} A_i) \\
&\Leftrightarrow x \in S \wedge \neg(\forall i \in I)(x \in A_i) \\
&\Leftrightarrow x \in S \wedge (\exists i \in I)(x \notin A_i) \\
&\Leftrightarrow (\exists i \in I)(x \in S \wedge x \notin A_i) \\
&\Leftrightarrow (\exists i \in I)(x \in (S - A_i)) \\
&\Leftrightarrow x \in \cup_{i\in I}(S - A_i).
\end{aligned}$$

□

Theorem 47. ***Generalized DeMorgan law***

If $A_1, A_2, ...A_n$ are sets in some universal set U, then $\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$ for $n \geq 2$.

(Complement of the intersection of sets = union of the complements of sets).

Solution. We prove by induction.

The statement S_n is $\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$.

The statement S_k is $\overline{\bigcap_{i=1}^{k} A_i} = \bigcup_{i=1}^{k} \overline{A_i}$.

The statement S_{k+1} is $\overline{\bigcap_{i=1}^{k+1} A_i} = \bigcup_{i=1}^{k+1} \overline{A_i}$.

If we try weak induction then we must prove $S_k \Rightarrow S_{k+1}$ for $k \geq 2$.

However, we realize this is not sufficient to prove S_{k+1}. We really need to assume S_{k-1} is also true.

This suggests using strong induction instead.

The statement S_{k-1} is $\overline{A_1 \cap A_2 \cap ... \cap A_{k-1}} = \overline{A_1} \cup \overline{A_2} \cup ... \cup \overline{A_{k-1}}$.
The statement S_k is $\overline{A_1 \cap A_2 \cap ... \cap A_{k-1} \cap A_k} = \overline{A_1} \cup \overline{A_2} \cup ... \cup \overline{A_{k-1}} \cup \overline{A_k} = \overline{A_1 \cap A_2 \cap ... \cap A_{k-1}} \cup \overline{A_k}$.
The statement S_{k+1} is $\overline{A_1 \cap A_2 \cap ... \cap A_k \cap A_{k+1}} = \overline{A_1} \cup \overline{A_2} \cup ... \cup \overline{A_k} \cup \overline{A_{k+1}}$.

So for strong induction we must prove $S_{k-1} \wedge S_k \Rightarrow S_{k+1}$ for $k \geq 2$.
Thus for the basis step we must prove S_2. Note we don't have to prove S_1 because set union and intersection are binary operations which require at least two sets.

For the induction step we must prove $S_{k-1} \wedge S_k \Rightarrow S_{k+1}$ for $k \geq 2$. □

Proof. We prove by induction(strong).

Basis: If $n = 2$, then the statement is $\overline{\bigcap_{i=1}^{2} A_i} = \bigcup_{i=1}^{2} \overline{A_i}$.
The left-hand side is $\overline{A_1 \cap A_2}$ and the right-hand side is $\overline{A_1} \cup \overline{A_2}$.
This statement is true for $n = 2$ due to DeMorgan's Law which states that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Induction: Suppose $k \geq 2$.
We must prove $S_{k-1} \wedge S_k \Rightarrow S_{k+1}$ for $k \geq 2$.
We use direct proof.
Suppose $S_{k-1} \wedge S_k, k \geq 2$.
Observe that

$$\begin{aligned}
\overline{A_1 \cap A_2 \cap ... \cap A_k \cap A_{k+1}} &= \overline{(A_1 \cap A_2 \cap ... \cap A_{k-1}) \cap (A_k \cap A_{k+1})} \text{ } \cap \text{ is associative} \\
&= \overline{(A_1 \cap A_2 \cap ... \cap A_{k-1})} \cup \overline{(A_k \cap A_{k+1})} \text{induction hypothesis } S_{k-1} \wedge S_k \\
&= (\overline{A_1} \cup \overline{A_2} \cup ... \cup \overline{A_{k-1}}) \cup \overline{(A_k \cap A_{k+1})} \text{induction hypothesis } S_{k-1} \\
&= (\overline{A_1} \cup \overline{A_2} \cup ... \cup \overline{A_{k-1}}) \cup (\overline{A_k} \cup \overline{A_{k+1}}) \text{ DeMorgan} \\
&= \overline{A_1} \cup \overline{A_2} \cup ... \cup \overline{A_k} \cup \overline{A_{k+1}} \text{ } \cup \text{ is associative}
\end{aligned}$$

Thus S_{k+1} is true so S_{k-1} and S_k imply S_{k+1} for $k \geq 2$.
It follows by induction that $\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$ for $n \geq 2$. □

Theorem 48. *Let $\{A_n : n \in \mathbb{N}\}$ be an infinite collection of sets from a universal set U.*
Let $k \in \mathbb{N}$. Then
1. $A_k \subset \cup_{n=1}^{\infty} A_n$.
2. $\cap_{n=1}^{\infty} A_n \subset A_k$.
3. $\overline{\cap_{n=1}^{\infty} A_n} = \cup_{n=1}^{\infty} \overline{A_n}$. *(Generalized DeMorgan law)*
4. $\overline{\cup_{n=1}^{\infty} A_n} = \cap_{n=1}^{\infty} \overline{A_n}$.
5. *if $B \subset U$, then $\cup_{n=1}^{\infty} A_n \cap B = \cup_{n=1}^{\infty} (A_n \cap B)$. (Generalized distributivity)*
6. *if $B \subset U$, then $\cap_{n=1}^{\infty} A_n \cup B = \cap_{n=1}^{\infty} (A_n \cup B)$.*

Proof. We prove 1.

Let $x \in A_k$.

To prove $x \in \cup_{n=1}^{\infty} A_n$, we must prove $(\exists n \in \mathbb{N})(x \in A_n)$.

Let $n = k$.

Since $k \in \mathbb{N}$, then $n \in \mathbb{N}$.

Since $x \in A_k$ and $k = n$, then $x \in A_n$.

Therefore, $x \in A_n$ for some $n \in \mathbb{N}$, as desired. $\square$

Proof. We prove 2.

Let $x \in \cap_{n=1}^{\infty} A_n$.

Then $x \in A_n$ for all $n \in \mathbb{N}$.

Since $k \in \mathbb{N}$, then $x \in A_k$, as desired. $\square$

Proof. We prove 3.

We must prove $\forall x.(x \in \overline{\cap_{n=1}^{\infty} A_n} \leftrightarrow x \in \cup_{n=1}^{\infty} \overline{A}_n)$ is true.

Observe that

$$\begin{aligned}
x \in \overline{\cap_{n=1}^{\infty} A_n} &\Leftrightarrow x \notin \cap_{n=1}^{\infty} A_n \\
&\Leftrightarrow \neg(x \in \cap_{n=1}^{\infty} A_n) \\
&\Leftrightarrow \neg(\forall n \in \mathbb{N})(x \in A_n) \\
&\Leftrightarrow (\exists n \in \mathbb{N})(x \notin A_n) \\
&\Leftrightarrow (\exists n \in \mathbb{N})(x \in \overline{A}_n) \\
&\Leftrightarrow x \in \cup_{n=1}^{\infty} \overline{A}_n.
\end{aligned}$$

$\square$

Proof. We prove 4.

We must prove $\forall x.(x \in \overline{\cup_{n=1}^{\infty} A_n} \leftrightarrow x \in \cap_{n=1}^{\infty} \overline{A}_n)$ is true.

Observe that

$$\begin{aligned}
x \in \overline{\cup_{n=1}^{\infty} A_n} &\Leftrightarrow x \notin \cup_{n=1}^{\infty} A_n \\
&\Leftrightarrow \neg(x \in \cup_{n=1}^{\infty} A_n) \\
&\Leftrightarrow \neg(\exists n \in \mathbb{N})(x \in A_n) \\
&\Leftrightarrow (\forall n \in \mathbb{N})(x \notin A_n) \\
&\Leftrightarrow (\forall n \in \mathbb{N})(x \in \overline{A}_n) \\
&\Leftrightarrow x \in \cap_{n=1}^{\infty} \overline{A}_n.
\end{aligned}$$

$\square$

Proof. We prove 5.

Let $B \subset U$.

We must prove $x \in \cup_{n=1}^{\infty} A_n \cap B \leftrightarrow x \in \cup_{n=1}^{\infty}(A_n \cap B)$ is true.

Observe that

$$\begin{aligned}
x \in \cup_{n=1}^{\infty} A_n \cap B &\Leftrightarrow x \in \cup_{n=1}^{\infty} A_n \wedge x \in B \\
&\Leftrightarrow (\exists n \in \mathbb{N})(x \in A_n) \wedge (x \in B) \\
&\Leftrightarrow (\exists n \in \mathbb{N})(x \in A_n \wedge x \in B) \\
&\Leftrightarrow (\exists n \in \mathbb{N})(x \in A_n \cap B) \\
&\Leftrightarrow x \in \cup_{n=1}^{\infty} (A_n \cap B).
\end{aligned}$$

□

Proof. We prove 6.

Let $B \subset U$.

We must prove $x \in \cap_{n=1}^{\infty} A_n \cup B \leftrightarrow x \in \cap_{n=1}^{\infty} (A_n \cup B)$ is true.

Observe that

$$\begin{aligned}
x \in \cap_{n=1}^{\infty} A_n \cup B &\Leftrightarrow x \in \cap_{n=1}^{\infty} A_n \vee x \in B \\
&\Leftrightarrow (\forall n \in \mathbb{N})(x \in A_n) \vee (x \in B) \\
&\Leftrightarrow (\forall n \in \mathbb{N})(x \in A_n \vee x \in B) \\
&\Leftrightarrow (\forall n \in \mathbb{N})(x \in A_n \cup B) \\
&\Leftrightarrow x \in \cap_{n=1}^{\infty} (A_n \cup B).
\end{aligned}$$

□

Theorem 49. *Let $\mathcal{S} = \{S_k : k \in \mathbb{N}\}$ be an infinite collection of subsets of a universal set U.*

Then $\cup_{k=1}^{\infty} S_k$ is the least upper bound of $\mathcal{S}$.

Solution. We know that the union of S is a superset of each S_n.

Thus, $(\forall n \in \mathbb{N})(S_n \subset \cup_{k=1}^{\infty} S_k)$.

To prove the union of S is the least upper bound of S, we need to show that any set B in U that satisfies $(\forall n \in \mathbb{N})(S_n \subset B)$ must contain $\cup_{k=1}^{\infty} S_k$.

In other words, we must show that if any set B in U satisfies $(\forall n \in \mathbb{N})(S_n \subset B)$, then $\cup_{k=1}^{\infty} S_k$ is the smallest such set.

Thus, we must prove the statement $(\forall B \in U)[(\forall n \in \mathbb{N})(S_n \subset B) \rightarrow (\cup_{k=1}^{\infty} S_k \subset B)]$. □

Proof. Let B be an arbitrary set in U.

To prove $(\forall n \in \mathbb{N})(S_n \subset B) \rightarrow (\cup_{k=1}^{\infty} S_k \subset B)$, we assume $(\forall n \in \mathbb{N})(S_n \subset B)$.

To prove $\cup_{k=1}^{\infty} S_k \subset B$, we let $x \in \cup_{k=1}^{\infty} S_k$.

We must prove $x \in B$.

Observe that $x \in \cup_{k=1}^{\infty} S_k \Leftrightarrow (\exists k \in \mathbb{N})(x \in S_k)$.

Since $x \in \cup_{k=1}^{\infty} S_k$, then there exists $k \in \mathbb{N}$ such that $x \in S_k$.

Hence, let k be a natural number such that $x \in S_k$ (existential instantiation).

Since $(\forall n \in \mathbb{N})(S_n \subset B)$, then in particular, if we let $n = k$, then $S_k \subset B$.

Since $x \in S_k$ and $S_k \subset B$, then $x \in B$, as desired. □

Theorem 50. *Let $\mathcal{S} = \{S_k : k \in \mathbb{N}\}$ be an infinite collection of subsets of a universal set U.*

Then $\cap_{k=1}^{\infty} S_k$ is the greatest lower bound of $\mathcal{S}$.

Solution. We know that the intersection of S is a subset of each S_n.

Thus, $(\forall n \in \mathbb{N})(\cap_{k=1}^{\infty} S_k \subset S_n)$.

To prove the intersection of S is the greatest lower bound of S, we need to show that any set B in U that satisfies $(\forall n \in \mathbb{N})(B \subset S_n)$ must be contained in $\cap_{k=1}^{\infty} S_k$.

In other words, we must show that if any set B in U satisfies $(\forall n \in \mathbb{N})(B \subset S_n)$, then $\cap_{k=1}^{\infty} S_k$ is the largest such set.

Thus, we must prove the statement $(\forall B \in U)[(\forall n \in \mathbb{N})(B \subset S_n) \to (B \subset \cap_{k=1}^{\infty} S_k)]$. $\square$

Proof. Let B be an arbitrary set in U.

To prove $(\forall n \in \mathbb{N})(B \subset S_n) \to (B \subset \cap_{k=1}^{\infty} S_k)$, we assume $(\forall n \in \mathbb{N})(B \subset S_n)$.

To prove $B \subset \cap_{k=1}^{\infty} S_k$, we assume $x \in B$.

We must prove $x \in \cap_{k=1}^{\infty} S_k$.

Observe that $x \in \cap_{k=1}^{\infty} S_k \Leftrightarrow (\forall k \in \mathbb{N})(x \in S_k)$.

Hence, we must prove $(\forall k \in \mathbb{N})(x \in S_k)$.

To prove this, we let k be an arbitrary natural number.

We must prove $x \in S_k$.

Since $(\forall n \in \mathbb{N})(B \subset S_n)$, then in particular, if we let $n = k$, then $B \subset S_k$.

Since $x \in B$ and $B \subset S_k$, then $x \in S_k$, as desired. $\square$

Theorem 51. *Let $\mathcal{S} = \{S_k : k \in \mathbb{N}\}$ be an infinite collection of sets.*

1. *If $\mathcal{S}$ is increasing, then $\cap_{k=1}^{\infty} S_k = S_1$.*
2. *If $\mathcal{S}$ is decreasing, then $\cup_{k=1}^{\infty} S_k = S_1$.*
3. *If $\mathcal{S}$ is mutually disjoint, then $\cap_{k=1}^{\infty} S_k = \emptyset$.*

Proof. To prove 1, we assume $\mathcal{S}$ is increasing.

To prove $\cap_{k=1}^{\infty} S_k = S_1$, we prove both $\cap_{k=1}^{\infty} S_k \subset_1$ and $S_1 \subset \cap_{k=1}^{\infty} S_k$.

We prove $\cap_{k=1}^{\infty} S_k \subset S_1$.

To prove this, we assume $x \in \cap_{k=1}^{\infty} S_k$.

We must prove $x \in S_1$.

Since $x \in \cap_{k=1}^{\infty} S_k$, then $(\forall k \in \mathbb{N})(x \in S_k)$.

Hence, in particular, if we let $k = 1$, then $x \in S_1$, as desired.

We prove $S_1 \subset \cap_{k=1}^{\infty} S_k$.

To prove this, we assume $x \in S_1$.

To prove $x \in \cap_{k=1}^{\infty} S_k$, we must prove $(\forall k \in \mathbb{N})(x \in S_k)$.

Let $k \in \mathbb{N}$.

We must prove $x \in S_k$.

Either $k = 1$ or $k > 1$.

We consider these cases separately.

Case 1: Suppose $k = 1$.

By assumption $x \in S_1$, so $x \in S_k$.

Case 2: Suppose $k > 1$.
Then $1 < k$.
Since S is increasing, then $(\forall i \in \mathbb{N})(\forall j \in \mathbb{N})(i < j \to S_i \subset S_j)$.
Hence, in particular, if we let $i = 1$ and $j = k$, then $1 < k \Rightarrow S_1 \subset S_k$.
Since $1 < k$ and $1 < k \Rightarrow S_1 \subset S_k$, then $S_1 \subset S_k$.
Since $x \in S_1$ and $S_1 \subset S_k$, then $x \in S_k$.
Both cases show $x \in S_k$, as desired. $\square$

Proof. To prove 2, we assume $\mathcal{S}$ is decreasing.
To prove $\cup_{k=1}^{\infty} S_k = S_1$, we prove $\cup_{k=1}^{\infty} S_k \subset S_1$ and $S_1 \subset \cup_{k=1}^{\infty} S_k$.
We prove $S_1 \subset \cup_{k=1}^{\infty} S_k$.
To prove this, we assume $x \in S_1$.
To prove $x \in \cup_{k=1}^{\infty} S_k$, we must prove $x \in S_k$ for some natural number k.
Let $k = 1$.
Since $x \in S_1$, then $x \in S_k$, as desired.
We prove $\cup_{k=1}^{\infty} S_k \subset S_1$.
To prove this we assume $x \in \cup_{k=1}^{\infty} S_k$.
We must prove $x \in S_1$.
Since $x \in \cup_{k=1}^{\infty} S_k$, then $x \in S_k$ for some natural number k.
Either $k = 1$ or $k > 1$.
We consider these cases separately.
Case 1: Suppose $k = 1$.
Since $x \in S_k$, then $x \in S_1$.
Case 2: Suppose $k > 1$.
Then $1 < k$.
Since S is decreasing, then $i < j \Rightarrow S_j \subset S_i$ for any natural numbers i, j.
Hence, in particular, if we let $i = 1$ and $j = k$, then $1 < k \Rightarrow S_k \subset S_1$.
Since $1 < k$ and $1 < k \Rightarrow S_k \subset S_1$, then $S_k \subset S_1$.
Since $x \in S_k$ and $S_k \subseteq S_1$, then $x \in S_1$.
Both cases show $x \in S_1$, as desired. $\square$

Proof. To prove 3), we assume $\mathcal{S}$ is mutually disjoint.
To prove $\cap_{k=1}^{\infty} S_k = \emptyset$, we use proof by contradiction.
Suppose $\cap_{k=1}^{\infty} S_k \neq \emptyset$.
Then there exists x in the domain of discourse such that $x \in \cap_{k=1}^{\infty} S_k$.
Let $x \in \cap_{k=1}^{\infty} S_k$.
Let $m, n \in \mathbb{N}$ such that $m \neq n$.
Since $x \in \cap_{k=1}^{\infty} S_k$, then $x \in S_k$ for every $k \in \mathbb{N}$.
Hence, in particular, $x \in S_m$ and $x \in S_n$.
Thus, $x \in S_m \cap S_n$, so $S_m \cap S_n \neq \emptyset$.
Since S is mutually disjoint ,then for every $i, j \in \mathbb{N}$, if $i \neq j$, then $S_i \cap S_j = \emptyset$.
Let $i = m, j = n$.
Then $m \neq n \Rightarrow S_m \cap S_n = \emptyset$.
Since $m \neq n$ and $m \neq n \Rightarrow S_m \cap S_n = \emptyset$, then we conclude $S_m \cap S_n = \emptyset$.
Thus, we have $S_m \cap S_n \neq \emptyset$ and $S_m \cap S_n = \emptyset$, a contradiction.
Therefore, $\cap_{k=1}^{\infty} S_k = \emptyset$, as desired. $\square$

Proposition 52. *Let $\mathcal{S}$ be an arbitrary collection of sets. Then*

1. $X \subset \cup\mathcal{S}$ for each $X \in \mathcal{S}$.

2. If A is a set and $S \subset A$ for all $S \in \mathcal{S}$, then $\cup\mathcal{S} \subset A$.

Proof. We prove 1.

Let $X \in \mathcal{S}$.
We must prove $X \subset \cup\mathcal{S}$.
Either $X = \emptyset$ or $X \neq \emptyset$.
We consider these cases separately.
Case 1: Suppose $S = \emptyset$.
Since the empty set is a subset of every set, then in particular, $\emptyset \subset \cup\mathcal{S}$.
Therefore, $X \subset \cup\mathcal{S}$.
Case 2: Suppose $X \neq \emptyset$.
Then there is at least one element in X.
Let $x \in X$.
Since there exists $X \in \mathcal{S}$ such that $x \in X$, then $x \in \cup\mathcal{S}$.
Therefore, $X \subset \cup\mathcal{S}$.
Thus, in all cases, $X \subset \cup\mathcal{S}$, as desired. □

Proof. We prove 2.

Let A be a set such that $S \subset A$ for all $S \in \mathcal{S}$.
We must prove $\cup\mathcal{S} \subset A$.
Either $\cup\mathcal{S} = \emptyset$ or $\cup\mathcal{S} \neq \emptyset$.
We consider these cases separately.
Case 1: Suppose $\cup\mathcal{S} = \emptyset$.
Since the empty set is a subset of every set, then in particular, $\emptyset \subset A$.
Therefore, $\cup\mathcal{S} \subset A$.
Case 2: Suppose $\cup\mathcal{S} \neq \emptyset$.
Then there is at least one element in $\cup\mathcal{S}$.
Let $x \in \cup\mathcal{S}$.
Then there exists $X \in \mathcal{S}$ such that $x \in X$.
Since $X \in \mathcal{S}$, then $X \subset A$.
Since $x \in X$ and $X \subset A$, then $x \in A$.
Therefore, $\cup\mathcal{S} \subset A$.
Thus, in all cases, $\cup\mathcal{S} \subset A$, as desired. □

Cardinality of Sets

Proposition 53. ***numeric equivalence is an equivalence relation.***

Let $\sim$ be the numeric equivalence relation defined on the collection of all subsets of a given universal set U.

Let A, B, C be sets. Then

1. $A \sim A$. (reflexive)

2. If $A \sim B$, then $B \sim A$. (symmetric)

3. If $A \sim B$ and $B \sim C$, then $A \sim C$. (transitive)

Proof. We prove $A \sim A$.

Let $f : A \to A$ be the map defined by $f(a) = a$ for all $a \in A$.
Then f is the identity map on A.
Since the identity map is bijective, then f is bijective.
Therefore, there exists a bijective map $f : A \to A$, so $A \sim A$. $\square$

Proof. We prove if $A \sim B$, then $B \sim A$.

Suppose $A \sim B$.
Then there exists a bijective map $f : A \to B$.
Since $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.
Thus, there exists a bijective map $f^{-1} : B \to A$, so $B \sim A$. $\square$

Proof. We prove if $A \sim B$ and $B \sim C$, then $A \sim C$.

Suppose $A \sim B$ and $B \sim C$.
Since $A \sim B$, then there exists a bijective map $f : A \to B$.
Since $B \sim C$, then there exists a bijective map $g : B \to C$.
Since f and g are bijective, then the composition $g \circ f : A \to C$ is bijective.
Therefore, there exists a bijective map $g \circ f : A \to C$, so $A \sim C$. $\square$

Proposition 54. *Let S be a set.*
Then $S \sim \emptyset$ iff $S = \emptyset$.

Proof. We prove if $S = \emptyset$, then $S \sim \emptyset$.

Suppose $S = \emptyset$.
Since $S \sim S$, then $S \sim \emptyset$. $\square$

Proof. Conversely, we prove if $S \sim \emptyset$, then $S = \emptyset$ by contrapostive.

Suppose $S \neq \emptyset$.
Then there exists $x \in S$.
Suppose there exists a function $f : S \to \emptyset$.
Then $f(x) \in \emptyset$.
But, this contradicts the fact that $\emptyset$ is empty.
Thus, there is no function $f : S \to \emptyset$.
Therefore, there is no bijective function $f : S \to \emptyset$.
Hence, $S \not\sim \emptyset$, as desired. $\square$

Proposition 55. *Let S be a set.*
Then S is finite iff either $S = \emptyset$ or there is a bijection $f : S \to \{1, ..., n\}$ for some $n \in \mathbb{N}$.

Proof. Let n be a nonnegative integer.

Then $n \in \mathbb{Z}$ and $n \geq 0$, so either $n > 0$ or $n = 0$.
We consider these cases separately.
Case 1: Suppose $n = 0$.
Then S is finite iff $S \sim \mathbb{N}_0 = \emptyset$.
Thus, S is finite iff $S \sim \emptyset$ iff $S = \emptyset$.
Therefore, S is finite iff $S = \emptyset$.
Case 2: Suppose $n > 0$.

Then S is finite iff $S \sim \mathbb{N}_n$.
Since $n \in \mathbb{Z}$ and $n > 0$, then $n \in \mathbb{N}$.
Therefore, S is finite iff there exists $n \in \mathbb{N}$ such that $S \sim \mathbb{N}_n$.
Thus, a set S is finite iff either $S = \emptyset$ or there exists $n \in \mathbb{N}$ such that $S \sim \mathbb{N}_n$.
Therefore, a set S is finite iff either $S = \emptyset$ or there exist $n \in \mathbb{N}$ and a bijective function $f : S \to N_n$. ☐

Lemma 56. *For every $n \in \mathbb{N}$, $n \in \mathbb{N}_n$.*

Proof. Let $n \in \mathbb{N}$.
Then $n \geq 1$.
Since $n \in \mathbb{N}$ and $1 \leq n$ and $n = n$, then $n \in \mathbb{N}_n$, as desired. ☐

Proposition 57. *For every $n \in \mathbb{N}$, $\mathbb{N}_n$ is finite.*

Proof. Let $n \in \mathbb{N}$.
Then $n \in \mathbb{N}_n$, so $\mathbb{N}_n \neq \emptyset$.
Let $I : \mathbb{N}_n \to \mathbb{N}_n$ be the identity function defined by $I(k) = k$ for each $k \in \mathbb{N}_n$.
Since I is bijective, then $\mathbb{N}_n \sim \mathbb{N}_n$.
Hence, there exists $n \in \mathbb{N}$ such that $\mathbb{N}_n \sim \mathbb{N}_n$.
Therefore, $\mathbb{N}_n$ is finite. ☐

Lemma 58. *For every $n \in \mathbb{N}$, if $n > 1$, then $\mathbb{N}_2 \subset \mathbb{N}_n$.*

Proof. Let $n \in \mathbb{N}$ such that $n > 1$.
We must prove $\mathbb{N}_2 \subset \mathbb{N}_n$.
Since $2 \in \mathbb{N}_2$, then $\mathbb{N}_2 \neq \emptyset$.
Let $k \in \mathbb{N}_2$.
Then $1 \leq k \leq 2$.
Since $n > 1$, then $n \geq 2$.
Since $1 \leq k \leq 2 \leq n$, then $1 \leq k \leq n$, so $k \in \mathbb{N}_n$.
Therefore, $\mathbb{N}_2 \subset \mathbb{N}_n$. ☐

Lemma 59. *Let A be a set such that $a \in A$.*
Then for all $n \in \mathbb{N}$, if $A \sim \mathbb{N}_n$, then $A - \{a\} \sim \mathbb{N}_{n-1}$.

Proof. Since $a \in A$, then $A \neq \emptyset$.
Let $n \in \mathbb{N}$ such that $A \sim \mathbb{N}_n$.
We must prove $A - \{a\} \sim \mathbb{N}_{n-1}$.
Since $n \in \mathbb{N}$, then $n \geq 1$, so either $n > 1$ or $n = 1$.
We consider these cases separately.
Case 1: Suppose $n = 1$.
We must prove $A - \{a\} \sim \mathbb{N}_0$.
Since $A \sim \mathbb{N}_n = \mathbb{N}_1$, then $A \sim \mathbb{N}_1$, so there exists a bijective function $f : A \to \mathbb{N}_1$.
Hence, f is injective.
Since $a \in A$, then $\{a\} \subset A$ and $A \neq \emptyset$.

Let $x \in A$.
Then $f(x) \in \mathbb{N}_1 = \{1\}$, so $f(x) = 1$.
Since $a \in A$, then $f(a) \in \mathbb{N}_1 = \{1\}$, so $f(a) = 1$.
Hence, $f(x) = 1 = f(a)$.
Since f is injective, then $x = a$, so $x \in \{a\}$.
Thus, $A \subset \{a\}$.
Since $A \subset \{a\}$ and $\{a\} \subset A$, then $A = \{a\}$.
Hence, $A - \{a\} = A - A = A \cap \overline{A} = \emptyset \sim \emptyset = \mathbb{N}_0$, so $A - \{a\} \sim \mathbb{N}_0$, as desired.
Case 2: Suppose $n > 1$.
We must prove $A - \{a\} \sim \mathbb{N}_{n-1}$.
Since $A \sim \mathbb{N}_n$, then $\mathbb{N}_n \sim A$, so there exists a bijective function $f : \mathbb{N}_n \to A$.
Hence, f is injective and surjective.
Since $n \in \mathbb{N}$ and $n > 1$, then by a previous lemma, $\mathbb{N}_2 \subset \mathbb{N}_n$.
Thus, $\{1, 2\} \subset \mathbb{N}_n$, so $1 \in \mathbb{N}_n$ and $2 \in \mathbb{N}_n$.
Since $1 \neq 2$ and f is injective, then $f(1) \neq f(2)$.
Since $f(1) \in A$ and $f(2) \in A$, then A has at least two distinct elements.
Since $a \in A$, then this implies there exists $x \in A$ such that $x \neq a$.
Thus, $x \in A - \{a\}$, so $A - \{a\} \neq \emptyset$.
Since $n > 1$, then $n - 1 > 0$.
Since $n \in \mathbb{N}$ and $n - 1 > 0$, then $n - 1 \in \mathbb{N}$, so $n - 1 \in \mathbb{N}_{n-1}$.
Hence, $\mathbb{N}_{n-1} \neq \emptyset$.
We prove there exists a bijective function $g : \mathbb{N}_{n-1} \to A - \{a\}$.
Let $g : \mathbb{N}_{n-1} \to A - \{a\}$ be a function defined by $g(k) = f(k)$ for each $k \in \mathbb{N}_n$ such that $f(k) \neq a$.

We prove g is injective.
Let $s, t \in \mathbb{N}_n$ such that $g(s) = g(t)$.
Then $g(s) = f(s)$ and $g(t) = f(t)$, so $f(s) = g(s) = g(t) = f(t)$.
Hence, $f(s) = f(t)$.
Since f is injective, then $s = t$, so g is injective.

We prove g is surjective.
Since $A - \{a\} \neq \emptyset$, let $b \in A - \{a\}$.
Then $b \in A$ and $b \neq a$.
Since $b \in A$ and f is surjective, then there exists $k \in \mathbb{N}_n$ such that $f(k) = b$.
Since $f(k) = b$ and $b \neq a$, then $f(k) \neq a$.
Since $k \in \mathbb{N}_n$ and $f(k) \neq a$, then $g(k) = f(k) = b$.
Thus, there exists $k \in \mathbb{N}_n$ such that $g(k) = b$, so g is surjective.
Hence, the function $g : \mathbb{N}_{n-1} \to A - \{a\}$ is bijective, so $\mathbb{N}_{n-1} \sim A - \{a\}$.
Therefore, $A - \{a\} \sim \mathbb{N}_{n-1}$, as desired. $\square$

Theorem 60. ***counting theorem***
Let $m, n \in \mathbb{N}$.
Then $\mathbb{N}_m \sim \mathbb{N}_n$ iff $m = n$.

Proof. We prove if $m = n$, then $\mathbb{N}_m \sim \mathbb{N}_n$.

Suppose $m = n$.

Since $\mathbb{N}_m \sim \mathbb{N}_m$ and $m = n$, then $\mathbb{N}_m \sim \mathbb{N}_n$. $\square$

Proof. Conversely, we prove if $\mathbb{N}_m \sim \mathbb{N}_n$, then $m = n$.

We must prove $(\forall m \in \mathbb{N})(\forall n \in \mathbb{N})(\mathbb{N}_m \sim \mathbb{N}_n \to m = n)$.

This is equivalent to $(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})(\mathbb{N}_m \sim \mathbb{N}_n \to m = n)$.

We prove by induction on n.

Let $S = \{n \in \mathbb{N} : \text{ for all } m \in \mathbb{N}, \text{ if } \mathbb{N}_m \sim \mathbb{N}_n, \text{ then } m = n\ \}$.

Basis:

To prove $1 \in S$, we must prove for all $m \in \mathbb{N}$, if $\mathbb{N}_m \sim \mathbb{N}_1$, then $m = 1$.

Let $m \in \mathbb{N}$.

We prove if $\mathbb{N}_m \sim \mathbb{N}_1$, then $m = 1$ by contrapositive.

Suppose $m \neq 1$.

We must prove $\mathbb{N}_m \not\sim \mathbb{N}_1$.

Let $f : \mathbb{N}_m \to \mathbb{N}_1$ be a function.

Since $1 \in \mathbb{N}_m$ and $\mathbb{N}_1 = \{1\}$, then $f(1) \in \{1\}$, so $f(1) = 1$.

Since $m \in \mathbb{N}_m$ and $\mathbb{N}_1 = \{1\}$, then $f(m) \in \{1\}$, so $f(m) = 1$.

Thus, $f(m) = 1 = f(1)$.

Since $f(m) = f(1)$ and $m \neq 1$, then f is not injective, so f is not bijective.

Hence, for every function $f : \mathbb{N}_m \to \mathbb{N}_1$, f is not bijective, so there is no function $f : \mathbb{N}_m \to \mathbb{N}_1$ such that f is bijective.

Therefore, $\mathbb{N}_m \not\sim \mathbb{N}_1$, so $1 \in S$.

Induction:

Let $k \in S$.

Then $k \in \mathbb{N}$ and for all $m \in \mathbb{N}$, if $\mathbb{N}_m \sim \mathbb{N}_k$, then $m = k$.

To prove $k + 1 \in S$, let $m \in \mathbb{N}$ such that $\mathbb{N}_m \sim \mathbb{N}_{k+1}$.

We must prove $m = k + 1$.

We first prove $\mathbb{N}_{m-1} \sim \mathbb{N}_k$.

Since $m \in \mathbb{N}_m$ and $k + 1 \in \mathbb{N}$, then by the previous lemma, if $\mathbb{N}_m \sim \mathbb{N}_{k+1}$, then $\mathbb{N}_m - \{m\} \sim \mathbb{N}_k$.

Since $\mathbb{N}_m \sim \mathbb{N}_{k+1}$, then $\mathbb{N}_m - \{m\} \sim \mathbb{N}_k$.

Since $m \in \mathbb{N}_m$ and $m \in \mathbb{N}$, then by the previous lemma, if $\mathbb{N}_m \sim \mathbb{N}_m$, then $\mathbb{N}_m - \{m\} \sim \mathbb{N}_{m-1}$.

Since $\mathbb{N}_m \sim \mathbb{N}_m$, then $\mathbb{N}_m - \{m\} \sim \mathbb{N}_{m-1}$, so $\mathbb{N}_{m-1} \sim \mathbb{N}_m - \{m\}$.

Since $\mathbb{N}_{m-1} \sim \mathbb{N}_m - \{m\}$ and $\mathbb{N}_m - \{m\} \sim \mathbb{N}_k$, then $\mathbb{N}_{m-1} \sim \mathbb{N}_k$, so $\mathbb{N}_k \sim \mathbb{N}_{m-1}$.

We next prove $m - 1 \in \mathbb{N}$.

Suppose for the sake of contradiction $m = 1$.

Then $\mathbb{N}_m \sim \mathbb{N}_1$.

Since $m \in \mathbb{N}_m$ and $1 \in \mathbb{N}$, then if $\mathbb{N}_m \sim \mathbb{N}_1$, then $\mathbb{N}_m - \{m\} \sim \mathbb{N}_0$.

Since $\mathbb{N}_m \sim \mathbb{N}_1$, then $\mathbb{N}_m - \{m\} \sim \mathbb{N}_0$, so $\mathbb{N}_m - \{m\} \sim \emptyset$.

Since $m \in \mathbb{N}_m$ and $m \in \mathbb{N}$, then if $\mathbb{N}_m \sim \mathbb{N}_m$, then $\mathbb{N}_m - \{m\} \sim \mathbb{N}_{m-1}$.

Since $\mathbb{N}_m \sim \mathbb{N}_m$, then $\mathbb{N}_m - \{m\} \sim \mathbb{N}_{m-1}$, so $\mathbb{N}_{m-1} \sim \mathbb{N}_m - \{m\}$.

Thus, $\mathbb{N}_{m-1} \sim \mathbb{N}_m - \{m\}$ and $\mathbb{N}_m - \{m\} \sim \emptyset$, so $\mathbb{N}_{m-1} \sim \emptyset$.
Since $\mathbb{N}_k \sim \mathbb{N}_{m-1}$ and $\mathbb{N}_{m-1} \sim \emptyset$, then $\mathbb{N}_k \sim \emptyset$, so $\mathbb{N}_k = \emptyset$.
Since $k \in \mathbb{N}$, then $k \in \mathbb{N}_k$, so $\mathbb{N}_k \neq \emptyset$.
Hence, we have $\mathbb{N}_k = \emptyset$ and $\mathbb{N}_k \neq \emptyset$, a contradiction.
Thus, $m \neq 1$.
Since $m \in \mathbb{N}$, then $m \geq 1$.
Since $m \geq 1$ and $m \neq 1$, then $m > 1$, so $m - 1 > 0$.
Since $m - 1$ is an integer and $m - 1 > 0$, then $m - 1 \in \mathbb{N}$.

Since $m-1 \in \mathbb{N}$ and $\mathbb{N}_{m-1} \sim \mathbb{N}_k$, then by the induction hypothesis, $m-1 = k$, so $m = k + 1$.
Hence, $k + 1 \in S$, so by PMI, $S = \mathbb{N}$.
Therefore, for all $n \in \mathbb{N}$ and for all $m \in \mathbb{N}$, if $\mathbb{N}_m \sim \mathbb{N}_n$, then $m = n$. □

Theorem 61. ***cardinality of a finite set is well defined.***
The cardinality of a finite set is unique.

Proof. Let S be a finite set.
Either $S = \emptyset$ or $S \neq \emptyset$.
We consider these cases separately.
Case 1: Suppose $S = \emptyset$.
Since the cardinality of the empty set is defined to be zero, then the cardinality of S is unique.
Case 2: Suppose $S \neq \emptyset$.
Since S is finite and $S \neq \emptyset$, then there exists $n \in \mathbb{N}$ such that $S \sim \mathbb{N}_n$.
Suppose there exists $m \in \mathbb{N}$ such that $S \sim \mathbb{N}_m$.
To prove the cardinality of S is unique, we must prove $m = n$.
Since $S \sim \mathbb{N}_m$, then $\mathbb{N}_m \sim S$.
Since $\mathbb{N}_m \sim S$ and $S \sim \mathbb{N}_n$, then $\mathbb{N}_m \sim \mathbb{N}_n$.
Therefore, $m = n$, as desired. □

Theorem 62. *Every subset of a finite set is finite.*

Proof. This statement means if A is a finite set and $B \subset A$, then B is finite.
Let $S = \{n \in \mathbb{Z}, n \geq 0 :$ for all sets A, B, if $A \sim \mathbb{N}_n$ and $B \subset A$, then B is finite $\}$.
We prove the statement by proving $S = \{0\} \cup \mathbb{N}$ by induction on n.
Basis:
For $n = 0$.
We must prove for every set A and B, if $A \sim \mathbb{N}_0$ and $B \subset A$, then B is finite.
Let A and B be sets such that $A \sim \mathbb{N}_0$ and $B \subset A$.
We must prove B is finite.
Since $A \sim \mathbb{N}_0 = \emptyset$, then $A \sim \emptyset$, so $A = \emptyset$.
Since $B \subset A$, then $B \subset \emptyset$.
Suppose $B \neq \emptyset$.
Then there exists $b \in B$.
Since $B \subset \emptyset$, then $b \in \emptyset$.

But, $\emptyset$ is empty, so $B = \emptyset$.
Therefore, B is finite, so $0 \in S$.
Induction:
Let $k \in S$.
Then $k \in \mathbb{Z}$ and $k \geq 0$ and for all sets A and B, if $A \sim \mathbb{N}_k$ and $B \subset A$, then B is finite.
To prove $k+1 \in S$, we must prove $k+1 \in \mathbb{Z}$ and $k+1 \geq 0$ and for all sets A and B, if $A \sim \mathbb{N}_{k+1}$ and $B \subset A$, then B is finite.
Since $k \in \mathbb{Z}$, then $k+1 \in \mathbb{Z}$.
Since $k \geq 0$, then $k+1 \geq 1 > 0$, so $k+1 > 0$.
Let A and B be sets such that $A \sim \mathbb{N}_{k+1}$ and $B \subset A$.
We must prove B is finite.
Either $B = A$ or $B \neq A$.
We consider these cases separately.
Case 1: Suppose $B = A$.
Since $k+1 \in \mathbb{Z}$ and $k+1 > 0$, then $k+1 \in \mathbb{N}$.
Since $k+1 \in \mathbb{N}$ and $A \sim \mathbb{N}_{k+1}$, then A is finite.
Since $B = A$, then B is finite.
Case 2: Suppose $B \neq A$.
Since $B = A$ iff $B \subset A$ and $A \subset B$, then $B \neq A$ iff either $B \not\subset A$ or $A \not\subset B$.
Since $B \neq A$, then either $B \not\subset A$ or $A \not\subset B$.
Since $B \subset A$, then $A \not\subset B$, so there exists $a \in A$ such that $a \notin B$.
Since $a \in A$ and $k+1 \in \mathbb{N}$, then by previous lemma, if $A \sim \mathbb{N}_{k+1}$, then $A - \{a\} \sim \mathbb{N}_k$.
Since $A \sim \mathbb{N}_{k+1}$, then $A - \{a\} \sim \mathbb{N}_k$.
Since $B \neq \emptyset$, let $x \in B$.
Since $B \subset A$, then $x \in A$.
Since $x \in B$ and $a \notin B$, then $x \neq a$, so $x \notin \{a\}$.
Thus, $x \in A$ and $x \notin \{a\}$, so $x \in A - \{a\}$.
Hence, $B \subset A - \{a\}$.
Since $A - \{a\} \sim \mathbb{N}_k$ and $B \subset A - \{a\}$, then by the induction hypothesis, B is finite.
Therefore, in either case, B is finite, so $k+1 \in S$.
Hence, by PMI, $S = \mathbb{N}$. ☐

Proof. Let A be a finite set.
Let B be a set such that $B \subset A$.
We must prove B is finite.
Either $B = \emptyset$ or $B \neq \emptyset$.
We consider these cases separately.
Case 1: Suppose $B = \emptyset$.
Since the empty set is finite, then B is finite.
Case 2: Suppose $B \neq \emptyset$.
Then there exists $b \in B$.
Since $B \subset A$, then $b \in A$, so $A \neq \emptyset$.
Since A is finite and $A \neq \emptyset$, then there exists $n \in \mathbb{N}$ such that $A \sim \mathbb{N}_n$.

We prove B is finite by induction on n.
Let $S = \{n \in \mathbb{N} : \text{ if } A \sim \mathbb{N}_n \text{ and } B \subset A, \text{ then } B \text{ is finite } \}$.
Basis:
Suppose $A \sim \mathbb{N}_1$ and $B \subset A$.
Then either $B = \emptyset$ or $B = A$.
If $B = \emptyset$, then B is finite since the empty set is finite.
If $B = A$, then B is finite since A is finite.
Therefore, in either case, B is finite, so $1 \in S$.
Induction:
Let $k \in S$.
Then $k \in \mathbb{N}$ and if $|A| = k$ and $B \subset A$, then B is finite.
To prove $k + 1 \in S$, suppose $|A| = k + 1$ and $B \subset A$.
We must prove B is finite.
Either $B = A$ or $B \neq A$.
We consider these cases separately.
Case 2a: Suppose $B = A$.
Since A is finite and $B = A$, then B is finite.
Case 2b: Suppose $B \neq A$.
Since $B = A$ iff $B \subset A$ and $A \subset B$, then $B \neq A$ iff either $B \not\subset A$ or $A \not\subset B$.
Since $B \neq A$, then either $B \not\subset A$ or $A \not\subset B$.
Since $B \subset A$, then $A \not\subset B$, so there exists $a \in A$ such that $a \notin B$.
Since $|A| = k + 1$, then A has $k + 1$ elements.
Since $a \in A$ and $\{a\}$ has 1 element, then $A - \{a\}$ has k elements, so $|A - \{a\}| = k$.
Since $B \neq \emptyset$, let $x \in B$.
Since $B \subset A$, then $x \in A$.
Since $x \in B$ and $a \notin B$, then $x \neq a$, so $x \notin \{a\}$.
Thus, $x \in A$ and $x \notin \{a\}$, so $x \in A - \{a\}$.
Hence, $B \subset A - \{a\}$.
Since $|A - \{a\}| = k$ and $B \subset A - \{a\}$, then by the induction hypothesis, B is finite.
Therefore, in either case, B is finite, so $k + 1 \in S$.
Hence, by PMI, $S = \mathbb{N}$.
Thus, for every $n \in \mathbb{N}$, if $|A| = n$ and $B \subset A$, then B is finite. $\square$

Theorem 63. ***Pigeonhole Principle***
Let $m, n \in \mathbb{N}$.
Let $A = \{1, 2, ..., m\}$.
Let $B = \{1, 2, ..., n\}$.
1. If $m > n$, then there is no injective function $f : A \to B$. (one hole has at least 2 pigeons flying into it)
2. If $m < n$, then there is no surjective function $f : A \to B$. (at least one pigeonhole remains empty)

Proof. Let $|A| = m$ and $|B| = n$.
Then $m, n \in \mathbb{Z}$.

We prove if $|A| > |B|$, then f is not one to one.
Suppose for the sake of contradiction that $|A| > |B|$ and f is one to one.
Then $m > n$ and if $a \neq b$, then $f(a) \neq f(b)$ for every $a, b \in A$.
Thus, the images of distinct elements in A are distinct.
Let $f(A)$ be the image of A under f.
Since there are m elements in A, then there are at least m elements in $f(A)$.
Hence, $|f(A)| \geq m$, so $m \leq |f(A)|$.
Since $f(A) \subset B$ and B is finite and $|B| = n$, then $|f(A)| \leq n$.
Thus, $m \leq |f(A)|$ and $|f(A)| \leq n$, so $m \leq n$.
Therefore, we have $m > n$ and $m \leq n$, a violation of the trichotomy law.
Hence, $|A| > |B|$ implies f is not one to one. □

Proof. We prove if $m < n$, then there is no surjective function $f : A \to B$ by induction on n.
Let $\mathbb{N}_m = \{1, 2, ..., m\}$.
Let $\mathbb{N}_n = \{1, 2, ..., n\}$.
Define predicate $p(n)$: if $m < n$, then there is no surjective function $f : \mathbb{N}_m \to \mathbb{N}_n$.
We prove $p(n)$ for all $n \in \mathbb{N}$ by induction on n.
Basis:
Since $m \in \mathbb{N}$, then $m \geq 1$, so $m < 1$ is false.
Hence, the conditional 'if $m < 1$, then there is no surjective function $f : \mathbb{N}_m \to \mathbb{N}_1$' is vacuously true.
Therefore, $p(1)$ is true.
Induction:
Let $k \in \mathbb{N}$ such that $p(k)$ is true.
Then if $m < k$, then there is no surjective function $f : \mathbb{N}_m \to \mathbb{N}_k$.
To prove $p(k + 1)$ is true, we must prove if $m < k + 1$, then there is no surjective function $f : \mathbb{N}_m \to \mathbb{N}_{k+1}$.
Suppose $m < k + 1$.
Since $m \in \mathbb{N}$ and $k + 1 \in \mathbb{N}$, then $m \leq k$, so either $m < k$ or $m = k$.

Suppose for the sake of contradiction that $|A| < |B|$ and f is onto.
Then $m < n$ and for every $b \in B$, there exists $a \in A$ such that $f(a) = b$.
Since there are n elements in B, then this implies there exist at least n elements in A.
Hence, $|A| \geq n$, so $m \geq n$.
Thus, we have $m \geq n$ and $m < n$, a violation of the trichotomy law.
Hence, $|A| < |B|$ implies f is not onto. □

Proposition 64. *Let A and B be finite sets and $|A| = |B|$.*
Let $f : A \to B$ be a function.
Then f is injective iff f is surjective.

Proof. Let n be the cardinality of A.

Then $n = |A| = |B|$.

Suppose f is injective.

Let $a_1, a_2 \in A$ such that $a_1 \neq a_2$.

Since f is injective, then $f(a_1) \neq f(a_2)$.

Hence, distinct elements of A map to distinct elements of B.

Since there are n elements of A, then this implies $f(A)$ contains n distinct elements of B.

Hence, $|f(A)| = n = |B|$.

Since B is finite and $f(A) \subset B$ and $|f(A)| = |B|$, then $f(A) = B$.

Therefore, f is surjective.

Conversely, suppose f is surjective.

Then $f(A) = B$.

Hence, $|f(A)| = |B| = n$.

Suppose for the sake of contradiction that f is not injective.

Then there exist $a_1, a_2 \in A$ such that $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.

Thus, $f(A)$ has less than n elements, so $|f(A)| < n$.

Hence, we have $|f(A)| = n$ and $|f(A)| < n$, a contradiction.

Therefore, f is injective. $\square$

Theorem 65. *Let A, B be finite sets.*

Then $|A \times B| = |A||B|$.

Proof. Either A and B are both empty, exactly one of A and B is empty or neither A nor B is empty.

We consider these cases separately.

Case 1: Suppose $A = B = \emptyset$.

Then $|A| = |B| = |\emptyset| = 0$.

Thus $|A \times B| = |\emptyset \times \emptyset| = |\emptyset| = 0 = 0 * 0 = |A||B|$.

Case 2: Suppose exactly one of A and B is empty.

Without loss of generality let A be empty and B not empty.

Since B is finite then $|B| = n$ for some $n \in \mathbb{Z}^+$.

Then $|A \times B| = |\emptyset \times B| = |\emptyset| = 0 = 0 * n = |\emptyset||B| = |A||B|$.

Case 3: Suppose both A and B are not empty.

Then $A \neq \emptyset$ and $B \neq \emptyset$.

Since A, B are finite then $|A| = m$ and $|B| = n$ for some $m, n \in \mathbb{Z}^+$.

Let $(a, b) \in A \times B$.

Then $a \in A$ and $b \in B$.

Hence a is one of the m elements in A and b is one of the n elements in B.

The number of possible different ordered pairs is m choices for the first element and n choices for the second element.

Therefore there are mn different ordered pairs in $A \times B$.

Hence $|A \times B| = mn = |A||B|$. $\square$

Theorem 66. $|\mathbb{Z}| = |\mathbb{N}|$.

Solution. We must prove the size of the set of integers is the same as the size of the set of natural numbers.

This means we must devise a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

So, we can either devise a bijective function $f : \mathbb{Z} \mapsto \mathbb{N}$ or $f : \mathbb{N} \mapsto \mathbb{Z}$, if such a bijective function exists.

Let's choose $f : \mathbb{N} \mapsto \mathbb{Z}$.

We know that a function defined on $\mathbb{N}$ is simply a sequence, so we'd like to find a formula relating the natural numbers to the integers.

We can write each integer, starting with 0 and list the integer and its negation.

So a sequence $a : \mathbb{N} \mapsto Z$ with $a = \{a_n\}_{n=1}^{\infty}$ can be defined whose n^{th} term is

$$a_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{1-n}{2} & \text{if } n \text{ is odd} \end{cases}$$

□

Proof. Define sequence $a : \mathbb{N} \mapsto \mathbb{Z}$ with $a = \{a_n\}_{n=1}^{\infty}$ whose n^{th} term is

$$a_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{1-n}{2} & \text{if } n \text{ is odd} \end{cases}$$

We must prove the sequence a is a bijective function.

We prove a is one to one.

We use contrapositive proof.

Let $m, n \in \mathbb{N}$.

Suppose $a_m = a_n$.

Either m is even or m is odd.

We consider these cases separately.

Case 1: Suppose m is even.

If n is odd, then $n \geq 1$ and $a_n = \frac{1-n}{2}$.

Thus, $a_n \leq 0$.

Since m is even, then $m \geq 2$ and $a_m = m/2$.

Thus, $a_m > 0$.

Since $a_m = a_n$ by assumption, then this means a positive integer equals a nonpositive integer.

But, this is impossible.

Therefore, if m is even, then n cannot be odd.

If n is even, then $a_n = n/2$.

Since m is even, then $a_m = m/2$.

Since $a_m = a_n$, then this implies $m/2 = n/2$, so $m = n$.

Thus, we have proved $a_m = a_n$ implies $m = n$ if m is even.

Case 2: Suppose m is odd.

If n is even, then $n \geq 2$ and $a_n = n/2$.

Thus, $a_n > 0$.

Since m is odd, then $m \geq 1$ and $a_m = \frac{1-m}{2}$.

Thus, $a_m \leq 0$.

Since $a_m = a_n$ by assumption, then this means a nonpositive integer equals a positive integer.

But, this is impossible.

Therefore, if m is odd, then n cannot be even.

If n is odd, then $a_n = \frac{1-n}{2}$.

Since m is odd, then $a_m = \frac{1-m}{2}$.

Since $a_m = a_n$ by assumption, then this implies $\frac{1-m}{2} = \frac{1-n}{2}$.

Thus, it follows that $m = n$.

Thus, we have proved $a_m = a_n$ implies $m = n$ if m is odd.

In either case we have proved $a_m = a_n$ implies $m = n$.

Therefore, we conclude a is one to one.

We prove a is onto.

Let a_n be any integer.

Either a_n is positive or nonpositive.

We consider these cases separately.

Case 1: Suppose a_n is positive.

Let $n \in N$ be even.

Then $a_n = n/2$.

Since $n \geq 2$, then a_n is positive.

Case 2: Suppose a_n is nonpositive.

Let $n \in N$ be odd.

Then $a_n = \frac{1-n}{2}$.

Since $n \geq 1$, then $a_n \leq 0$.

Thus, a_n is nonpositive.

In either case we have proved a_n is the image of at least one $n \in \mathbb{N}$.

Hence, a is an onto function.

Since a is both one to one and onto, then a is a bijective function from $\mathbb{N}$ onto $\mathbb{Z}$.

Thus, $|\mathbb{N}| = |\mathbb{Z}|$. $\square$

Theorem 67. *A set is countably infinite if and only if its elements can be arranged in an infinite sequence.*

Solution. Let S be a set.

We must prove S is countably infinite if and only if its elements can be arranged in an infinite sequence.

We must prove this biconditional:

1) Prove if S is countably infinite, then its elements can be arranged in an infinite sequence $a_1, a_2, a_3, a_4, ...$

2) Prove if the elements of S can be written as an infinite sequence(list) $a_1, a_2, a_3, a_4, ...$, then S is countably infinite.

S is countably infinite if and only if $|\mathbb{N}| = |S|$, by definition of countably infinite.

This means there is a bijective function $f : \mathbb{N} \mapsto S$.

We know a sequence, by definition, is a mapping from $\mathbb{N}$. $\square$

Proof. Let S be a set.

We prove if S is countably infinite, then its elements can be arranged in an infinite sequence.

We use direct proof.

Suppose S is countably infinite.

Then there exists a bijection $f : \mathbb{N} \mapsto S$, by definition of countably infinite.

So we can write each element of S as an infinite sequence: $f(1), f(2), f(3), f(4), ...$

Conversely, we prove if the elements of S can be written in a sequence, then S is countably infinite.

We use direct proof.

Suppose the elements of S can be written as an infinite sequence: $a_1, a_2, a_3, a_4, ...$

Then there exists a function $f : \mathbb{N} \mapsto S$ that is bijective where $f(n) = a_n$.

Hence, S is countably infinite, by definition of countably infinite. $\square$

Theorem 68. *The cartesian product of two countable sets is countable.*

Proof. Let A, B be countable sets.

We prove $A \times B$ is countable.

Since A is countable $\exists$ a one to one correspondence between A and $\mathbb{N}$.

We can therefore list the elements of A in an infinite sequence $A = \{a_1, a_2, ..., a_k, ...\}$.

Similarly we can list the elements of B in an infinite sequence $B = \{b_1, b_2, ..., b_k, ...\}$.

Let $f : A \times B \mapsto \mathbb{N}$ be a function defined by $f(a_k, b_k) = k$.

We prove f is bijective.

Let $(a_m, b_m), (a_n, b_n) \in A \times B$.

Suppose $f(a_m, b_m) = f(a_n, b_n)$.

Then $m = n$.

Thus $a_m = a_n$ and $b_m = b_n$ so $(a_m, b_m) = (a_n, b_n)$.

Hence $f(a_m, b_m) = f(a_n, b_n) \rightarrow (a_m, b_m) = (a_n, b_n)$ so f is one to one.

Let $n \in \mathbb{N}$.

Observe that $(a_n, b_n) \in A \times B$ and $f(a_n, b_n) = n$.

Hence $\exists\ (a_n, b_n) \in A \times B$ such that $f(a_n, b_n) = n$ so f is onto.

Since f is one to one and onto then f is a one to one correspondence from $A \times B$ to $\mathbb{N}$.

Therefore $A \times B$ is countable. $\square$

Theorem 69. *The Cartesian product of n countable sets is countable.*

Solution. We restate this as:

If $A_1, A_2, ...A_n$ are countably infinite sets, then the product $A_1 \times A_2 \times ... \times A_n$ is countably infinite.

We can use proof by induction.

Our statement S_n is:

if $A_1, A_2, ...A_n$ are countably infinite, then $A_1 \times A_2 \times ... \times A_n$ is countably infinite. $\square$

Proof. We prove by induction.

Let S_n be the statement: if $A_1, A_2, ...A_n$ are countable, then $A_1 \times A_2 \times ... \times A_n$ is countable.

Basis:

Let $n = 2$.

Then the statement S_2 is if A_1 and A_2 are countable, then $A_1 \times A_2$ is countable.

Suppose A_1 and A_2 are both countably infinite sets.

We know the Cartesian product of two countably infinite sets is countably infinite.

Hence, $A_1 \times A_2$ is countably infinite.

Induction:

We must prove $S_k \Rightarrow S_{k+1}$ for $k \geq 2$.

Suppose S_k is true.

Then it is true that if $A_1, A_2, ...A_k$ are countably infinite, then $A_1 \times A_2 \times ... \times A_k$ is countably infinite.

We must prove this implies if $A_1, A_2, ..., A_{k+1}$ are countably infinite, then $A_1 \times A_2 \times ... \times A_{k+1}$ is countably infinite.

Suppose $A_1, A_2, ...A_k$ are countably infinite.

Then $A_1 \times A_2 \times ... \times A_k$ is countably infinite.

Let A_{k+1} be a countably infinite set.

Consider the product $A_1 \times A_2 \times ... \times A_{k+1}$ of countably infinite sets.

Observe that

$$\begin{aligned} A_1 \times A_2 \times ... \times A_{k+1} &= A_1 \times A_2 \times ... \times A_k \times A_{k+1} \\ &= (A_1 \times A_2 \times ... \times A_k) \times A_{k+1} \end{aligned}$$

By the induction hypothesis, this is a product of two countably infinite sets, so it is countably infinite.

It follows by induction that if $A_1, A_2, ...A_n$ are countably infinite, then $A_1 \times A_2 \times ... \times A_n$ is countably infinite.

Hence, the Cartesian product of n countably infinite sets is countably infinite. $\square$

Theorem 70. *The union of two countable disjoint sets is countable.*

Proof. Let A and B be disjoint sets.

Suppose A and B are countably infinite.

We must prove $A \cup B$ is countably infinite.

Then we can write each as an infinite sequence: $A = \{a_1, a_2, a_3, a_4, ...\}$ and $B = \{b_1, b_2, b_3, b_4, ...\}$.

The union can be written as an infinite sequence: $A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, ...\}$.

Hence, $A \cup B$ is countably infinite. □

Theorem 71. *A subset of a countable set is countable.*

Solution. Let S be a countable set.

Let T be a subset of S.

Either T is finite or infinite.

We know any finite set is countable, so if T is finite, then T is countable.

If T is infinite, then we need to devise a way to prove that T is countably infinite.

We can devise a bijection from $\mathbb{N}$ to T, but that approach doesn't appear fruitful.

We can also try to write out the elements of T as an infinite linear sequence. □

Proof. Let S be a countable set.

Let T be a subset of S.

We prove T is countable.

Either S is finite or infinite.

We consider these cases separately.

Case 1: Suppose S is finite.

A subset of a finite set is finite, so T is finite.

Since any finite set is countable, then T is countable.

Case 2: Suppose S is infinite.

Then T is either finite or infinite.

We consider these cases separately.

Case 2a: Suppose T is finite.

Since any finite set is countable, then T is countable.

Case 2b: Suppose T is infinite.

Since S is countable and infinite, then S is countably infinite, so S can be written as an ordered infinite linear sequence: $s_1, s_2, s_3, s_4, s_5, ...$.

We can write the elements of T in list form by proceeding through the elements of S, in order, and selecting those that belong to T.

Hence, T can be written in list form, and since S is infinite, then $T's$ list will be infinite.

Therefore, T is countably infinite, so T is countable. □

Theorem 72. *A superset of an uncountable set is uncountable.*

Solution. We restate this as follows:

Let S be an uncountable set.

Let T be a superset of S.

We must prove T is uncountable. □

Proof. Let S be an uncountable set.
Let T be a superset of S.
Then $S \subset T$.
Suppose for the sake of contradiction that T is countable.
Every subset of a countable set is countable.
Since $S \subset T$ and T is countable, then this implies S is countable.
Thus we have S is uncountable and S is countable, a contradiction.
Therefore T is not countable, so T is uncountable. $\square$

Theorem 73. ***Cantor's Theorem***
The power set of a set is strictly larger than the set itself.
Let S be a set.
Then $|S| < |\mathscr{P}(S)|$.

Proof. Then S is either finite or infinite.
Case 1: Suppose S is finite.
Then $|S| = n$ for some $n \in \mathbb{Z}$ and $n \geq 0$.
Thus $|\mathscr{P}(S)| = 2^{|S|} = 2^n$.
Since $n < 2^n$ for all $n \geq 0$ then $|S| < |\mathscr{P}(S)|$.
Case 2: Suppose S is infinite.
Let $f : S \to \mathscr{P}(S)$ be a function defined by $f(x) = \{x\}$.
Let $a, b \in S$.
Suppose $f(a) = f(b)$.
Then $\{a\} = \{b\}$ so $a = b$.
Hence $f(a) = f(b)$ implies $a = b$ so f is injective.
Since there does not exist a surjective function from S to $\mathscr{P}(S)$, then f cannot be surjective.
Since f is one to one but not onto then $|S| < |\mathscr{P}(S)|$.
Therefore $|S| < |\mathscr{P}(S)|$. $\square$