Set Theory Examples

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Sets

Specification of Sets

Example 1. Using the roster method to list the names of the elements of a set.
Let S be the set of numbers $1, 2, 3, 4$.
Then $S = \{1, 2, 3, 4\}$.
Observe that S contains 4 elements, and $2 \in S$, but $5 \notin S$.

Example 2. Order does not matter in a set.
Let S be the set containing only the following objects: red, white, blue.
Then $S = \{$ red, white, blue $\}$.
Since the order of objects listed in a set does not matter, then $S = \{$ blue, red, white $\}$.

Example 3. Duplicates are not allowed in a set.
Let $S = \{1, 1, 2\}$.
Since duplicates in a set are not allowed, then $S = \{1, 2\}$.

Example 4. Using the rule method to specify a set by some property.
Let S be the set of all positive integers less than 10.
Then $S = \{n : n$ is a positive integer and n is less than 10 $\}$.
Therefore, $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
Observe that $9 \in S$, but $0 \notin S$ and $10 \notin S$.

Example 5. A rule to describe a set must be meaningful.
Let S be the set of all boku.
Since 'boku' is a nonsense word, then S does not define a set.

Example 6. A rule to describe a set must be specific and definitive.
Let $S = \{x |\ x$ is a large states in the United States $\}$.
Since 'a large state' is a subjective criterion for membership, S is not well-defined.
Therefore, S is not a set.

Example 7. Using the roster method with ... to specify a set.
Let $S = \{1, 3, 5, ..., 97, 99\}$.
Observe that S is a finite set, and $47 \in S$ and $2 \notin S$.

Example 8. Using the roster method with ... to specify a set.

Let $S = \{1, 3, 5, ..., 97, 99\}$.

Observe that S is a finite set, and $47 \in S$ and $2 \notin S$.

Example 9. set of all natural numbers $\mathbb{N}$

The set of all natural numbers, denoted $\mathbb{N}$, is the set $\{1, 2, 3, ...\}$.

Therefore, $\mathbb{N} = \{1, 2, 3, ...\}$.

Observe that $-1 \notin \mathbb{N}$.

Example 10. set of all integers $\mathbb{Z}$

The set of all integers, denoted $\mathbb{Z}$, is the set $\{0, \pm 1, \pm 2, \pm 3, ...\}$.

Therefore, $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$.

Observe that $-\frac{1}{2} \notin \mathbb{Z}$.

Example 11. universal set in number theory

The domain of discourse in elementary number theory is the set of all natural numbers or the set of all integers.

Therefore, the universal set in this context is either $\mathbb{N}$ or $\mathbb{Z}$.

Example 12. set of all rational numbers $\mathbb{Q}$

A rational number is a ratio (i.e. quotient) of integers.

The set of all rational numbers, denoted $\mathbb{Q}$, is the set $\{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$.

Therefore, $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$.

Let $\mathbb{Q}$ be the set of all rational numbers.

Since $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$, then the universal set is $\mathbb{Z}$, and the expression is $\frac{m}{n}$, and the predicate defined over $\mathbb{Z}$ is $(\exists m \in \mathbb{Z})(\exists n \in \mathbb{Z})(n \neq 0)$.

Thus, if $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, then $\frac{m}{n} \in \mathbb{Q}$ iff $n \neq 0$.

Hence, if $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, then $\frac{m}{n} \notin \mathbb{Q}$ iff $n = 0$.

Let q be a rational number.

Then $q \in \mathbb{Q}$, so there exists $m \in \mathbb{Z}$ and there exists $n \in \mathbb{Z}$ with $n \neq 0$ such that $q = \frac{m}{n}$.

Since $1 \in \mathbb{Z}$ and $2 \in \mathbb{Z}$ and $2 \neq 0$, then $\frac{1}{2} \in \mathbb{Q}$.

Since $0 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$ and $1 \neq 0$, then $\frac{0}{1} \in \mathbb{Q}$.

Since $1 \in \mathbb{Z}$ and $0 \in \mathbb{Z}$ and $0 = 0$, then $\frac{1}{0} \notin \mathbb{Q}$.

Example 13. set of all real numbers $\mathbb{R}$

The set of all real numbers is denoted $\mathbb{R}$.

Therefore, $\mathbb{R} = \{x : x \text{ is a real number}\}$.

Observe that $0 \in \mathbb{R}$ and $1 \in \mathbb{R}$ and $-\sqrt[4]{3} \in \mathbb{R}$.

Example 14. universal set in real analysis

The domain of discourse in calculus and real analysis is the set of all real numbers.

Therefore, the universal set in this context is $\mathbb{R}$.

Example 15. set of all nonzero real numbers $\mathbb{R}^*$

The set of all nonzero real numbers is denoted $\mathbb{R}^*$.
The universal set is $\mathbb{R}$.
Let $x \in \mathbb{R}$.
Let $p(x) : x \neq 0$ be a predicate defined over $\mathbb{R}$.
Then $\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\}$.
Observe that $\frac{-\pi}{2} \in \mathbb{R}^*$ and $1 \in \mathbb{R}^*$.

Example 16. set of all positive real numbers $\mathbb{R}^+$

The set of all positive real numbers is denoted $\mathbb{R}^+$.
The universal set is $\mathbb{R}$.
Let $x \in \mathbb{R}$.
Let $p(x) : x > 0$ be a predicate defined over $\mathbb{R}$.
Then $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.
Observe that $1 \in \mathbb{R}^+$ and $\sqrt{2} \in \mathbb{R}^+$ and $\frac{\pi}{3} \in \mathbb{R}^+$ and $e \in \mathbb{R}^+$.

Example 17. set of all complex numbers $\mathbb{C}$

The set of all complex numbers is denoted $\mathbb{C}$.
Define the **imaginary unit** i by $i^2 = -1$.
Then $\mathbb{C} = \{x + yi : \ x \in \mathbb{R} \text{ and } y \in \mathbb{R} \ \}$.
Since $0 = 0 + 0i$, then $0 \in \mathbb{C}$.
Since $1 = 1 + 0i$, then $1 \in \mathbb{C}$.
Observe that $3 - \frac{\pi i}{4} \in \mathbb{C}$.

Example 18. universal set in complex analysis

The domain of discourse in complex analysis is the set of all complex numbers.

Therefore, the universal set in this context is $\mathbb{C}$.

Example 19. set of all nonzero complex numbers $\mathbb{C}^*$

The set of all nonzero complex numbers is denoted $\mathbb{C}^*$.
The universal set is $\mathbb{C}$.
Let $z \in \mathbb{C}$.
Let $p(z) : z \neq 0$ be a predicate defined over $\mathbb{C}$.
Then $\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}$.
Since $1 = 1 + 0i$ and $0 = 0 + 0i$ and $1 \neq 0$, then $1 \in \mathbb{C}^*$.
Since $0 = 0 + 0i$ and $2 \neq 0$ and $-\frac{\pi}{5} \neq 0$, then $2 - \frac{\pi}{5}i \in \mathbb{C}^*$.

Example 20. The universal set specified in a set is important.

Let $A = \{x \in \mathbb{Q} | x^2 \geq 2\}$.
Let $B = \{x \in \mathbb{R} | x^2 \geq 2\}$.

Observe that $\mathbb{Q}$ is the universal set of A, but $\mathbb{R}$ is the universal set of B.
Since $(\sqrt{2})^2 = 2$, then $(\sqrt{2})^2 \geq 2$.
Since $\sqrt{2} \in \mathbb{R}$ and $(\sqrt{2})^2 \geq 2$, then $\sqrt{2} \in B$.
Since $\sqrt{2}$ is irrational, then $\sqrt{2} \notin \mathbb{Q}$, so $\sqrt{2} \notin A$.
Since $\sqrt{2} \notin A$ and $\sqrt{2} \in B$, then A and B are not the same set.

Therefore, even though both sets A and B are described by the same rule $x^2 \geq 2$, A and B are not the same set, since A and B have different universal sets.

Example 21. closed unit interval
Let $I = \{x|$ is a real number and $0 \leq x \leq 1\}$.
Then $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\} = [0, 1]$ is the closed unit interval.

Example 22. Let S be the set of all natural numbers greater than 7.
Then $\mathbb{N}$ is the universal set.
Let $n \in \mathbb{N}$.
Define a predicate $p(n) : n > 7$ over $\mathbb{N}$.
Then $S = \{n \in \mathbb{N} : n > 7\}$.
Hence, a natural number n is an element of S iff $n > 7$.
Thus, if $n \in \mathbb{N}$, then $n \in S$ iff $n > 7$.
Each element of S is a natural number greater than 7.
Thus, for each $s \in S$, $s \in \mathbb{N}$ and $s > 7$, so $(\forall s \in S)(s \in \mathbb{N} \wedge s > 7)$.
Since $8 \in \mathbb{N}$ and $8 > 7$, then $8 \in S$.
Since $7 > 7$ is false, then $7 \notin S$.

Example 23. set equality example
Let $A = \{x|$x was the first president of the United States$\}$.
Let $B = \{$George Washington$\}$.
Then $A = B$.

Example 24. subset example
Let $A = \{$blue, orange$\}$.
Let $B = \{$red, green, orange, white, blue$\}$.
Then $A \subseteq B$.

Example 25. The subset relation is not symmetric.
We show the conditional 'if $A \subseteq B$, then $B \subseteq A$' is false.

Proof. Every integer is a rational number, so $\mathbb{Z} \subseteq \mathbb{Q}$.

Since $\frac{1}{2}$ is a rational number, but $\frac{1}{2}$ is not an integer, then $\frac{1}{2} \in \mathbb{Q}$, but $\frac{1}{2} \notin \mathbb{Z}$, so not every rational number is an integer.

Hence, $\mathbb{Q} \not\subseteq \mathbb{Z}$.

Therefore, $\mathbb{Z} \subseteq \mathbb{Q}$, but $\mathbb{Q} \not\subseteq \mathbb{Z}$, so the subset relation is not symmetric. □

Example 26. proper subset
Let $A = \{1, 4, 7, 10, 13, 16\}$.
Let $B = \{1, 3, 5, 7, 9\}$.

Let $C = \{4, 10, 16\}$.
Since $3 \in B$, but $3 \notin A$, then $B \not\subseteq A$, so B is not a proper subset of A.
Since $C \subseteq A$ and $C \neq A$, then C is a proper subset of A, so $C \subset A$.

Example 27. powerset example
Compute the power set of $\{1, 2, 3, 4\}$.

Solution. Let $S = \{1, 2, 3, 4\}$.
Observe that $\mathscr{P}(S) = \{\emptyset,$
$\{1\}, \{2\}, \{3\}, \{4\},$
$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\},$
$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\},$
$\{1, 2, 3, 4\}\}$.

Since $\emptyset \subseteq S$, then $\emptyset \in \mathscr{P}(S)$, so $\{\emptyset\} \subseteq \mathscr{P}(S)$. □

Set Operations

Example 28. set union
Let $A = \{1, 4, 7, 10, 13, 16\}$.
Let $B = \{1, 3, 5, 7, 9\}$.
Then $A \cup B = \{x : x \in A \text{ or } x \in B\} = \{1, 3, 4, 5, 7, 9, 10, 13, 16\}$.

Example 29. set intersection
Let $A = \{1, 4, 7, 10, 13, 16\}$.
Let $B = \{1, 3, 5, 7, 9\}$.
Then $A \cap B = \{x : x \in A \text{ and } x \in B\} = \{1, 7\}$.

Example 30. absolute complement
Let $U = \{1, 2, 3, 4, 5\}$ be a universal set.
Let $S = \{1, 3\}$ be a subset of U.
The complement of S in U is the set $\overline{S} = \{x \in U : x \notin S\} = \{2, 4, 5\}$.

Example 31. set difference
Let $A = \{1, 4, 7, 10, 13, 16\}$.
Let $B = \{1, 3, 5, 7, 9\}$.
Then $A - B = \{x \in A : x \notin B\} = \{4, 10, 13, 16\}$.

Example 32. symmetric difference
Let $A = \{1, 4, 7, 10, 13, 16\}$.
Let $B = \{1, 3, 5, 7, 9\}$.
Observe that

$$\begin{aligned} A \triangle B &= (A - B) \cup (B - A) \\ &= \{4, 10, 13, 16\} \cup \{3, 5, 9\} \\ &= \{3, 4, 5, 9, 10, 13, 16\}. \end{aligned}$$

Therefore, $A \triangle B = \{3, 4, 5, 9, 10, 13, 16\}$.

Cartesian Product

Example 33. differences between set and ordered pair equality
The ordered pairs $(2, 3)$ and $(3, 2)$ are distinct, so $(2, 3) \neq (3, 2)$.
However, the sets $\{2, 3\}$ and $\{3, 2\}$ are the same, so $\{2, 3\} = \{3, 2\}$.

Example 34. Ordered pairs can have duplicates, but sets do not have duplicates.
The ordered pair (a, a) consists of two identical objects.
However, the sets $\{a, a\}$ and $\{a\}$ are the same, so $\{a, a\} = \{a\}$.
The expression typically used is $\{a\}$, not $\{a, a\}$.

Example 35. cartesian product
Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$.
The cartesian product of A and B is the set $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$.

Example 36. Cartesian product is not commutative.
$\mathbb{R}^2 \times \mathbb{R} = \{((x, y), z) : x, y, z \in \mathbb{R}\}$.
$\mathbb{R} \times \mathbb{R}^2 = \{(x, (y, z)) : x, y, z \in \mathbb{R}\}$.
Therefore, $\mathbb{R}^2 \times \mathbb{R} \neq \mathbb{R} \times \mathbb{R}^2$.

Example 37. Cartesian product is not associative.
Let $A = \{1, 2\}$ and $B = \{3, 4\}$ and $C = \{5\}$.
Then $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$ and $B \times C = \{(3, 5), (4, 5)\}$.
Thus, $(A \times B) \times C = \{((1, 3), 5), ((1, 4), 5), ((2, 3), 5), ((2, 4), 5)\}$ and $A \times (B \times C) = \{(1, (3, 5)), (1, (4, 5)), (2, (3, 5)), (2, (4, 5))\}$.
Since $((1, 3), 5) \in (A \times B) \times C$, but $((1, 3), 5) \notin A \times (B \times C$, then $(A \times B) \times C \neq A \times (B \times C)$.

TODO: Revise below stuff, as needed.

Indexed Sets

Example 38. family of sets indexed by $\mathbb{N}$
Let $\mathcal{S} = \{S_i : i = 1, 2, 3, ...\} = \{S_1, S_2, S_3, ...\} = \{S_i : i \in \mathbb{N}\} = \{S_i\}_{i \in \mathbb{N}}$.
Then $\mathcal{S}$ is a family of sets indexed by $\mathbb{N}$, and each S_i is a set in the collection.
$\mathcal{S}$ is a countably infinite collection of sets.

Example 39. mutually disjoint family of sets
Let $A_i = \{i\}$ for $i = 1, 2, 3, ...$
Then $A = \{A_i : i \in \mathbb{N}\} = \{\{1\}, \{2\}, \{3\}, ...\}$ is a family of sets indexed by $\mathbb{N}$.
Each A_i is a **singleton set**.
Let $i, j \in \mathbb{N}$ such that $i \neq j$.
Then $\{i\} \cap \{j\} = \emptyset$, so $A_i \cap A_j = \emptyset$.
Therefore, A is a mutually disjoint family of sets.

Example 40. increasing family of sets

Let $B_k = \{1, 2, 3, ..., k\}$ for each $k = 1, 2, 3, ...$

Then $B_k = \{n \in \mathbb{N} : 1 \leq n \leq k\}$ for all $k \in \mathbb{N}$ and

$B = \{B_k : k \in \mathbb{N}\} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, ...\}$ is a family of sets indexed by $\mathbb{N}$.

Let $i, j \in \mathbb{N}$ such that $i < j$.

Observe that $B_i = \{n \in \mathbb{N} : 1 \leq n \leq i\}$ and $B_j = \{n \in \mathbb{N} : 1 \leq n \leq j\}$.

Let $x \in B_i$.

Then $x \in \mathbb{N}$ and $1 \leq x \leq i$, so $1 \leq x$ and $x \leq i$.

Since $x \leq i$ and $i < j$, then $x < j$, so $x \leq j$.

Thus, $1 \leq x$ and $x \leq j$, so $1 \leq x \leq j$.

Since $x \in \mathbb{N}$ and $1 \leq x \leq j$, then $x \in B_j$.

Hence, $x \in B_i$ implies $x \in B_j$, so $B_i \subset B_j$.

Therefore, B is an increasing family of sets.

Example 41. decreasing family of nested intervals

Let $C_i = [i, \infty)$ for each $i = 1, 2, 3, ...$

Then $C_i = [i, \infty) = \{x \in \mathbb{R} : i \leq x\}$ for all $i \in \mathbb{N}$.

Let $C = \{C_i : i \in \mathbb{N}\} = \{[1, \infty), [2, \infty), [3, \infty), [4, \infty), ...\}$ be a family of intervals indexed by $\mathbb{N}$.

Let $i, j \in \mathbb{N}$ such that $i < j$.

Observe that $C_i = [i, \infty) = \{x \in \mathbb{R} : i \leq x\}$ and $C_j = [j, \infty) = \{x \in \mathbb{R} : j \leq x\}$.

Let $x \in C_j$.

Then $x \in \mathbb{R}$ and $j \leq x$.

Since $i < j$ and $j \leq x$, then $i < x$.

Thus, $x \in \mathbb{R}$ and $i < x$, so $x \in C_i$.

Hence, $x \in C_j$ implies $x \in C_i$, so $C_j \subset C_i$.

Therefore, $C_i \supset C_j$.

Hence, C is a decreasing family of intervals, or **family of nested intervals**.

Cardinality of Sets

Example 42. $\mathbb{N}_0 = \emptyset$.

Proof. Suppose for the sake of contradiction that $\mathbb{N}_0 \neq \emptyset$.

Then there exists $m \in \mathbb{N}_0$, so $m \in \mathbb{N}$ and $1 \leq m \leq 0$.

Since $1 \leq m \leq 0$, then $1 \leq 0$, a contradiction.

Therefore, $\mathbb{N}_0 = \emptyset$. □

Example 43. $\mathbb{N}_5 = \{k \in \mathbb{N} : 1 \leq k \leq 5\} = \{1, 2, 3, 4, 5\}$.

Example 44. cardinality of a singleton set is 1

The cardinality of the singleton set $\{a\}$ is 1 since $\{a\} \sim \mathbb{N}_1 = \{1\}$.

Example 45. The set $\mathbb{N}$ **is infinite.**

The set of natural numbers is infinite.

Proof. Define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(k) = 2k$ for all $k \in \mathbb{N}$.
We prove f is injective.
Let $k, m \in \mathbb{N}$ such that $f(k) = f(m)$.
Then $2k = 2m$, so $k = m$.
Therefore, f is injective.

We prove f is not surjective by contradiction.
Suppose f is surjective.
Then for each $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $f(k) = m$.
Let $m = 1$.
Then there exists $k \in \mathbb{N}$ such that $f(k) = 1$.
Thus, $2k = 1$, so $k = \frac{1}{2}$.
But, $\frac{1}{2} \notin \mathbb{N}$, so $k \notin \mathbb{N}$.
Therefore, we have $k \in \mathbb{N}$ and $k \notin \mathbb{N}$, a contradiction.
Hence, f is not surjective.

Since $f : \mathbb{N} \to \mathbb{N}$ is injective, but not surjective, then $\mathbb{N}$ is infinite. □

Example 46. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$.
Define a function $f : A \to B$ by $f(1) = c$, $f(2) = a$, $f(3) = d$, $f(4) = b$.
Then clearly f is a bijection, so $A \sim B$.
Therefore, $|A| = |B|$.

Example 47. $\mathbb{N}$ is countable
The set of natural numbers is countable.

Example 48. $\mathbb{N}$ is countably infinite
Since $\mathbb{N}$ is countable and $\mathbb{N}$ is infinite, then $\mathbb{N}$ is countably infinite.

Example 49. $\mathbb{Z}$ is countable
$|\mathbb{Z}| = |\mathbb{N}|$.

Therefore the set of integers is countable.

Example 50. $\mathbb{Q}$ is countable
$|\mathbb{Q}| = |\mathbb{N}|$.

Therefore the set of rational numbers is countable.
Since $\mathbb{Q}$ can be arranged in an infinite sequence, then $\mathbb{Q}$ is countable.

Example 51. $\mathbb{R}$ is uncountable
$|\mathbb{R}| = |\mathscr{P}(\mathbb{N})|$.

$|\mathbb{R}| = |\mathscr{P}(\mathbb{N})| = 2^{\aleph_0}$.
Since there are no surjections onto $\mathbb{R}$ then there are no bijections $f : \mathbb{N} \to \mathbb{R}$.
Thus $|\mathbb{N}| \neq |\mathbb{R}|$. Hence, $\mathbb{R}$ is uncountable.
We know $|S| < |\mathscr{P}(S)|$ for any set S.
Thus $|\mathscr{P}(\mathbb{N})| > |\mathbb{N}|$ so $|\mathbb{R}| > |\mathbb{N}|$.
Therefore the set of real numbers is larger than the set of natural numbers.