Set Theory Exercises

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Sets

Exercise 1. Let $S = \{x : x < 30 \text{ and } x = n^2 \text{ for some } n \in \mathbb{N}\}$.
Then $S = \{1, 4, 9, 16, 25\}$.

Solution. Since $S = \{x : x < 30 \text{ and } x = n^2 \text{ for some } n \in \mathbb{N}\}$, then $S = \{n^2 : n^2 < 30 \text{ for some } n \in \mathbb{N}\}$.

Thus, S is the set of squares of natural numbers less than 30, so $S = \{1, 4, 9, 16, 25\}$. □

Proof. We prove $\{1, 4, 9, 16, 25\} \subset S$.
Since $1 \in \mathbb{N}$ and $1^2 = 1 < 30$, then $1 \in S$.
Since $2 \in \mathbb{N}$ and $2^2 = 4 < 30$, then $4 \in S$.
Since $3 \in \mathbb{N}$ and $3^2 = 9 < 30$, then $9 \in S$.
Since $4 \in \mathbb{N}$ and $4^2 = 16 < 30$, then $16 \in S$.
Since $5 \in \mathbb{N}$ and $5^2 = 25 < 30$, then $25 \in S$.
Since $1 \in S$ and $4 \in S$ and $9 \in S$ and $16 \in S$ and $25 \in S$, then $\{1, 4, 9, 16, 25\} \subset S$.

We prove $n^2 \notin S$ for every natural number $n \geq 6$.
Let $n \in \mathbb{N}$ such that $n \geq 6$.
Then $n^2 \geq 6^2 = 36 > 30$, so $n^2 > 30$.
Since $n \in \mathbb{N}$ and $n^2 > 30$, then $n^2 \notin S$.
Thus, if $n \in \mathbb{N}$ and $n \geq 6$, then $n^2 \notin S$, so if $n \in \mathbb{N}$ and $n^2 \in S$, then $n < 6$.

We prove $S \subset \{1, 4, 9, 16, 25\}$.
Since $1 \in S$, then $S \neq \emptyset$.
Let $x \in S$.
Then $x < 30$ and $x = n^2$ for some $n \in \mathbb{N}$.
Since $n \in \mathbb{N}$, then $n \geq 1$.
Since $n \in \mathbb{N}$ and $n^2 \in S$, then $n < 6$.
Since $n \geq 1$ and $n < 6$, then $1 \leq n < 6$.
Since $n \in \mathbb{N}$ and $1 \leq n < 6$, then either $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$ or $n = 5$.
We consider these cases separately.

Case 1: Suppose $n = 1$.
Since $1^2 = 1 < 30$, then $x = 1$.
Case 2: Suppose $n = 2$.
Since $2^2 = 4 < 30$, then $x = 4$.
Case 3: Suppose $n = 3$.
Since $3^2 = 9 < 30$, then $x = 9$.
Case 4: Suppose $n = 4$.
Since $4^2 = 16 < 30$, then $x = 16$.
Case 5: Suppose $n = 5$.
Since $5^2 = 25 < 30$, then $x = 25$.
Hence, either $x = 1$ or $x = 4$ or $x = 9$ or $x = 16$ or $x = 25$, so $x \in \{1, 4, 9, 16, 25\}$.
Therefore, $S \subset \{1, 4, 9, 16, 25\}$.
Since $S \subset \{1, 4, 9, 16, 25\}$ and $\{1, 4, 9, 16, 25\} \subset S$, then $S = \{1, 4, 9, 16, 25\}$, as desired. □

Exercise 2. Let $S = \{x : x = n + 2 \text{ for some } n \in \mathbb{N} \text{ such that } n < 6\ \}$.
Then $S = \{3, 4, 5, 6, 7\}$.

Solution. Since $S = \{x : x = n + 2 \text{ for some } n \in \mathbb{N} \text{ such that } n < 6\ \}$, then $S = \{n + 2 : \text{ there exists } n \in \mathbb{N} \text{ such that } n < 6\ \}$.
If $n \in \mathbb{N}$ and $n < 6$, then $n \in \{1, 2, 3, 4, 5\}$, so $n + 2 \in \{3, 4, 5, 6, 7\}$.
Thus, $S = \{3, 4, 5, 6, 7\}$. □

Proof. We prove $\{3, 4, 5, 6, 7\} \subset S$.
Since $1 \in \mathbb{N}$ and $1 < 6$ and $1 + 2 = 3$, then $3 \in S$.
Since $2 \in \mathbb{N}$ and $2 < 6$ and $2 + 2 = 4$, then $4 \in S$.
Since $3 \in \mathbb{N}$ and $3 < 6$ and $3 + 2 = 5$, then $5 \in S$.
Since $4 \in \mathbb{N}$ and $4 < 6$ and $4 + 2 = 6$, then $6 \in S$.
Since $5 \in \mathbb{N}$ and $5 < 6$ and $5 + 2 = 7$, then $7 \in S$.
Since $3 \in S$ and $4 \in S$ and $5 \in S$ and $6 \in S$ and $7 \in S$, then $\{3, 4, 5, 6, 7\} \subset S$.

We prove $S \subset \{3, 4, 5, 6, 7\}$.
Since $3 \in S$, then $S \neq \emptyset$.
Let $x \in S$.
Then $x = n + 2$ for some $n \in \mathbb{N}$ such that $n < 6$.
Since $n \in \mathbb{N}$, then $n \geq 1$.
Since $n \geq 1$ and $n < 6$, then $1 \leq n < 6$, so $n \in \{1, 2, 3, 4, 5\}$.
Hence, $x = n + 2 \in \{3, 4, 5, 6, 7\}$.
Since $x \in S$ implies $x \in \{3, 4, 5, 6, 7\}$, then $S \subset \{3, 4, 5, 6, 7\}$.
Since $S \subset \{3, 4, 5, 6, 7\}$ and $\{3, 4, 5, 6, 7\} \subset S$, then $S = \{3, 4, 5, 6, 7\}$. □

Exercise 3. Let $S = \{n^2 + 2 : n \in \mathbb{N}\}$.
Let $T = \{3, 6, 11, 18, 27, 33, 38, 51\}$.
a. Find an element of S that is not in T.
b. Find an element of T that is not in S.

Solution. a. Since $8 \in \mathbb{N}$ and $8^2 + 2 = 66$, then $66 \in S$, but $66 \notin T$.

b. Observe that $33 \in T$, but $33 \notin S$.

We prove $33 \notin S$.

Suppose for the sake of contradiction $33 \in S$.

Then $33 = n^2 + 2$ for some $n \in \mathbb{N}$, so $n^2 = 31$.

Hence, $31 = nn$, so $n|31$.

Since 31 is prime, then either $n = 1$ or $n = 31$.

We consider these cases separately.

Case 1: Suppose $n = 1$.

Then $31 = n^2 = 1^2 = 1$, a contradiction.

Thus, $n \neq 1$.

Case 2: Suppose $n = 31$.

Then $31 = n^2 = 31^2$, so $1 = 31$, a contradiction.

Thus, $n \neq 31$.

Hence, neither $n = 1$ nor $n = 31$, so there is no $n \in \mathbb{N}$ such that $n^2 + 2 = 33$.

Therefore, $33 \notin S$. ☐

Exercise 4. The subset relation is not symmetric.

Solution. Let $A = \{a\}$ and $B = \{a, b\}$.

Since $a \in A$ and $a \in B$, then $A \subset B$.

Since $b \in B$, but $b \notin A$, then $B \not\subset A$.

Since $A \subset B$ and $B \not\subset A$, then $A \subset B$ does not imply $B \subset A$.

Therefore $\subset$ is not symmetric. ☐

Exercise 5. Let $A = \{k \in \mathbb{N} : k \leq 20\}$.

Let $B = \{3k - 1 : k \in \mathbb{N}\}$.

Let $C = \{2k + 1 : k \in \mathbb{N}\}$.

Compute:

1. $A \cap B \cap C$.
2. $(A \cap B) - C$.
3. $(A \cap C) - B$.

Solution. Observe that $A \cap B \cap C = \{5, 11, 17\}$ and $(A \cap B) - C = \{2, 8, 14, 20\}$ and $(A \cap C) - B = \{3, 7, 9, 13, 15, 19\}$. ☐

Exercise 6. For any sets A and B, $A \subseteq (A \cup B) \cap (A \cup \overline{B})$.

Solution. Let A and B be arbitrary sets.

To prove $A \subseteq (A \cup B) \cap (A \cup \overline{B})$, we must prove that the statement $\forall x.[x \in A \rightarrow x \in (A \cup B) \cap (A \cup \overline{B})]$ is true.

Since this is a universally quantified statement, we assume x is an arbitrary object in the domain of discourse.

We must prove $x \in A \rightarrow x \in (A \cup B) \cap (A \cup \overline{B})$ is true.

We use direct proof, so we assume $x \in A$.

We must show that $x \in A \cup B$ and $x \in A \cup \overline{B}$.

We can work backwards from the conclusion to determine how to prove this.

Observe that $x \in (A \cup B) \cap (A \cup \overline{B}) \Leftrightarrow (x \in A \cup B) \wedge (x \in A \cup \overline{B}) \Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in \overline{B}) \Leftrightarrow (x \in A) \vee (x \in B \wedge x \in \overline{B})$.

Thus, the condensed proof is:

$x \in A \Rightarrow x \in A \vee (x \in B \wedge x \in \overline{B}) \Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in \overline{B}) \Leftrightarrow (x \in A \cup B) \wedge (x \in A \cup \overline{B}) \Leftrightarrow x \in (A \cup B) \cap (A \cup \overline{B})$. □

Proof. Let A and B be arbitrary sets.

To prove $A \subseteq (A \cup B) \cap (A \cup \overline{B})$, let $x \in A$.

We must prove $x \in A \cup B$ and $x \in A \cup \overline{B}$.

Since $x \in A$, then either $x \in A$ or $x \in B$.

Hence, $x \in A \cup B$.

Similarly, since $x \in A$, then $x \in A$ or $x \in \overline{B}$.

Hence, $x \in A \cup \overline{B}$.

Therefore $x \in A \cup B$ and $x \in A \cup \overline{B}$, as desired. □

Solution. An alternate solution is to use algebraic properties of sets. □

Proof. Let A and B be arbitrary sets.

Observe that $(A \cup B) \cap (A \cup \overline{B}) = A \cup (B \cap \overline{B}) = A \cup \emptyset = A$.

Since every set is a subset of itself, then $A \subseteq A$.

Hence, A is a subset of $(A \cup B) \cap (A \cup \overline{B})$, so $A \subseteq (A \cup B) \cap (A \cup \overline{B})$. □

Exercise 7. Let A, B be sets.

Then $A - (B - A) = A$.

Proof. We prove $A - (B - A) \subset A$.

Let $x \in A - (B - A)$.

Then $x \in A$, so $A - (B - A) \subset A$.

We prove $A \subset A - (B - A)$.

Let $y \in A$.

Then either $y \notin B$ or $y \in A$.

Hence, it is not the case that $y \in B$ and $y \notin A$.

Thus, $y \notin B - A$.

Since $y \in A$ and $y \notin B - A$, then $y \in A - (B - A)$.

Hence, $A \subset A - (B - A)$.

Since $A - (B - A) \subset A$ and $A \subset A - (B - A)$, then $A - (B - A) = A$, as desired. □

Lemma 8. *Let A, B be sets.*

Then $A - (A - B) = A \cap B$.

Proof. We prove $A - (A - B) \subset A \cap B$.

Let $x \in A - (A - B)$.

Then $x \in A$ and $x \notin A - B$.

Since $x \notin A - B$, then either $x \notin A$ or $x \in B$.

Since $x \in A$, then we conclude $x \in B$.

Thus, $x \in A$ and $x \in B$, so $x \in A \cap B$.
Hence, $A - (A - B) \subset A \cap B$.

We prove $A \cap B \subset A - (A - B)$.
Let $y \in A \cap B$.
Then $y \in A$ and $y \in B$.
Since $y \in B$, then either $y \notin A$ or $y \in B$.
Hence, it cannot be the case that $y \in A$ and $y \notin B$, so $y \notin A - B$.
Since $y \in A$ and $y \notin A - B$, then $y \in A - (A - B)$.
Thus, $A \cap B \subset A - (A - B)$.

Since $A-(A-B) \subset A\cap B$ and $A\cap B \subset A-(A-B)$, then $A-(A-B) = A\cap B$, as desired. □

Proof. Observe that

$$\begin{aligned} A - (A - B) &= A - (A \cap \overline{B}) \\ &= A \cap (\overline{A \cap \overline{B}}) \\ &= A \cap (\overline{A} \cup B) \\ &= (A \cap \overline{A}) \cup (A \cap B) \\ &= \emptyset \cup (A \cap B) \\ &= A \cap B. \end{aligned}$$

□

Exercise 9. Let A, B be sets.
Then $(A \cap B) - B = \emptyset$.

Proof. Observe that

$$\begin{aligned} (A \cap B) - B &= (A \cap B) \cap \overline{B} \\ &= A \cap (B \cap \overline{B}) \\ &= A \cap \emptyset \\ &= \emptyset. \end{aligned}$$

□

Proof. Suppose $(A \cap B) - B \neq \emptyset$.
Then there exists $x \in (A \cap B) - B$, so $x \in A \cap B$ and $x \notin B$.
Since $x \in A \cap B$, then $x \in A$ and $x \in B$.
But, we have $x \in B$ and $x \notin B$, a contradiction.
Therefore, $(A \cap B) - B = \emptyset$. □

Exercise 10. Let A, B be sets.
Then $(A \cup B) - B = A - B$.

Proof. Observe that

$$\begin{aligned}
(A \cup B) - B &= (A \cup B) \cap \overline{B} \\
&= (A \cap \overline{B}) \cup (B \cap \overline{B}) \\
&= (A \cap \overline{B}) \cup \emptyset \\
&= A \cap \overline{B} \\
&= A - B.
\end{aligned}$$

□

Exercise 11. Under what conditions does $A - (A - B) = B$?

Solution. Let A, B be sets.
We just proved that $A - (A - B) = A \cap B$.
So, when does $A \cap B = B$?
We know that $A \subset B$ iff $A \cap B = A$.
Hence, $B \subset A$ iff $B \cap A = B$.
Thus, $B \subset A$ iff $B = B \cap A = A \cap B = A - (A - B)$.
Therefore, $B \subset A$ iff $B = A - (A - B)$.
Hence, $A - (A - B) = B$ iff $B \subset A$. □

Exercise 12. Under what conditions does $A - (A - B) = B$?

Solution. We let A, B be various sets to determine when the condition is true.
Based on examples, we conjecture the statement is true when $B \subset A$. Thus, if $B \subset A$, then $A - (A - B) = B$.
We shall prove this to be true. □

Proof. Let A, B be sets such that $B \subset A$.
We prove $A - (A - B) = B$.

We first prove $A - (A - B) \subset B$.
Let $x \in A - (A - B)$.
Then $x \in A$ and $x \notin A - B$.
Since $x \in A - B$ iff $x \in A$ and $x \notin B$, then $x \notin A - B$ iff $x \notin A$ or $x \in B$.
Hence, $x \notin A$ or $x \in B$.
But, $x \in A$, so it follows that $x \in B$.
Therefore, $x \in A - (A - B)$ implies $x \in B$, so $A - (A - B) \subset B$.

We next prove $B \subset A - (A - B)$.
Let $y \in B$.
Since $B \subset A$, then $y \in A$.
Since $y \in A$ and $y \in B$, then $y \notin A - B$.
Hence, $y \in A$ and $y \notin A - B$, so $y \in A - (A - B)$.
Therefore, $y \in B$ implies $y \in A - (A - B)$, so $B \subset A - (A - B)$.

Since $A-(A-B)\subset B$ and $B\subset A-(A-B)$, then $A-(A-B)=B$, as desired. □

Exercise 13. Let A, B be sets.
Then $A\cap(B-A)=\emptyset$.

Proof. We prove by contradiction.
Suppose $A\cap(B-A)\neq\emptyset$.
Then there exists $x\in A\cap(B-A)$, so $x\in A$ and $x\in B-A$.
Since $x\in B-A$, then $x\in B$ and $x\notin A$.
Thus, $x\in A$ and $x\notin A$, a contradiction.
Therefore, $A\cap(B-A)=\emptyset$, as desired. □

Exercise 14. Let A, B be sets.
Then $(A-B)\cap(B-A)=\emptyset$.

Proof. Observe that

$$\begin{aligned}
(A-B)\cap(B-A) &= (A\cap\overline{B})\cap(B\cap\overline{A}) \\
&= A\cap(\overline{B}\cap B)\cap\overline{A} \\
&= (A\cap\overline{A})\cap(\overline{B}\cap B) \\
&= (A\cap\overline{A})\cap(B\cap\overline{B}) \\
&= \emptyset\cap\emptyset \\
&= \emptyset.
\end{aligned}$$

□

Exercise 15. For arbitrary sets A, B, and C, $A\cap(B\cup C)\subseteq(A\cap B)\cup C$.

Solution. To prove $A\cap(B\cup C)\subseteq(A\cap B)\cup C$, we must prove that $\forall x.[x\in A\cap(B\cup C)\to x\in(A\cap B)\cup C]$ is true.

To prove this universal quantified statement, we assume x is an arbitrary object in the domain of discourse.

To prove $x\in A\cap(B\cup C)\to x\in(A\cap B)\cup C$, we use direct proof.
Hence, we assume $x\in A\cap(B\cup C)$ is true.
We must prove $x\in(A\cap B)\cup C$.
To prove $x\in(A\cap B)\cup C$, we must prove $x\in A\cap B$ or $x\in C$.
Observe that $x\in(A\cap B)\cup C\Leftrightarrow x\in(A\cap B)\vee x\in C\Leftrightarrow(x\in A\wedge x\in B)\vee(x\in C)\Leftrightarrow(x\in A\vee x\in C)\wedge(x\in B\vee x\in C)\Leftrightarrow(x\in A\cup C)\wedge(x\in B\cup C)\Leftrightarrow x\in(A\cup C)\cap(B\cup C)$.

Now, our assumption is that $x\in A\cap(B\cup C)$ which differs from $x\in(A\cup C)\cap(B\cup C)$.

Somehow, we must deduce $x\in A\cup C$ from $x\in A$.
We do this by disjunction introduction.
Since $x\in A$, then $x\in A\vee x\in C$, so $x\in A\cup C$. □

Proof. To prove $A \cap (B \cup C) \subseteq (A \cap B) \cup C$, let $x \in A \cap (B \cup C)$.
We must prove $x \in (A \cap B) \cup C$.
Observe that

$$\begin{aligned} x \in A \cap (B \cup C) &\Leftrightarrow (x \in A) \wedge (x \in B \cup C) \\ &\Rightarrow (x \in A \vee x \in C) \wedge (x \in B \cup C) \\ &\Leftrightarrow (x \in A \vee x \in C) \wedge (x \in B \vee x \in C) \\ &\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in C) \\ &\Leftrightarrow (x \in A \cap B) \vee (x \in C) \\ &\Leftrightarrow x \in (A \cap B) \cup C \end{aligned}$$

□

Proof. An alternate proof follows.
The proof is based on the tautology: $p \rightarrow (q \vee r) \Leftrightarrow (p \wedge \neg q) \rightarrow r$.
This is the case because $p \rightarrow (q \vee r) \Leftrightarrow \neg p \vee (q \vee r) \Leftrightarrow (\neg p \vee q) \vee r \Leftrightarrow \neg(p \wedge \neg q) \vee r \Leftrightarrow (p \wedge \neg q) \rightarrow r$.
To prove $A \cap (B \cup C) \subseteq (A \cap B) \cup C$, we assume $x \in A \cap (B \cup C)$.
We must prove $x \in A \cap B$ or $x \in C$.
Suppose $x \notin C$.
Since $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$.
Since $x \in B \cup C$, then either $x \in B$ or $x \in C$.
Since $x \notin C$ and either $x \in B$ or $x \in C$, then by disjunctive syllogism, $x \in B$.
Since $x \in A$ and $x \in B$, then $x \in A \cap B$. □

Exercise 16. $A - B = A$ if and only if $A \cap B = \emptyset$.

Proof. We prove $A \cap B = \emptyset$ implies $A - B = A$.
To prove $A - B = A$, we assume $A \cap B = \emptyset$.
To prove $A - B = A$, we prove both $A - B \subset A$ and $A \subset A - B$.
We prove $A \subset A - B$.
Let $a \in A$.
We must prove $a \in A - B$.
Since $A \cap B = \emptyset$, then there is no x such that $x \in A \cap B$.
Hence, $\neg(\exists x)(x \in A \cap B)$.
Observe that $\neg(\exists x)(x \in A \cap B) \Leftrightarrow \neg(\exists x)(x \in A \wedge x \in B) \Leftrightarrow (\forall x)(x \notin A \vee x \notin B)$.
Thus, for every x, either $x \notin A$ or $x \notin B$.
In particular, if we let $x = a$, then either $a \notin A$ or $a \notin B$.
Since $a \in A$, and either $a \notin A$ or $a \notin B$, then by disjunctive syllogism, $a \notin B$.
Hence, $a \in A$ and $a \notin B$, so $a \in A - B$, as desired.
We prove $A - B \subset A$.
To prove this, we assume $a \in A - B$.
We must prove $a \in A$.
Since $a \in A - B$, then $a \in A$ and $a \notin B$.

Hence, $a \in A$, as desired.
Since $A - B \subset A$ and $A \subset A - B$, then $A - B = A$, as desired.

Conversely, we prove $A - B = A$ implies $A \cap B = \emptyset$.
To prove this, we assume $A - B = A$.
To prove $A \cap B = \emptyset$, we use proof by contradiction.
Suppose $A \cap B \neq \emptyset$.
Then there exists an element in $A \cap B$.
Let x be an arbitrary element in $A \cap B$.
Then $x \in A \cap B$.
Hence, $x \in A$ and $x \in B$.
Since $x \in A$ and $A = A - B$, then $x \in A - B$.
Therefore, $x \in A$ and $x \notin B$, so $x \notin B$.
Thus, we have $x \in B$ and $x \notin B$, a contradiction.
Hence, $A \cap B = \emptyset$. □

Lemma 17. *Let A, B be sets.*
Then $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$.

Proof. Observe that

$$\begin{aligned}
(A - B) \cup (A \cap B) \cup (B - A) &= (A \cap \overline{B}) \cup (A \cap B) \cup (B \cap \overline{A}) \\
&= (A \cap (\overline{B} \cup B)) \cup (B \cap \overline{A}) \\
&= (A \cap U) \cup (B \cap \overline{A}) \\
&= A \cup (B \cap \overline{A}) \\
&= (A \cup B) \cap (A \cup \overline{A}) \\
&= (A \cup B) \cap U \\
&= A \cup B
\end{aligned}$$

□

Exercise 18. Let A and C be subsets of a set U.
Then
1. $\overline{(U - A) \cup C} = A - C = A - A \cap C$.
2. $(C - A) \cap U = C - A = C - A \cap C$.

Proof. We prove 1: $\overline{(U - A) \cup C} = A - C = A - A \cap C$.
Observe that

$$\begin{aligned}
\overline{(U - A) \cup C} &= \overline{\overline{A} \cup C} \\
&= \overline{\overline{A}} \cap \overline{C} \\
&= A \cap \overline{C} \\
&= A - C.
\end{aligned}$$

Hence, $\overline{(U - A) \cup C} = A - C$.

Observe that

$$\begin{aligned}
A - A \cap C &= A \cap \overline{A \cap C} \\
&= A \cap (\overline{A} \cup \overline{C}) \\
&= (A \cap \overline{A}) \cup (A \cap \overline{C}) \\
&= \emptyset \cup (A \cap \overline{C}) \\
&= A \cap \overline{C} \\
&= A - C.
\end{aligned}$$

Thus, $A - A \cap C = A - C$.

Therefore, $\overline{(U - A) \cup C} = A - C = A - A \cap C$. □

Proof. We prove 2: $(C - A) \cap U = C - A = C - A \cap C$.

Observe that

$$\begin{aligned}
(C - A) \cap U &= C - A \\
&= C - (C \cap A) \\
&= C - (A \cap C).
\end{aligned}$$

Therefore, $(C - A) \cap U = C - A = C - A \cap C$. □

Exercise 19. Let A, B, C be sets.

Then $A - (B \cup C) = (A - B) \cap (A - C)$.

Proof. Observe that

$$\begin{aligned}
A - (B \cup C) &= A \cap (\overline{B \cup C}) \\
&= A \cap (\overline{B} \cap \overline{C}) \\
&= (A \cap A) \cap (\overline{B} \cap \overline{C}) \\
&= A \cap (A \cap \overline{B}) \cap \overline{C} \\
&= A \cap (\overline{B} \cap A) \cap \overline{C} \\
&= (A \cap \overline{B}) \cap (A \cap \overline{C}) \\
&= (A - B) \cap (A - C).
\end{aligned}$$

□

Proposition 20. *Let A, B, C be sets.*

Then $A \cap (B - C) = (A \cap B) - (A \cap C)$.

Proof. Observe that

$$\begin{aligned}
(A \cap B) - (A \cap C) &= (A \cap B) \cap \overline{A \cap C} \\
&= (A \cap B) \cap (\overline{A} \cup \overline{C}) \\
&= (A \cap B \cap \overline{A}) \cup (A \cap B \cap \overline{C}) \\
&= (A \cap \overline{A} \cap B) \cup (A \cap B \cap \overline{C}) \\
&= (\emptyset \cap B) \cup (A \cap B \cap \overline{C}) \\
&= \emptyset \cup (A \cap B \cap \overline{C}) \\
&= A \cap B \cap \overline{C} \\
&= A \cap (B \cap \overline{C}) \\
&= A \cap (B - C)
\end{aligned}$$

□

Exercise 21. Let A, B, C, D be sets.
Then $(A \cup B) \cap (C \cap D) = (A \cap C \cap D) \cup (B \cap C \cap D)$.

Proof. Observe that

$$\begin{aligned}
(A \cup B) \cap (C \cap D) &= (C \cap D) \cap (A \cup B) \\
&= ((C \cap D) \cap A) \cup ((C \cap D) \cap B) \\
&= (A \cap (C \cap D)) \cup (B \cap (C \cap D)) \\
&= (A \cap C \cap D) \cup (B \cap C \cap D)
\end{aligned}$$

□

Exercise 22. Let A, X, Y be sets.
Let $A \cap X \subseteq A \cap Y$ and $A \cup X \subseteq A \cup Y$.
Then $X \subseteq Y$.

Solution. Our hypothesis is A, X, Y are sets such that $A \cap X \subseteq A \cap Y$ and $A \cup X \subseteq A \cup Y$.
To prove conclusion $X \subseteq Y$, we let $x \in X$.
We must prove $x \in Y$.
How can we use the hypotheses to prove $x \in Y$?
If we know $x \in A \cap Y$, then $x \in Y$.
Thus, we assume $x \in A$.
But x is either in A or not.
In other words, $x \in A$ or $x \in \overline{A}$, but not both.
This establishes a mutually exclusive, yet exhaustive cases to consider. □

Proof. Let A, X, Y be sets such that $A \cap X \subseteq A \cap Y$ and $A \cup X \subseteq A \cup Y$.
To prove $X \subseteq Y$, let $x \in X$.
We must prove $x \in Y$.
Either $x \in A$ or $x \in \overline{A}$, but not both.
We consider these cases separately.

Case 1: Suppose $x \in A$.
Since $x \in A$ and $x \in X$, then $x \in A \cap X$.
Since $A \cap X \subseteq A \cap Y$, then $x \in A \cap Y$.
Hence, $x \in Y$, as desired.
Case 2: Suppose $x \in \overline{A}$.
Then $x \notin A$.
Since $x \in A$, then either $x \in A$ or $x \in X$, so $x \in A \cup X$.
Since $A \cup X \subseteq A \cup Y$, then $x \in A \cup Y$.
Hence, either $x \in A$ or $x \in Y$.
But, $x \notin A$, so $x \in Y$, as desired. □

Exercise 23. Let A, B, X be sets.
If $A \cup X \subseteq B \cup X$ and $A \cup \overline{X} \subseteq B \cup \overline{X}$, then $A \subseteq B$.

Proof. Let A, B and X be arbitrary sets such that $A \cup X \subseteq B \cup X$ and $A \cup \overline{X} \subseteq B \cup \overline{X}$.
To prove $A \subseteq B$, let $x \in A$.
We must prove $x \in B$.
Since $x \in A$, then either $x \in A$ or $x \in X$.
Hence, $x \in A \cup X$.
Since $A \cup X \subseteq B \cup X$, then $x \in B \cup X$.
Similarly, since $x \in A$, then $x \in A$ or $x \in \overline{X}$.
Hence, $x \in A \cup \overline{X}$.
Since $A \cup \overline{X} \subseteq B \cup \overline{X}$, then $x \in B \cup \overline{X}$.
Either $x \in X$ or $x \in \overline{X}$, but not both.
We consider these cases separately.
Case 1: Suppose $x \in \overline{X}$.
Then $x \notin X$.
Since $x \in B \cup X$, then either $x \in B$ or $x \in X$.
Since $x \notin X$, then we conclude $x \in B$, as desired.
Case 2: Suppose $x \in X$.
Then $x \notin \overline{X}$.
Since $x \in B \cup \overline{X}$, then either $x \in B$ or $x \in \overline{X}$.
Since $x \notin \overline{X}$, then we conclude $x \in B$, as desired. □

Exercise 24. Let A, B, X be subsets of a universal set U.
If $A \cap X = B \cap X$ and $A \cap \overline{X} = B \cap \overline{X}$, then $A = B$.

Solution. To prove $A = B$, we assume $A \cap X = B \cap X$ and $A \cap \overline{X} = B \cap \overline{X}$.
We can use set identities to prove elegantly.
We just play around with various set identities, such as set distributivity and observe that $X \cup \overline{X} = U$.
We could also prove $A \subset B$ and $B \subset A$, although that is a longer, less elegant proof. □

Proof. Let A, B, and X be arbitrary subsets of a universal set U such that $A \cap X = B \cap X$ and $A \cap \overline{X} = B \cap \overline{X}$.

Then

$$\begin{aligned} A &= A \cap U \\ &= A \cap (X \cup \overline{X}) \\ &= (A \cap X) \cup (A \cap \overline{X}) \\ &= (B \cap X) \cup (B \cap \overline{X}) \\ &= B \cap (X \cup \overline{X}) \\ &= B \cap U \\ &= B \end{aligned}$$

□

Proposition 25. *Let A, B, C be sets.*
Then $(A \cup B) \cap C \subset A \cup (B \cap C)$.

Solution. There are multiple approaches to prove this.
Let $x \in (A \cup B) \cap C$.
To prove $x \in A \cup (B \cap C)$, we must prove either $x \in A$ or $x \in B \cap C$.
Since this is a statement of the form $P \to Q \vee R$, we can prove $P \wedge \neg Q \to R$.
Let $P : x \in (A \cup B) \cap C$.
Let $Q : x \in A$.
Let $R : x \in B \cap C$.
Then the statement form to prove is: $P \to Q \vee R$.
Since we know $P \to Q \vee R \Leftrightarrow (P \wedge \neg Q) \to R$, then we can assume $x \notin A$ and prove $x \in B$ and $x \in C$.
Thus,
Let $R_1 : x \in B$.
Let $R_2 : x \in C$.
Let $P' : P \wedge \neg Q$.
We now must prove: $P' \to (R_1 \wedge R_2)$.
We know that $P' \to (R_1 \wedge R_2) \Leftrightarrow (P' \to R_1) \wedge (P' \to R_2)$.
Thus, we must divide into cases:
One case to prove $x \in B$ and a separate case to prove $x \in C$.
Thus, we assume $x \notin A$ and prove $x \in B \cap C$.
Thus, we prove both $x \in B$ and $x \in C$ separately.
To prove $x \in C$:
Since $x \in (A \cup B) \cap C$, then $x \in A \cup B$ and $x \in C$, so $x \in C$.
To prove $x \in B$:
Since $x \in (A \cup B) \cap C$, then $x \in A \cup B$ and $x \in C$, so $x \in A \cup B$.
Hence, either $x \in A$ or $x \in B$.
By assumption, $x \notin A$, so $x \in B$. □

Proof. Let $x \in (A \cup B) \cap C$.
To prove $x \in A \cup (B \cap C)$, we must prove either $x \in A$ or $x \in B \cap C$.
Suppose $x \notin A$.
To prove $x \in B \cap C$, we prove $x \in B$ and $x \in C$.
Case 1: We prove $x \in C$.
Since $x \in (A \cup B) \cap C$, then $x \in C$.
Case 2: We prove $x \in B$.
Since $x \in (A \cup B) \cap C$, then $x \in A \cup B$.
Hence, either $x \in A$ or $x \in B$.
By assumption, $x \notin A$, so $x \in B$. $\square$

Exercise 26. For any sets A, B, C, if $A \subset B$, then $C - B \subset C - A$.
Is the converse true?
That is, does $C - B \subset C - A$ imply $A \subset B$?
If not , provide a counterexample.

Proof. Let A, B, C be sets such that $A \subset B$.
To prove $C - B \subset C - A$, let $x \in C - B$.
To prove $x \in C - A$, we must prove $x \in C$ and $x \notin A$.
Since $x \in C - B$, then $x \in C$ and $x \notin B$.
To prove $x \notin A$, we prove by contradiction.
Suppose $x \in A$.
Since $A \subset B$, then $x \in B$.
Hence, we have $x \in B$ and $x \notin B$, a contradiction.
Therefore, $x \notin A$.
Since $x \in C$ and $x \notin A$, then $x \in C - A$, as desired. $\square$

Solution. The converse is false.
That is, $C - B \subset C - A$ does not imply $A \subset B$.
Here is a counterexample.
Let $A = \{2, 4\}$ and $B = \{1, 2\}$ and $C = \{1, 2, 3\}$.
Then $C - B = \{3\}$ and $C - A = \{1, 3\}$ and $C - B \subset C - A$.
However, $A \not\subset B$ since $4 \in A$, but $4 \notin B$. $\square$

Proposition 27. *Let set A be a subset of a universal set U that has the property that $A \subset B$ for all $B \subset U$.*
Then $A = \emptyset$.

Solution. We can try an example, say let $U = \{1, 2, 3\}$.
We compute all subsets of U, ie, the power set of U.
Thus, $2^U = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
If B is an arbitrary set in 2^U, what set must be contained in all such sets B?
Well, it is obvious that the only set that satisfies this is the empty set.
Now, let's prove this.
The hypothesis is: for all sets B in U, $A \subset B$.
The conclusion is: $A = \emptyset$.

Since the conclusion is simple and the hypothesis is complicated, let's try proof by contrapositive, since direct proof leads nowhere.

Thus, we assume the negation of the conclusion.

Hence, we assume $A \neq \emptyset$.

We prove the negation of the hypothesis.

Thus, we prove there exists some set B in U such that $A \not\subset B$.

Since $A \neq \emptyset$, then A is not empty.

Hence, there is some element a in A.

To prove $A \not\subset B$, we must prove there exists some element in A that is not in B.

Now, which set should B be?

B must be a subset of U and not contain a.

A good choice, in fact the only choice, is B must be the empty set.

This is because the empty set is a subset of every set.

So, in particular, $\emptyset \subset U$.

Therefore, let $B = \emptyset$.

Since B is empty, then there is no element in B.

Hence, a cannot be in B, so $a \notin B$.

Thus, $a \in A$ and $a \notin B$, so $A \not\subset B$. □

Proof. We prove by contrapositive.

Suppose A is not empty.

We must prove there is some set B in U such that $A \not\subset B$.

Since the empty set $\emptyset$ is a subset of every set, then in particular, the empty set is a subset of U.

Let $B = \emptyset$.

Then B is a subset of U.

Since $A \neq \emptyset$, then there is some element a in A.

Since $B = \emptyset$, then there is no element in B.

Hence, a cannot be in B.

Thus, $a \in A$ and $a \notin B$, so $A \not\subset B$. □

Exercise 28. The power set of the empty set is the set $\{\emptyset\}$.

Proof. Let A be the power set of the empty set $\emptyset$ and let $B = \{\emptyset\}$.

Then $A = \{X : X \subset \emptyset\}$.

We must prove $A = B$.

We first prove $B \subset A$.

Since $\emptyset \in B$ and $\emptyset \subset \emptyset$, then $\emptyset \in A$, so $B \subset A$.

We next prove $A \subset B$.

Let $X \in A$.

Then $X \subset \emptyset$.

The only subset of the empty set is the empty set itself.

Thus, the empty set is a subset of itself and no other set is a subset of the empty set.

Thus, $X = \emptyset$, so $X \in B$.

Since $X \in A$ implies $X \in B$, then $A \subset B$.

Since $A \subset B$ and $B \subset A$, then $A = B$, as desired. $\square$

Proposition 29. *If A, B are sets such that $(A \cap \overline{B}) \cup (B \cap \overline{A}) = B$, then $A = \emptyset$.*

Solution. We note that this is symmetric difference of A and B.

Thus, we're trying to prove $A \triangle B = B$ implies $A = \emptyset$.

In other words, we wish to show $\emptyset$ is a left identity element for $\triangle$.

Hypothesis is: $(A \cap \overline{B}) \cup (B \cap \overline{A}) = B$.

Conclusion is: $A = \emptyset$.

Using direct proof doesn't lead anywhere, so try indirect proof.

In this case, let's try proof by contradiction.

Suppose $A \neq \emptyset$.

Then there is some element in A, so let $x \in A$.

Since $(A \cap \overline{B}) \cup (B \cap \overline{A}) = B$, then $x \in (A \cap \overline{B}) \cup (B \cap \overline{A})$ if and only if $x \in B$.

We can divide into mutually exhaustive cases.

We know either $x \in B$ or $x \notin B$.

Note that we cannot divide the proof into the cases Either $x \in A \cap \overline{B}$ or $x \in B \cap \overline{A}$ because we're not guaranteed that these are exhaustive and cover all scenarios.

We consider these cases separately.

Case 1: Suppose $x \in B$.

Since $x \in B$ if and only if $x \in (A \cap \overline{B}) \cup (B \cap \overline{A})$, then this implies either $x \in A \cap \overline{B}$ or $x \in B \cap \overline{A}$.

Suppose $x \in A \cap \overline{B}$.

Then $x \in A$ and $x \in \overline{B}$.

Since $x \in \overline{B}$, then $x \notin B$.

But, we have $x \in B$ and $x \notin B$, a contradiction.

Therefore, $x \notin A \cap \overline{B}$.

Suppose $x \in B \cap \overline{A}$.

Then $x \in B$ and $x \in \overline{A}$.

Since $x \in \overline{A}$, then $x \notin A$.

But, we have $x \in A$ and $x \notin A$, a contradiction.

Therefore, $x \notin B \cap \overline{A}$.

Hence, x is neither in $A \cap \overline{B}$ nor in $B \cap \overline{A}$, so $x \notin (A \cap \overline{B}) \cup (B \cap \overline{A})$.

Since $x \in (A \cap \overline{B}) \cup (B \cap \overline{A})$ if and only if $x \in B$, then this implies $x \notin B$.

Therefore, we have $x \in B$ and $x \notin B$, a contradiction.

Thus, x cannot be in B.

Case 2: Suppose $x \notin B$.

Since $x \notin B$, then $x \in \overline{B}$.

Since $x \in A$ and $x \in \overline{B}$, then $x \in A \cap \overline{B}$.

Since $x \notin B$ and $x \in B$ if and only if $x \in (A \cap \overline{B}) \cup (B \cap \overline{A})$, then this implies $x \notin (A \cap \overline{B}) \cup (B \cap \overline{A})$.

Thus, x is neither in $A \cap \overline{B}$ nor in $B \cap \overline{A}$, so, $x \notin A \cap \overline{B}$ and $x \notin B \cap \overline{A}$.
But, we have $x \in A \cap \overline{B}$ and $x \notin A \cap \overline{B}$, a contradiction.
Therefore, x cannot be in B. □

Proof. We use proof by contradiction.
Suppose $A \neq \emptyset$.
Then there is some element in A, so let $x \in A$.
Either $x \in B$ or $x \notin B$.
We consider these cases separately.
Case 1: Suppose $x \in B$.
Since $x \in B$ if and only if $x \in (A \cap \overline{B}) \cup (B \cap \overline{A})$, then this implies either $x \in A \cap \overline{B}$ or $x \in B \cap \overline{A}$.
Since $x \in B$, then $x \notin \overline{B}$.
Hence, x cannot be in $A \cap \overline{B}$.
Therefore, $x \in B \cap \overline{A}$.
Thus, $x \in \overline{A}$.
Hence, $x \in A$ and $x \in \overline{A}$, so $x \in A \cap \overline{A}$.
But, $A \cap \overline{A} = \emptyset$, so $x \in \emptyset$, a contradiction.
Therefore, x cannot be in B.
Case 2: Suppose $x \notin B$.
Since $x \notin B$, then $x \in \overline{B}$.
Since $x \in A$ and $x \in \overline{B}$, then $x \in A \cap \overline{B}$.
Since $A \cap \overline{B} \subset (A \cap \overline{B}) \cup (B \cap \overline{A})$, then this implies $x \in (A \cap \overline{B}) \cup (B \cap \overline{A})$.
Since $(A \cap \overline{B}) \cup (B \cap \overline{A}) = B$, then $x \in B$.
Since $x \in \overline{B}$ and $x \in B$, then $x \in \overline{B} \cap B$.
But, $\overline{B} \cap B = \emptyset$, so $x \in \emptyset$, a contradiction.
Therefore, x cannot be in B.
Therefore, in all cases, a contradiction is reached, and so $A = \emptyset$. □

Proposition 30. *For any sets A and B, if $(A - B) \cup (B - A) = A \cup B$, then $A \cap B = \emptyset$.*

Solution. We note that this is symmetric difference of A and B.
Thus, we're trying to prove $A \triangle B = A \cup B$ implies $A \cap B = \emptyset$.
Direct proof doesn't seem to lead anywhere, so let's try indirect proof. We try proof by contradiction. □

Proof. Let A and B be sets such that $(A - B) \cup (B - A) = A \cup B$.
To prove $A \cap B = \emptyset$, we use proof by contradiction.
Suppose $A \cap B \neq \emptyset$.
Then $A \cap B$ is not empty, so there exists at least one element in $A \cap B$.
Let x be an element of $A \cap B$.
Then $x \in A$ and $x \in B$.
Since $P \wedge Q \Rightarrow P \vee Q$ is a tautology, where $P : x \in A$ and $Q : x \in B$, then $x \in A$ or $x \in B$.
Hence, $x \in A \cup B$.
Since $x \in A$ and $x \in B$, then $x \notin A - B$.

Since $x \in B$ and $x \in A$, then $x \notin B - A$.
Thus, $x \notin A - B$ and $x \notin B - A$, so x is not in either $A - B$ or $B - A$.
Therefore, $x \notin (A - B) \cup (B - A)$.
Hence, $x \in A \cup B$ and $x \notin (A - B) \cup (B - A)$, so there is an element in $A \cup B$, but not in $(A - B) \cup (B - A)$.
Therefore, $A \cup B \neq (A - B) \cup (B - A)$.
Thus, we have $(A - B) \cup (B - A) = A \cup B$ and $(A - B) \cup (B - A) \neq A \cup B$, a contradiction. Hence, $A \cap B = \emptyset$. ☐

Proposition 31. *If A and B are sets, then $\mathscr{P}(A) \cup \mathscr{P}(B) \subset \mathscr{P}(A \cup B)$.*

Proof. Let A and B be sets.
To prove $\mathscr{P}(A) \cup \mathscr{P}(B) \subset \mathscr{P}(A \cup B)$, let $X \in \mathscr{P}(A) \cup \mathscr{P}B$.
To prove $X \in \mathscr{P}(A \cup B)$, we must prove $X \subset A \cup B$.
Thus, assume $a \in X$.
We must prove $a \in A \cup B$.
Since $X \in \mathscr{P}(A) \cup \mathscr{P}B$, then either $X \in \mathscr{P}(A)$ or $X \in \mathscr{P}(B)$.
Hence, either $X \subset A$ or $X \subseteq B$.
We consider these cases separately.
Case 1: Suppose $X \subset A$.
Since $a \in X$ and $X \subset A$, then $a \in A$.
Case 2: Suppose $X \subset B$.
Since $a \in X$ and $X \subset B$, then $a \in B$.
Hence, in all cases, either $a \in A$ or $a \in B$, so that $a \in A \cup B$.
Therefore, $a \in X$ implies $a \in A \cup B$, so $X \subset A \cup B$.
Hence, $X \in \mathscr{P}(A \cup B)$, as desired. ☐

Proposition 32. *If A and B are arbitrary sets such that $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$, then either $A \subseteq B$ or $B \subseteq A$.*

Proof. Let A and B be arbitrary sets such that $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$.
We must prove either $A \subseteq B$ or $B \subseteq A$.
Since $A \cup B \subseteq A \cup B$, then $A \cup B \in \mathscr{P}(A \cup B)$.
Since $\mathscr{P}(A \cup B) = \mathscr{P}(A) \cup \mathscr{P}(B)$, then $A \cup B \in \mathscr{P}(A) \cup \mathscr{P}(B)$.
Hence, either $A \cup B \in \mathscr{P}(A)$ or $A \cup B \in \mathscr{P}(B)$.
Thus, either $A \cup B \subseteq A$ or $A \cup B \subseteq B$.
We consider these cases separately.
Case 1: Suppose $A \cup B \subseteq A$.
Since $A \subseteq A \cup B$, then $A \cup B = A$.
Hence, $B \cup A = A$.
Since $X \cup Y = Y$ iff $X \subseteq Y$, then in particular, if we let $X = B$ and $Y = A$, then $B \cup A = A$ iff $B \subseteq A$.
Since $B \cup A = A$, then we conclude $B \subseteq A$, as desired.
Case 2: Suppose $A \cup B \subseteq B$.
Since $B \subseteq A \cup B$, then $A \cup B = B$.
Hence, $B \cup A = B$.

Since $X \cup Y = Y$ iff $X \subseteq Y$, then in particular, if we let $X = A$ and $Y = B$, then $A \cup B = B$ iff $A \subseteq B$.

Since $A \cup B = B$, then we conclude $A \subseteq B$, as desired. $\square$

Exercise 33. Is $\mathscr{P}(A \cup B) = \mathscr{P}(A) \cup \mathscr{P}(B)$?

Solution. We can show that the assertion is false.

Here is a counter example.

Let $A = \{x, y\}$ and $B = \{y, z\}$.

Then $\mathscr{P}(A) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$ and $\mathscr{P}(B) = \{\emptyset, \{y\}, \{z\}, \{y, z\}\}$ and $A \cup B = \{x, y, z\}$ and

$\mathscr{P}(A \cup B) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$ and

$\mathscr{P}(A) \cup \mathscr{P}(B) = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{z\}, \{y, z\}\}$.

Observe that $\{x, z\} \in \mathscr{P}(A \cup B)$, but $\{x, z\} \notin \mathscr{P}(A) \cup \mathscr{P}(B)$.

Hence, $\mathscr{P}(A \cup B) \neq \mathscr{P}(A) \cup \mathscr{P}(B)$. $\square$

Proposition 34. *Let A and B be sets.*

Then $\mathscr{P}(A) \subset \mathscr{P}(B)$ iff $A \subset B$.

Proof. We prove if $\mathscr{P}(A) \subset \mathscr{P}(B)$, then $A \subset B$.

Suppose $\mathscr{P}(A) \subset \mathscr{P}(B)$.

Since $A \subset A$, then $A \in \mathscr{P}(A)$.

Since $A \in \mathscr{P}(A)$ and $\mathscr{P}(A) \subset \mathscr{P}(B)$, then $A \in \mathscr{P}(B)$.

Hence, $A \subset B$.

Conversely, we prove if $A \subset B$, then $\mathscr{P}(A) \subset \mathscr{P}(B)$.

Suppose $A \subset B$.

Let $X \in \mathscr{P}(A)$.

Then $X \subset A$.

Since $X \subset A$ and $A \subset B$, then by transitivity of $\subset$, $X \subset B$.

Hence, $X \in \mathscr{P}(B)$.

Therefore, if $X \in \mathscr{P}(A)$ then $X \in \mathscr{P}(B)$, so $\mathscr{P}(A) \subset \mathscr{P}(B)$. $\square$

Proposition 35. *Let A and B be sets.*

If $\mathscr{P}(A) = \mathscr{P}(B)$ then $A = B$.

Proof. Suppose $\mathscr{P}(A) = \mathscr{P}(B)$.

Since $A \subset A$, then $A \in \mathscr{P}(A)$.

But, $\mathscr{P}(A) = \mathscr{P}(B)$, so $A \in \mathscr{P}(B)$.

Therefore, $A \subset B$.

Since $B \subset B$, then $B \in \mathscr{P}(B)$.

But, $\mathscr{P}(B) = \mathscr{P}(A)$, so $B \in \mathscr{P}(A)$.

Therefore, $B \subset A$.

Since $A \subset B$ and $B \subset A$, then $A = B$, as desired. $\square$

Exercise 36. Let A, B, X be sets.

If $A \cap X = B \cap X$ and $A \cup X = B \cup X$, then $A = B$.

Solution. To prove $A = B$, we assume $A \cap X = B \cap X$ and $A \cup X = B \cup X$.

We can use set identities to prove elegantly.

Since the conclusion asserts $A = B$ and we're dealing with set unions and set intersections, this suggest possible use of the set absorption law which states:

For every set S and T, $S \cup (S \cap T) = S$. □

Proof. Let A, B, and X be arbitrary sets such that $A \cap X = B \cap X$ and $A \cup X = B \cup X$.

Then

$$\begin{aligned}
A &= A \cup (A \cap X) \\
&= A \cup (B \cap X) \\
&= (A \cup B) \cap (A \cup X) \\
&= (A \cup B) \cap (B \cup X) \\
&= (B \cup A) \cap (B \cup X) \\
&= B \cup (A \cap X) \\
&= B \cup (B \cap X) \\
&= B
\end{aligned}$$

□

Exercise 37. Let $A = \{5, -7\}$ and $B = \{x \in \mathbb{R} : x^2 + 2x - 35 = 0\}$.

Then $A = B$.

Solution. To prove $A = B$ we may use the theorem that states $A = B$ iff $A \subset B$ and $B \subset A$.

Thus we must prove both $A \subset B$ and $B \subset A$. □

Proof. To prove $A = B$ we prove $A \subset B$ and $B \subset A$.

We prove $A \subset B$.

To prove $A \subset B$, we assume $a \in A$.

To prove $a \in B$ we must prove $a \in R$ and satisfies $a^2 + 2a - 35 = 0$.

Since $a \in A$, then either $a = 5$ or $a = -7$.

Case 1: Suppose $a = 5$.

Then $5 \in \mathbb{R}$ and $5^2 + 2 * 5 - 35 = 0$.

Case 2: Suppose $a = -7$.

Then $-7 \in \mathbb{R}$ and $(-7)^2 + 2(-7) - 35 = 0$.

In either case, $a \in B$, as desired.

We prove $B \subset A$.

To prove $B \subset A$, we assume $x \in B$.

We must prove $x \in A$.

Thus, $x \in \mathbb{R}$ and satisfies $x^2 + 2x - 35 = 0$.

Observe that $0 = x^2 + 2x - 35 = (x + 7)(x - 5)$.

Hence, either $x + 7 = 0$ or $x - 5 = 0$.

Thus, either $x = -7$ or 5, so $x \in \{-7, 5\}$. Hence, $x \in A$. □

Exercise 38. For all sets X and Y, $X \cap (Y \cup \overline{X}) \subset Y$.

Proof. Let X and Y be arbitrary sets.
To prove $X \cap (Y \cup \overline{X}) \subset Y$, we assume $a \in X \cap (Y \cup \overline{X})$.
We must prove $a \in Y$.
By assumption, $a \in X$ and $a \in Y \cup \overline{X}$.
Therefore, $a \in X$ and either $a \in Y$ or $a \in \overline{X}$.
Hence, either $a \in Y$ or $a \notin X$.
Since $a \in X$ and either $a \in Y$ or $a \notin X$, then by disjunctive syllogism, $a \in Y$, as desired. □

Exercise 39. For all sets X, Y, and Z, if $X \subset Y$, then $X \cap Z \subset Y \cap Z$.

Proof. Let X, Y, and Z be arbitrary sets.
To prove the conditional $X \subset Y \to X \cap Z \subset Y \cap Z$, we assume $X \subset Y$, our hypothesis.
We must prove $X \cap Z \subset Y \cap Z$.
To prove $X \cap Z \subset Y \cap Z$, we assume $b \in X \cap Z$.
We must prove $b \in Y$ and $b \in Z$.
By assumption, $b \in X$ and $b \in Z$.
Since $b \in X$ and $X \subset Y$, then $b \in Y$.
Thus, we have $b \in Y$ and $b \in Z$, as desired. □

Exercise 40. For all sets X, Y, and Z, if $X \subseteq Y$ and $X \subseteq Z$, then $X \subseteq Y \cap Z$.

Solution. Let X, Y and Z be arbitrary sets.
To prove the conditional $X \subseteq Y \wedge X \subseteq Z \to X \subseteq Y \cap Z$, we assume $X \subseteq Y$ and $X \subseteq Z$.
We must prove $X \subseteq Y \cap Z$.
To prove $X \subseteq Y \cap Z$, we must prove the statement $\forall a.(a \in X \to a \in Y \cap Z)$ is true.
Since this is universally quantified statement, we let a be an arbitrary object in the domain of discourse.
To prove $a \in X \to a \in Y \cap Z$, we use direct proof.
Thus, we assume $a \in X$.
We must prove $a \in Y \cap Z$.
To prove $a \in Y \cap Z$, we must prove $a \in Y$ and $a \in Z$. □

Proof. Let X, Y and Z be arbitrary sets.
Assume $X \subseteq Y$ and $X \subseteq Z$.
Let a be an arbitrary object in the domain of discourse.
To prove $X \subseteq Y \cap Z$, we must prove $a \in X \to a \in Y \cap Z$.
To prove this, we assume $a \in X$.
We must prove $a \in Y$ and $a \in Z$.
Since $a \in X$ and $X \subseteq Y$, then $a \in Y$.
Since $a \in X$ and $X \subseteq Z$, then $a \in Z$.
Hence, $a \in Y$ and $a \in Z$, as desired. □

Exercise 41. For all sets X, Y, and Z, if $X \subseteq Z$ and $Y \subseteq Z$, then $X \cup Y \subseteq Z$.

Solution. Let X, Y and Z be arbitrary sets.
The hypothesis is $X \subseteq Z$ and $Y \subseteq Z$.
The conclusion is $X \cup Y \subseteq Z$.
To prove the conditional $X \subseteq Z \wedge Y \subseteq Z \rightarrow X \cup Y \subseteq Z$ true, we assume the hypothesis is true.
Thus, we assume $X \subseteq Z$ and $Y \subseteq Z$.
We must prove $X \cup Y \subseteq Z$.
To prove the conclusion is true, we must prove $\forall a.(a \in X \cup Y \rightarrow a \in Z)$ is true.
Since the conclusion is a universally quantified statement, we let a be an arbitrary object in the domain of discourse.
To prove $a \in X \cup Y \rightarrow a \in Z$, we use direct proof.
Thus, we assume $a \in X \cup Y$.
We must prove $a \in Z$. □

Proof. Let X, Y and Z be arbitrary sets.
We assume $X \subseteq Z$ and $Y \subseteq Z$.
Let a be an arbitrary object in the domain of discourse.
To prove $X \cup Y \subseteq Z$, we must prove $a \in X \cup Y \rightarrow a \in Z$.
To prove this, we assume $a \in X \cup Y$.
We must prove $a \in Z$.
Since $a \in X \cup Y$, then either $a \in X$ or $a \in Y$.
We consider these cases separately.
Case 1: Suppose $a \in X$.
Since $a \in X$ and $X \subseteq Z$, by hypothesis, then $a \in Z$.
Case 2: Suppose $a \in Y$.
Since $a \in Y$ and $Y \subseteq Z$, by hypothesis, then $a \in Z$.
Both cases exhaustively show $a \in Z$, our desired conclusion. □

Exercise 42. For arbitrary sets A and B, if $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$.

Solution. The hypothesis is $A \subseteq B$.
The conclusion is $\overline{B} \subseteq \overline{A}$.
To prove the conditional $A \subseteq B \rightarrow \overline{B} \subseteq \overline{A}$, we assume $A \subseteq B$.
We must prove $\overline{B} \subseteq \overline{A}$.
To prove $\overline{B} \subseteq \overline{A}$, we must prove the statement $\forall x.(x \in \overline{B} \rightarrow x \in \overline{A})$ is true.
Let x be an arbitrary object in the domain of discourse.
We must prove $x \in \overline{B} \rightarrow x \in \overline{A}$ is true. □

Proof. We assume $A \subseteq B$.
To prove $\overline{B} \subseteq \overline{A}$, we must prove the statement $\forall x.(x \in \overline{B} \rightarrow x \in \overline{A})$ is true.
Let x be an arbitrary object in the domain of discourse.
We must prove $x \in \overline{B} \rightarrow x \in \overline{A}$ is true.
To prove $x \in \overline{A}$, we assume $x \in \overline{B}$.
We use proof by contradiction.

Suppose $x \notin \overline{A}$.
Then $x \in A$.
Since $x \in A$ and $A \subseteq B$, then $x \in B$.
By assumption, $x \in \overline{B}$, so $x \notin B$.
Hence, we have $x \in B$ and $x \notin B$, a contradiction.
Therefore, $x \in \overline{A}$. □

Exercise 43. For arbitrary sets A and B, if $\overline{B} \subseteq \overline{A}$, then $A \subseteq B$.

Proof. To prove the conditional $\overline{B} \subseteq \overline{A} \to A \subseteq B$, we assume $\overline{B} \subseteq \overline{A}$.
We must prove $A \subseteq B$.
Since $\overline{B} \subseteq \overline{A}$, then we know that $\overline{\overline{A}} \subseteq \overline{\overline{B}}$, based on the previous exercise.
Since $\overline{\overline{A}} = A$ and $\overline{\overline{B}} = B$, then we have $A \subseteq B$, as desired. □

Exercise 44. For arbitrary sets A and B, if $A \subset B$, then $\overline{A} \cup B = U$.

Solution. There are multiple approaches to this proof. One approach uses algebraic set properties. □

Proof. To prove the conditional $A \subset B \to \overline{A} \cup B = U$, we assume $A \subset B$.
We must prove $\overline{A} \cup B = U$.
Since $A \subset B$, then we know $A - B = \emptyset$.
Since $A - B = A \cap \overline{B}$, then $A \cap \overline{B} = \emptyset$.
Taking the complement of both sets, we obtain $\overline{A \cap \overline{B}} = \overline{\emptyset}$.
Thus, $\overline{A} \cup \overline{\overline{B}} = U$, so $\overline{A} \cup B = U$, as desired. □

Exercise 45. For arbitrary sets A and B, if $\overline{A} \cup B = U$, then $A \subset B$.

Proof. To prove $A \subset B$, let $x \in A$.
We must prove $x \in B$.
Since U is a universal set, then $x \in U$.
Since $U = \overline{A} \cup B$, then $x \in \overline{A} \cup B$.
Hence, either $x \in \overline{A}$ or $x \in B$.
Since $x \in A$, then $x \notin \overline{A}$.
Therefore, we conclude $x \in B$, as desired. □

Exercise 46. For arbitrary sets A and B, if $(\overline{A} \cup B) \cap (A \cup \overline{B}) = U$, then $A = B$.

Proof. To prove $A = B$, we prove $A \subset B$ and $B \subset A$.
To prove $A \subset B$, let $x \in A$.
We must prove $x \in B$.
Since $x \in U$, then $x \in (\overline{A} \cup B) \cap (A \cup \overline{B})$.
Thus, $x \in \overline{A} \cup B$ and $x \in A \cup \overline{B}$.
Hence, $x \in \overline{A} \cup B$, so either $x \in \overline{A}$ or $x \in B$.
Since $x \in A$, then $x \notin \overline{A}$.
Therefore, $x \in B$, so $A \subset B$, as desired.
To prove $B \subset A$, let $y \in B$.
We must prove $y \in A$.

Since $y \in U$, then $y \in (\overline{A} \cup B) \cap (A \cup \overline{B})$.
Thus, $y \in \overline{A} \cup B$ and $y \in A \cup \overline{B}$.
Hence, $y \in A \cup \overline{B}$, so either $y \in A$ or $y \in \overline{B}$.
Since $y \in B$, then $y \notin \overline{B}$.
Therefore, $y \in A$, so $B \subset A$, as desired.
Since $A \subset B$ and $B \subset A$, then $A = B$, as desired. ☐

Exercise 47. Let X be a set.
A set Y is a complement of X iff $X \cup Y = U$ and $X \cap Y = \emptyset$.
Prove every set has at most one complement.

Solution. To prove every set has at most one complement, let X be an arbitrary set.
We must prove X has at most one complement.
Define predicates:
1. $p(X,Y) : X \cup Y = U$
2. $q(X,Y) : X \cap Y = \emptyset$.
To prove there exists at most one complement, we let Y_1, Y_2 be arbitrary sets such that $p(X,Y_1)$ and $q(X,Y_1)$ and $p(X,Y_2)$ and $q(X,Y_2)$ hold.
We must show $Y_1 = Y_2$.
To prove this we prove $Y_1 \subset Y_2$ and $Y_2 \subset Y_1$.
Note, we are only proving at most one complement exists.
We aren't proving there exists a complement. ☐

Proof. To prove every set has at most one complement, let X be an arbitrary set.
To prove there exists at most one complement of X, let Y_1 and Y_2 be arbitrary sets such that $X \cup Y_1 = U$ and $X \cap Y_1 = \emptyset$ and $X \cup Y_2 = U$ and $X \cap Y_2 = \emptyset$.
We must prove $Y_1 = Y_2$.
Since $X \cup Y_1 = U$ and $X \cup Y_2 = U$, then $X \cup Y_1 = X \cup Y_2$.
Since $X \cap Y_1 = \emptyset$ and $X \cap Y_2 = \emptyset$, then $X \cap Y_1 = X \cap Y_2$.

We prove $Y_1 \subset Y_2$.
Suppose $a \in Y_1$.
We must prove $a \in Y_2$.
Either $a \in X$ or $a \notin X$.
We consider these cases separately.
Case 1: Suppose $a \in X$.
Since $a \in X$ and $a \in Y_1$, then $a \in X \cap Y_1 = X \cap Y_2$.
Thus, $a \in X \cap Y_2$, so $a \in Y_2$.
Case 2: Suppose $a \notin X$.
Since $a \in Y_1$ and $Y_1 \subset X \cup Y_1$, then $a \in X \cup Y_1 = X \cup Y_2$.
Thus, $a \in X \cup Y_2$, so either $a \in X$ or $a \in Y_2$.
Since $a \notin X$, then we conclude $a \in Y_2$.
In all cases, $a \in Y_2$, so $Y_1 \subset Y_2$.

We prove $Y_2 \subset Y_1$.
Suppose $b \in Y_2$.
We must prove $b \in Y_1$.
Either $b \in X$ or $b \notin X$.
We consider these cases separately.
Case 1: Suppose $b \in X$.
Since $b \in X$ and $b \in Y_2$, then $b \in X \cap Y_2 = X \cap Y_1$.
Thus, $b \in X \cap Y_1$, so $b \in Y_1$.
Case 2: Suppose $b \notin X$.
Since $b \in Y_2$ and $Y_2 \subset X \cup Y_2$, then $b \in X \cup Y_2 = X \cup Y_1$.
Thus, $b \in X \cup Y_1$, so either $b \in X$ or $b \in Y_1$.
Since $b \notin X$, then we conclude $b \in Y_1$.
In all cases, $b \in Y_1$, so $Y_2 \subset Y_1$.

Since $Y_1 \subset Y_2$ and $Y_2 \subset Y_1$, then $Y_1 = Y_2$, as desired. □

Solution. To prove there exists at most one complement of X, let Y_1 and Y_2 be arbitrary sets such that $X \cup Y_1 = U$ and $X \cap Y_1 = \emptyset$ and $X \cup Y_2 = U$ and $X \cap Y_2 = \emptyset$.
We must prove $Y_1 = Y_2$.
Observe that $X \cup Y_1 = X \cup Y_2 = U$ and $X \cap Y_1 = X \cap Y_2 = \emptyset$.
We can play around with set identities.
Observe that $Y_1 \cap U = Y_1$ and $Y_2 \cap U = Y_2$.
Thus, $Y_1 = Y_1 \cap U = Y_1 \cap (X \cup Y_2) = (Y_1 \cap X) \cup (Y_1 \cap Y_2) = \emptyset \cup (Y_1 \cap Y_2) = Y_1 \cap Y_2$.
Also, $Y_2 = Y_2 \cap U = Y_2 \cap (X \cup Y_1) = (Y_2 \cap X) \cup (Y_2 \cap Y_1) = \emptyset \cup (Y_2 \cap Y_1) = Y_2 \cap Y_1$.
Hence, we re-write in a proof by transitive format to derive the proof. □

Proof. To prove there exists at most one complement of X, let Y_1 and Y_2 be arbitrary sets such that $X \cup Y_1 = U$ and $X \cap Y_1 = \emptyset$ and $X \cup Y_2 = U$ and $X \cap Y_2 = \emptyset$.
We must prove $Y_1 = Y_2$.
Observe that $X \cup Y_1 = X \cup Y_2 = U$ and $X \cap Y_1 = X \cap Y_2 = \emptyset$.
Then

$$\begin{aligned}
Y_1 &= Y_1 \cap U \\
&= Y_1 \cap (X \cup Y_2) \\
&= (Y_1 \cap X) \cup (Y_1 \cap Y_2) \\
&= (X \cap Y_1) \cup (Y_1 \cap Y_2) \\
&= (X \cap Y_2) \cup (Y_1 \cap Y_2) \\
&= (X \cup Y_1) \cap Y_2 \\
&= U \cap Y_2 \\
&= Y_2\text{, as desired.}
\end{aligned}$$

□

Exercise 48. Let A and B sets in a universal set U.

A subset C of U is a complement of A relative to B iff $A \cup C = A \cup B$ and $A \cap C = \emptyset$. Prove that given any sets A and B, A has a unique relative complement in B.

Solution. Let A and B be arbitrary sets in universal set U.

To prove A has a unique relative complement in B, we must prove:

1. Existence: There exists a subset C of U such that $A \cup C = A \cup B$ and $A \cap C = \emptyset$.

2. Uniqueness: If sets C_1 and C_2 of U exist such that $A \cup C_1 = A \cup B$ and $A \cap C_1 = \emptyset$ and $A \cup C_2 = A \cup B$ and $A \cap C_2 = \emptyset$, then $C_1 = C_2$. □

Proof. **Existence:**

Let $C = \{x \in U : x \in B \wedge x \notin A\}$.

Then C is a subset of U.

We prove $A \cup C = A \cup B$ and $A \cap C = \emptyset$.

We prove $A \cup C = A \cup B$.

We first prove $A \cup C \subset A \cup B$.

Suppose $x \in A \cup C$.

To prove $x \in A \cup B$, we must prove either $x \in A$ or $x \in B$.

So, assume $x \notin A$.

We must prove $x \in B$.

Since $x \in A \cup C$, then either $x \in A$ or $x \in C$.

But, $x \notin A$, so $x \in C$.

Hence, $x \in B$.

Therefore, $x \in A \cup C$ implies $x \in A \cup B$, so $A \cup C \subset A \cup B$.

We now prove $A \cup B \subset A \cup C$.

Suppose $x \in A \cup B$.

To prove $x \in A \cup C$, we must prove either $x \in A$ or $x \in C$.

So, assume $x \notin A$.

We must prove $x \in C$.

Since $x \in A \cup B$, then either $x \in A$ or $x \in B$.

But, $x \notin A$, so $x \in B$.

Since $x \in B$ and $x \notin A$, then $x \in C$.

Therefore, $x \in A \cup B$ implies $x \in A \cup C$, so $A \cup B \subset A \cup C$.

Since $A \cup C \subset A \cup B$ and $A \cup B \subset A \cup C$, then $A \cup C = A \cup B$.

To prove $A \cap C = \emptyset$, suppose $A \cap C \neq \emptyset$.

Then there is some $x \in A \cap C$.

Hence, $x \in A$ and $x \in C$, so $x \in B$ and $x \notin A$.

Thus, $x \in A$ and $x \notin A$, a contradiction.

Therefore, $A \cap C = \emptyset$.

Since there exists a set C such that $A \cup C = A \cup B$ and $A \cap C = \emptyset$, then there is at least one set that is a relative complement of A in B.

Uniqueness: Let C_1 and C_2 be subsets of U such that $A \cup C_1 = A \cup B$ and $A \cap C_1 = \emptyset$ and $A \cup C_2 = A \cup B$ and $A \cap C_2 = \emptyset$.

We must prove $C_1 = C_2$.

Observe that

$$\begin{aligned}
C_1 &= C_1 \cup \emptyset \\
&= C_1 \cup (A \cap C_2) \\
&= (C_1 \cup A) \cap (C_1 \cup C_2) \\
&= (A \cup C_1) \cap (C_1 \cup C_2) \\
&= (A \cup C_2) \cap (C_1 \cup C_2) \\
&= (A \cap C_1) \cup C_2 \\
&= \emptyset \cup C_2 \\
&= C_2, \text{ as desired.}
\end{aligned}$$

Therefore, there is at most one set that is a relative complement of A in B.

Since there is at least one set that is a relative complement of A in B and there is at most one set that is a relative complement of A in B, then we conclude there is exactly one set that is a relative complement of A in B. Hence, the relative complement of A in B is unique. □

Cartesian Product

Exercise 49. The unit circle in $\mathbb{R}^2$ is not a Cartesian product of two subsets of $\mathbb{R}$.

Solution. Let S be the unit circle in $\mathbb{R}^2$.
Then $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
Suppose that S is a Cartesian product of two subsets of $\mathbb{R}$.
Then there exist subsets A and B of $\mathbb{R}$ such that $S = A \times B$.
Thus, $S \subset A \times B$ and $A \times B \subset S$.
Let $(x, y) \in S$.
Then $x, y \in \mathbb{R}$ and $x^2 + y^2 = 1$.
Thus, $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.
Hence, $x \in [-1, 1]$ and $y \in [-1, 1]$.
Since $(x, y) \in S$ and $S \subset A \times B$, then $(x, y) \in A \times B$.
Hence, $x \in A$ and $y \in B$.
Thus, $x \in [-1, 1]$ and $x \in A$, so $x \in [-1, 1]$ implies $x \in A$.
Therefore, $[-1, 1] \subset A$.
Since $y \in [-1, 1]$ and $y \in B$, then $y \in [-1, 1]$ implies $y \in B$.
Therefore, $[-1, 1] \subset B$.

Consider the ordered pair $(0, 0)$.
Since $0 \in [-1, 1]$ and $[-1, 1] \subset A$, then $0 \in A$.
Since $0 \in [-1, 1]$ and $[-1, 1] \subset B$, then $0 \in B$.
Hence, $(0, 0) \in A \times B$.
Since $A \times B = S$, then $(0, 0) \in S$, so $0^2 + 0^2 = 1$.
Thus, $0 = 1$, a contradiction.

Therefore, S cannot be a Cartesian product of two subsets of $\mathbb{R}$. $\square$

Solution. Here is an alternate solution to show that the unit circle is not a Cartesian product.

Let S be the unit circle in $\mathbb{R}^2$.

Then $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Suppose that S is a Cartesian product of two subsets of $\mathbb{R}$.

Then there exist subsets A and B of $\mathbb{R}$ such that $S = A \times B$.

Since $(1, 0) \in S$ and $(0, 1) \in S$ and $S = A \times B$, then $(1, 0) \in A \times B$ and $(0, 1) \in A \times B$.

Since $(1, 0) \in A \times B$, then $1 \in A$.

Since $(0, 1) \in A \times B$, then $1 \in B$.

Thus, $(1, 1) \in A \times B$, so $(1, 1) \in S$.

Hence, $1^2 + 1^2 = 1$, so $2 = 1$, a contradiction.

Therefore, S is not a Cartesian product of two subsets of $\mathbb{R}$. $\square$

Exercise 50. The triangle $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq x\}$ is not a Cartesian product of two subsets of $\mathbb{R}$.

Solution. Let $S = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq x\}$ be the triangle and its interior in the first quadrant of the plane.

Suppose that S is a Cartesian product of two subsets of $\mathbb{R}$.

Then there exist subsets A and B of $\mathbb{R}$ such that $S = A \times B$.

Since $(0, 0) \in S$ and $(1, 1) \in S$ and $S = A \times B$, then $(0, 0) \in A \times B$ and $(1, 1) \in A \times B$. Hence, $(0, 1) \in A \times B$ and $(1, 0) \in A \times B$, so $(0, 1) \in A \times B = S$.

Thus, $(0, 1) \in S$, so $0 \leq 1 \leq 0$, a contradiction. Therefore, S is not a Cartesian product of two subsets of $\mathbb{R}$. $\square$

Exercise 51. Let $A \times B \subset \mathbb{R}^2$ be a Cartesian product for certain subsets A and B of $\mathbb{R}$.

Then $(x, x) \in A \times B$ and $(y, y) \in A \times B$ iff $(x, y) \in A \times B$ and $(y, x) \in A \times B$.

Proof. We first prove if $(x, x) \in A \times B$ and $(y, y) \in A \times B$ then $(x, y) \in A \times B$ and $(y, x) \in A \times B$.

Suppose $(x, x) \in A \times B$ and $(y, y) \in A \times B$.

Since $(x, x) \in A \times B$, then $x \in A$ and $x \in B$.

Since $(y, y) \in A \times B$, then $y \in A$ and $y \in B$.

Since $x \in A$ and $y \in B$, then $(x, y) \in A \times B$.

Since $y \in A$ and $x \in B$, then $(y, x) \in A \times B$.

We prove if $(x, y) \in A \times B$ and $(y, x) \in A \times B$, then $(x, x) \in A \times B$ and $(y, y) \in A \times B$.

Suppose $(x, y) \in A \times B$ and $(y, x) \in A \times B$.

Since $(x, y) \in A \times B$, then $x \in A$ and $y \in B$.

Since $(y, x) \in A \times B$, then $y \in A$ and $x \in B$.

Since $x \in A$ and $x \in B$, then $(x, x) \in A \times B$.

Since $y \in A$ and $y \in B$, then $(y, y) \in A \times B$. $\square$

Exercise 52. Let A and B be subsets of $\mathbb{R}$.

Then $(A \times \mathbb{R}) \cap (\mathbb{R} \times B) = A \times B$.

Proof. We first prove $(A \times \mathbb{R}) \cap (\mathbb{R} \times B) \subset A \times B$.
Let $(x, y) \in (A \times \mathbb{R}) \cap (\mathbb{R} \times B)$.
Then $(x, y) \in A \times \mathbb{R}$ and $(x, y) \in \mathbb{R} \times B$.
Since $(x, y) \in A \times \mathbb{R}$, then $x \in A$ and $y \in \mathbb{R}$.
Since $(x, y) \in \mathbb{R} \times B$, then $x \in \mathbb{R}$ and $y \in B$.
Since $x \in A$ and $y \in B$, then $(x, y) \in A \times B$.
Therefore, $(A \times \mathbb{R}) \cap (\mathbb{R} \times B) \subset A \times B$.

We next prove $A \times B \subset (A \times \mathbb{R}) \cap (\mathbb{R} \times B)$.
Let $(a, b) \in A \times B$.
Then $a \in A$ and $b \in B$.
Since $a \in A$ and $A \subset \mathbb{R}$, then $a \in \mathbb{R}$.
Since $b \in B$ and $B \subset \mathbb{R}$, then $b \in \mathbb{R}$.
Since $a \in A$ and $b \in \mathbb{R}$, then $(a, b) \in A \times \mathbb{R}$.
Since $a \in \mathbb{R}$ and $b \in B$, then $(a, b) \in \mathbb{R} \times B$.
Since $(a, b) \in A \times \mathbb{R}$ and $(a, b) \in \mathbb{R} \times B$, then $(a, b) \in (A \times \mathbb{R}) \cap (\mathbb{R} \times B)$.
Therefore, $A \times B \subset (A \times \mathbb{R}) \cap (\mathbb{R} \times B)$.

Since $(A \times \mathbb{R}) \cap (\mathbb{R} \times B) \subset A \times B$ and $A \times B \subset (A \times \mathbb{R}) \cap (\mathbb{R} \times B)$, then $(A \times \mathbb{R}) \cap (\mathbb{R} \times B) = A \times B$, as desired. □

Exercise 53. Prove or disprove the conjecture: If A, B and C are sets and $A \times C = B \times C$, then $A = B$.

Solution. We try direct proof,but if we try this we must assume $A \times C$ is not the empty set. However, $A \times C$ could very well be the empty set.

In fact, if $A \times C = \emptyset = B \times C$, then $A = B$ or $A \neq B$, we just can't tell. So, the possibility exists that $A \neq B$ when $A \times C = \emptyset$.

So, let's just come up with a counter example. This should be easy. □

Proof. The conjecture is false.

Here is a counterexample: Let $A = \{1, 2, 3\}$, $B = \{a, b\}$, and $C = \emptyset$. Then $A \times C = B \times C = \emptyset$, but $A \neq B$. □

Exercise 54. Show that the empty set is the only set W that has the property that there exist distinct sets A and B such that $A \times W = B \times W$.

Solution. Hypothesis: $A \neq B$ and $A \times W = B \times W$
Conclusion: $W = \emptyset$.
To prove $\emptyset$ is the unique set that satisfies this property, we must prove:
1. Existence: Prove $W = \emptyset$ satisfies $A \times W = B \times W$ for distinct sets A and B.
Thus, we must prove $A \times \emptyset = B \times \emptyset$.
But, we know that $A \times \emptyset = \emptyset$ and $B \times \emptyset = \emptyset$ for any sets A and B.
Thus, $A \times \emptyset = B \times \emptyset = \emptyset$.
2. Uniqueness: Prove $\emptyset$ is the only such set W that satisfies this.

To prove uniqueness, we must prove $A \neq B$ and $A \times W = B \times W$ imply $W = \emptyset$.

We use proof by contradiction.

Thus, we assume $W \neq \emptyset$ and derive a contradiction.

We use existing theorems/propositions as well. $\square$

Proof. Existence:

Since $A \times \emptyset = B \times \emptyset = \emptyset$ for any sets A and B, we may, for example, let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$.

Uniqueness:

Assume $A \neq B$ and $A \times W = B \times W$.

Suppose for the sake of contradiction that $W \neq \emptyset$.

Then $W \neq 0$ and $A \times W = B \times W$ imply $A = B$ by a previous theorem.

But, $A \neq B$ by hypothesis.

Hence, $W = \emptyset$. $\square$

Exercise 55. Let A, B, C, D be sets such that $A \subset B$ and $C \subset D$.

Then $A \times C \subset B \times D$.

Solution. Our hypothesis is:

A, B, C, D are sets

$A \subset B$

$C \subset D$.

The conclusion is: $A \times C \subset B \times D$

To prove $A \times C \subset B \times D$, we let $(x, y) \in A \times C$.

We must prove $(x, y) \in B \times D$. $\square$

Proof. To prove $A \times C \subset B \times D$, let (x, y) be an arbitrary element of $A \times C$.

We must prove $(x, y) \in B \times D$.

Since $(x, y) \in A \times C$, then $x \in A$ and $y \in C$.

Since $x \in A$ and $A \subset B$, then $x \in B$.

Since $y \in C$ and $C \subset D$, then $y \in D$.

Since $x \in B$ and $y \in D$, then $(x, y) \in B \times D$, as desired. $\square$

Exercise 56. Let A, B, C be sets.

$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof. Let $x \in (A \times B) \cap (C \times D)$ be arbitrary.

Then $x \in A \times B$ and $x \in C \times D$.

Thus, there exist $a \in A$ and $b \in B$ such that $x = (a, b)$ and there exist $c \in C$ and $d \in D$ such that $x = (c, d)$.

Hence, $(a, b) = (c, d)$, so $a = c$ and $b = d$.

Since $c = a$ and $a \in A$, then $c \in A$.

Thus, $c \in A$ and $c \in C$, so $c \in A \cap C$.

Since $d = b$ and $b \in B$, then $d \in B$.

Thus, $d \in B$ and $d \in D$, so $d \in B \cap D$.

Hence, $x = (c, d) \in (A \cap C) \times (B \cap D)$.

Therefore, $(A \times B) \cap (C \times D) \subset (A \cap C) \times (B \cap D)$.

Let $y \in (A \cap C) \times (B \cap D)$ be arbitrary.
Then there exist $s \in A \cap C$ and $t \in B \cap D$ such that $y = (s, t)$.
Hence, $s \in A$ and $s \in C$ and $t \in B$ and $t \in D$.
Thus, $(s, t) \in A \times B$ and $(s, t) \in C \times D$.
Consequently, $y = (s, t) \in (A \times B) \cap (C \times D)$.
Therefore, $(A \cap C) \times (B \cap D) \subset (A \times B) \cap (C \times D)$.
Since $(A \times B) \cap (C \times D) \subset (A \cap C) \times (B \cap D)$ and $(A \cap C) \times (B \cap D) \subset (A \times B) \cap (C \times D)$, then $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. □

Exercise 57. Let A, B, C, D be sets such that $A \times B \subset C \times D$.
Then $A \subset C$ and $B \subset D$.

Solution. Define
$H : A \times B \subset C \times D$.
$C_1 : A \subset C$.
$C_2 : B \subset D$.
The argument is: $H \rightarrow (C_1 \wedge C_2)$.
Observe that $H \rightarrow (C_1 \wedge C_2) \Leftrightarrow (H \rightarrow C_1) \wedge (H \rightarrow C_2)$.
Therefore, we must prove both:
1. if $A \times B \subset C \times D$, then $A \subset C$.
2. if $A \times B \subset C \times D$, then $B \subset D$. □

Proof. We first prove $A \subset C$.
Since A is not empty, let x be an arbitrary element of A.
We must prove $x \in C$.
Since B is not empty, then let y be an arbitrary element of B.
Since $x \in A$ and $y \in B$, then $(x, y) \in A \times B$.
Since $A \times B \subset C \times D$, then this implies $(x, y) \in C \times D$.
Hence, $x \in C$, as desired.
We now prove $B \subset D$.
Since B is not empty, let x be an arbitrary element of B.
We must prove $x \in D$.
Since A is not empty, then let y be an arbitrary element of A.
Since $y \in A$ and $x \in B$, then $(y, x) \in A \times B$.
Since $A \times B \subset C \times D$, then this implies $(y, x) \in C \times D$.
Hence, $x \in D$, as desired. □

Exercise 58. Let A, B, C be sets.
$(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$.

Proof. Let $x \in (A \times B) \cup (C \times D)$ be arbitrary.
Then either $x \in A \times B$ or $x \in C \times D$.
We consider these cases separately.
Case 1: Suppose $x \in A \times B$.
Then there exist $a \in A$ and $b \in B$ such that $x = (a, b)$.
Since $a \in A$, then either $a \in A$ or $a \in C$, so $a \in A \cup C$.
Since $b \in B$, then either $b \in B$ or $b \in D$, so $b \in B \cup D$.

Thus, $x = (a, b) \in (A \cup C) \times (B \cup D)$.
Case 2: Suppose $x \in C \times D$.
Then there exist $c \in C$ and $d \in D$ such that $x = (c, d)$.
Since $c \in C$, then either $c \in C$ or $c \in A$, so $c \in C \cup A$.
Hence, $c \in A \cup C$.
Since $d \in D$, then either $d \in D$ or $d \in B$, so $d \in D \cup B$.
Hence, $d \in B \cup D$.
Thus, $x = (c, d) \in (A \cup C) \times (B \cup D)$.
Therefore, in either case $x \in (A \times B) \cup (C \times D)$ implies $x \in (A \cup C) \times (B \cup D)$.
Hence, $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$. ☐

Exercise 59. Let A, B, C be sets.
Then $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$ is false.

Solution. We note that even though $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$, it is not the case that $(A \cup C) \times (B \cup D) \subset (A \times B) \cup (C \times D)$.
Therefore, it cannot be the case that $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.
Thus, we must devise a counter example.
Let $A = \{1, 2\}$ and
$B = \{3, 4\}$ and
$C = \{5, 6\}$ and
$D = \{7, 8\}$.
Then $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$ and
$C \times D = \{(5, 7), (5, 8), (6, 7), (6, 8)\}$ and
$A \cup C = \{1, 2, 5, 6\}$ and
$B \cup D = \{3, 4, 7, 8\}$.
Thus, $(A \times B) \cup (C \times D) = \{(1, 3), (1, 4), (2, 3), (2, 4), (5, 7), (5, 8), (6, 7), (6, 8)\}$.
The elements in $(A \cup C) \times (B \cup D)$ are:
$(1, 3), (1, 4), (1, 7), (1, 8)$,
$(2, 3), (2, 4), (2, 7), (2, 8)$,
$(5, 3), (5, 4), (5, 7), (5, 8)$,
$(6, 3), (6, 4), (6, 7), (6, 8)$.
Observe that $(1, 7) \in (A \cup C) \times (B \cup D)$, but $(1, 7) \notin (A \times B) \cup (C \times D)$. ☐

Exercise 60. For all sets A and B, $(A \cap B) \cup (A \cap \overline{B}) \subset A$.

Proof. Let A and B be arbitrary sets.
To prove $(A \cap B) \cup (A \cap \overline{B}) \subset A$, assume $x \in A \cap B) \cup (A \cap \overline{B})$.
We must prove $x \in A$.
By our assumption, we know that either $x \in A \cap B$ or $x \in A \cap \overline{B}$, that is, either $x \in A$ and $x \in B$, or else $x \in A$ and $x \in \overline{B}$.
We divide the argument into exhaustive cases:
Case 1: Suppose that $x \in A$ and $x \in B$.
Then, in particular, $x \in A$, so the desired conclusion is obtained in this case.
Case 2: Suppose that $x \in A$ and $x \in \overline{B}$.
Then, again, $x \in A$, so the desired conclusion is obtained in this case.
Under either of the only two possible cases, we have $x \in A$, the desired conclusion. ☐

Exercise 61. For all sets A and B, $A \subset (A \cap B) \cup (A \cap \overline{B})$.

Proof. Let A and B be arbitrary sets.

To prove $A \subset (A \cap B) \cup (A \cap \overline{B})$, assume $x \in A$.

We must prove $x \in (A \cap B) \cup (A \cap \overline{B})$.

To do this, we must prove either $x \in (A \cap B)$ or $x \in (A \cap \overline{B})$, that is, either $x \in A$ and $x \in B$, or else $x \in A$ and $x \in \overline{B}$.

We note that, necessarily, either $x \in B$ or $x \notin B$.

Having noted this, we consider two cases.

Case 1: Suppose that $x \in B$.

Then, since $x \in A$, we have $x \in A$ and $x \in B$, one of the two alternatives in our desired conclusion.

Case 2: Suppose that $x \notin B$.

Equivalently, $x \in \overline{B}$.

Then, since $x \in A$, we have $x \in A$ and $x \in \overline{B}$, the other of the two alternatives in our desired conclusion. □

Exercise 62. Prove: $(A - B) \cap (B - A) = \emptyset$.

Solution. Our conclusion is $C : (A - B) \cap A = \emptyset$.

To prove C, we can use algebraic properties of sets.

We know $A - B = A \cap \overline{B}$. □

Proof. Observe that

$$\begin{aligned}(A - B) \cap (B - A) &= (A \cap \overline{B}) \cap (B \cap \overline{A}) \\ &= A \cap (\overline{B} \cap B) \cap \overline{A} \\ &= A \cap \emptyset \cap \overline{A} \\ &= \emptyset \cap \overline{A} \\ &= \emptyset\end{aligned}$$

□

Exercise 63. Prove: $(A \cup B) - B = A - B$.

Solution. We can use algebraic properties of sets(ie, previous propositions/theorems of set theory). □

Proof. Observe that

$$\begin{aligned}(A \cup B) - B &= (A \cup B) \cap \overline{B} \\ &= (A \cap \overline{B}) \cup (B \cap \overline{B}) \\ &= (A \cap \overline{B}) \cup \emptyset \\ &= A \cap \overline{B} \\ &= A - B\end{aligned}$$

□

Exercise 64. Prove: $(A \cap B) - B = \emptyset$.

Solution. We use algebraic set identities. $\square$

Proof. Observe that

$$\begin{aligned}(A \cap B) - B &= (A \cap B) \cap \overline{B} \\ &= A \cap (B \cap \overline{B}) \\ &= A \cap \emptyset \\ &= \emptyset\end{aligned}$$

$\square$

Exercise 65. Prove: $A \cup B = (A \cap B) \cup (A - B) \cup (B - A)$.

Solution. We can use algebraic properties of set to prove this. Just play around with set identities until we devise a way to prove this. $\square$

Proof. Let U be the universal set and $A \subseteq U$ and $B \subseteq U$.
Observe that

$$\begin{aligned}(A \cap B) \cup (A - B) \cup (B - A) &= (A \cap B) \cup (A \cap \overline{B}) \cup (B \cap \overline{A}) \\ &= A \cap (B \cup \overline{B}) \cup (B \cap \overline{A}) \\ &= A \cap (U) \cup (B \cap \overline{A}) \\ &= A \cup (B \cap \overline{A}) \\ &= (A \cup B) \cap (A \cup \overline{A}) \\ &= (A \cup B) \cap (U) \\ &= A \cup B\end{aligned}$$

$\square$

Exercise 66. $B \subset \overline{A}$ if and only if $A \cap B = \emptyset$.

Solution. Let A, B be any sets.
How are A, B related?
A, B can be in only one of 5 possible relationships:
1. A and B are disjoint
2. A and B intersect but are not subsets of each other.
3. A is a proper subset of B.
4. B is a proper subset of A.
5. A=B

The hypothesis $B \subset \overline{A}$ corresponds to scenario 1.
This is consistent with a Venn diagram of A and B as disjoint sets.
We prove this biconditional using two proofs. $\square$

Proof. We prove $A \cap B = \emptyset$ implies $B \subset \overline{A}$.
Suppose $A \cap B = \emptyset$.
Let $x \in B$.
Since $A \cap B$ is empty then $x \notin A \cap B$.
Hence $x \notin A$ or $x \notin B$.
Since $x \notin A$ or $x \notin B$ and $x \in B$ then $x \notin A$.
Thus $x \in \overline{A}$ by defn of $\overline{A}$.
Therefore $x \in B \to x \in \overline{A}$ so $B \subset \overline{A}$.

Conversely, we prove $B \subset \overline{A}$ implies $A \cap B = \emptyset$.
Suppose $B \subset \overline{A}$.
For the sake of contradiction suppose $A \cap B \neq \emptyset$.
Then there exists an element in $A \cap B$.
Let $x \in A \cap B$.
Then $x \in A$ and $x \in B$.
Since $x \in B$ and $B \subset \overline{A}$ then $x \in \overline{A}$.
Thus $x \notin A$ by defn of $\overline{A}$.
Thus we have $x \in A$ and $x \notin A$, a contradiction.
Hence $A \cap B = \emptyset$. □

Exercise 67. For all sets A, B, C if $A \subset B$ then $(A \cap C) \subset (B \cap C)$.

Proof. Let A, B, C be sets.
Suppose $A \subset B$.
Let $x \in A \cap C$.
Then $x \in A$ and $x \in C$ so $x \in A$.
Since $x \in A$ and $A \subset B$ then $x \in B$.
Since $x \in B$ and $x \in C$ then $x \in B \cap C$.
Hence $x \in A \cap C$ implies $x \in B \cap C$ so $A \cap C \subset B \cap C$. □

Exercise 68. For all sets A, B, C if $A \subset B$ then $(A \cup C) \subset (B \cup C)$.

Proof. Let A, B, C be sets.
Suppose $A \subset B$.
Let $x \in A \cup C$.
Then $x \in A$ or $x \in C$.
Case 1: Suppose $x \in A$.
Since $x \in A$ and $A \subset B$ then $x \in B$.
Thus $x \in B$ or $x \in C$ so $x \in B \cup C$.
Case 2: Suppose $x \in C$.
Then $x \in B$ or $x \in C$.
Thus $x \in B \cup C$.
Therefore $x \in B \cup C$ in general.
Hence $x \in A \cup C$ implies $x \in B \cup C$ so $A \cup C \subset B \cup C$. □

Exercise 69. For all sets A, B, C if $A \subset B$ then $\overline{B} \subset \overline{A}$.

Proof. Let A, B be sets.

Suppose $A \subset B$.

Let $x \in \overline{B}$.

Then $x \notin B$.

Since $A \subset B$ then $x \in A \to x \in B$.

Since $x \in A \to x \in B$ and $x \notin B$ then $x \notin A$.

Thus $x \in \overline{A}$.

Hence $x \in \overline{B} \to x \in \overline{A}$ so $\overline{B} \subset \overline{A}$. □

Exercise 70. $A \cap B = \emptyset$ iff $A \cup B = A \triangle B$.

Proof. We prove $A \cap B = \emptyset \Rightarrow A \cup B = A \triangle B$.

Observe that $A \triangle B = (A \cup B) - (A \cap B) = (A \cup B) - \emptyset = (A \cup B) \cap U = A \cup B$.

Conversely, we prove $A \cup B = A \triangle B \Rightarrow A \cap B = \emptyset$.

Suppose $A \cup B = A \triangle B$.

For the sake of contradiction assume $A \cap B \neq \emptyset$.

Let $x \in U$ such that $x \in A \cap B$.

Then $x \in A$ and $x \in B$.

Since $x \in A$ then $x \in A$ or $x \in B$ so $x \in A \cup B$.

Hence $x \in A \triangle B$ by hypothesis.

Since $A \triangle B = (A \cup B) - (A \cap B)$ then $x \notin (A \cap B)$ by definition of set difference.

Thus we have $x \in A \cap B$ and $x \notin A \cap B$, a contradiction. Therefore $A \cap B = \emptyset$. □

Exercise 71. Prove $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$ for any sets A, B, C.

Solution. We try some examples of different sets for A,B,C to verify the claim.

When we do this, the claim checks out.

We also draw a diagram to visualize to see if the claim seems correct, using Venn diagrams.

It appears it does.

Thus, we have more evidence that the claim seems to be true so we can prove this.

We play around with each of the two expressions using set identities to see if they equal some third expression.

We find that they do, so we write up a proof accordingly in a proof by transitive format because the claim is of the form $\forall x.p(x)$. □

Proof. Let A, B, and C be arbitrary sets.

Then

$$\begin{aligned}
(A\cap B)\triangle(A\cap C) &= [(A\cap B)\cap\overline{A\cap C}]\cup[(A\cap C)\cap\overline{A\cap B}]\\
&= [(A\cap B)\cap(\overline{A}\cup\overline{C})]\cup[(A\cap C)\cap(\overline{A}\cup\overline{B})]\\
&= [(A\cap B\cap\overline{A})\cup(A\cap B\cap\overline{C})]\cup[(A\cap C\cap\overline{A})\cup(A\cap C\cap\overline{B})]\\
&= (\emptyset\cup(A\cap B\cap\overline{C}))\cup(\emptyset\cup(A\cap C\cap\overline{B}))\\
&= (A\cap B\cap\overline{C})\cup(A\cap C\cap\overline{B})\\
&= A\cap[(B\cap\overline{C})\cup(C\cap\overline{B})]\\
&= A\cap(B\triangle C)
\end{aligned}$$

□

Exercise 72. Prove or disprove the conjecture: There exists a set X for which $\mathbb{R}\subseteq X$ and $\emptyset\in X$.

Solution. We can construct $X = R\cup\{\emptyset\}$. □

Proof. The conjecture is true.

Let $X=\mathbb{R}\cup\{\emptyset\}$.

If $r\in\mathbb{R}$, then $r\in\mathbb{R}\cup\{\emptyset\}=X$, so $R\subseteq X$.

Likewise, $\emptyset\in\mathbb{R}\cup\{\emptyset\}=X$ because $\emptyset\in\{\emptyset\}$. □

Exercise 73. Prove or disprove the conjecture: For all sets A and B, if $A-B=\emptyset$, then $B\neq\emptyset$.

Solution. If we let A and B both be the empty set, then $A-B=\emptyset$, but B does equal $\emptyset$. □

Proof. The conjecture is false.

Suppose $A=B=\emptyset$. Then $A-B=\emptyset$, but it is not true that $B\neq\emptyset$. □

Exercise 74. Prove or disprove the conjecture: If A, B, and C are sets, then $A-(B\cap C)=(A-B)\cap(A-C)$.

Solution. We can consider drawing a Venn diagram of sets A, B, C. We can have a configuration of sets in which they intersect.

If we draw such a configuration, the Venn diagram shows that this conjecture is false. So, we can devise a counter example to demonstrate that this conjecture is false. □

Proof. The conjecture is false because of the following counterexample.

Let $A=\{a,b,c,d\}$, $B=\{b,e,a,t\}$ and $C=\{d,e,a,f\}$.

Then $A-(B\cap C) = A-\{a,e\} = \{b,c,d\}$ and $(A-B)\cap(A-C) = \{c,d\}\cap\{b,c\}=\{c\}$, so $A-(B\cap C)\neq(A-B)\cap(A-C)$. □

Exercise 75. Let A, B, and C be sets. If $B\subset C$, then $A\times B\subset A\times C$.

Proof. To prove $A \times B \subset A \times C$, let $t \in A \times B$.
We must prove $t \in A \times C$.
Since $t \in A \times B$, then there exist $x \in A$ and $y \in B$ such that $t = (x, y)$.
Since $y \in B$ and $B \subset C$, then $y \in C$.
Hence, $x \in A$ and $y \in C$, so $(x, y) \in A \times C$.
Therefore, $t \in A \times C$, so $A \times B \subset A \times C$, as desired. $\square$

Exercise 76. If $(A \cup C) \subset (A \cup B)$ and $(A \cap C) \subset (A \cap B)$, what can we deduce about C?

Solution. Our hypothesis H is P_1 and P_2
where
$P_1 : (A \cup C) \subset (A \cup B)$
$P_2 : (A \cap C) \subset (A \cap B)$.
We use logic to deduce valid conclusions given these two premises.
Let $x \in C$.
Then $x \in A$ or $x \in C$, so $x \in A \cup C$.
Since $x \in A \cup C$ and $(A \cup C) \subset (A \cup B)$, then $x \in A \cup B$.
Thus, $x \in A$ or $x \in B$.
We consider these cases separately.
Case 1: Suppose $x \in B$.

Then $x \in C \rightarrow x \in B$.
Case 2: Suppose $x \in A$.

Then $x \in A$ and $x \in C$, so $x \in A \cap C$.
Since $x \in A \cap C$ and $(A \cap C) \subset (A \cap B)$, then $x \in A \cap B$.
Thus, by definition of $\cap$, $x \in B$.
Hence, $x \in C \rightarrow x \in B$.
Both cases show $x \in C \rightarrow x \in B$.
Since x is arbitrary, then $\forall x. x \in C \rightarrow x \in B$.
Therefore, $C \subset B$. $\square$

Exercise 77. Prove $(A \cup B) \cap (B \cup C) \cap (C \cup A) = (A \cap B) \cup (B \cap C) \cup (C \cap A)$.

Solution. Let $G = (A \cup B) \cap (B \cup C) \cap (C \cup A)$.
Let $H = (A \cap B) \cup (B \cap C) \cup (C \cap A)$.
To prove $G = H$, we prove $G \subseteq H$ and $H \subseteq G$.

Let $x \in H$.
Then $x \in A \cap B$ or $x \in B \cap C$ or $x \in C \cap A$.
We consider these cases separately.
Case 1: Suppose $x \in A \cap B$.

Then $x \in A$ and $x \in B$.
Since $x \in A$ and $x \in B$ then $x \in A \cup B$.
Since $x \in B$ then $x \in B$ or $x \in C$. Thus, $x \in B \cup C$.

Since $x \in A$ then $x \in C$ or $x \in A$. Thus, $x \in C \cup A$.
Hence, $x \in A \cup B$ and $x \in B \cup C$ and $x \in C \cup A$, so $x \in G$.
Therefore, $x \in A \cap B \to x \in G$.
Case 2: Suppose $x \in B \cap C$.

Then $x \in B$ and $x \in C$.
Since $x \in B$ then $x \in A$ or $x \in B$. Thus, $x \in A \cup B$.
Since $x \in B$ and $x \in C$ then $x \in B \cup C$.
Since $x \in C$ then $x \in C$ or $x \in A$. Thus, $x \in C \cup A$.
Hence, $x \in A \cup B$ and $x \in B \cup C$ and $x \in C \cup A$, so $x \in G$.
Therefore, $x \in B \cap C \to x \in G$.
Case 3: Suppose $x \in C \cap A$.

Then $x \in C$ and $x \in A$.
Since $x \in A$ then $x \in A$ or $x \in B$. Thus, $x \in A \cup B$.
Since $x \in C$ then $x \in B$ or $x \in C$. Thus, $x \in B \cup C$.
Since $x \in C$ and $x \in A$ then $x \in C \cup A$.
Hence, $x \in A \cup B$ and $x \in B \cup C$ and $x \in C \cup A$, so $x \in G$.
Therefore, $x \in C \cap A \to x \in G$.
Each case implies $x \in H \to x \in G$.
Since x is arbitrary, then $\forall x. x \in H \to x \in G$.
Therefore, $H \subseteq G$.
Let $y \in G$.
Then $y \in A \cap B$ or $y \in B \cap C$ or $y \in C \cap A$.
We consider these cases separately.
Case 1: Suppose $y \in A \cap B$.

Then $y \in A$ and $y \in B$.
Since $y \in A$ and $y \in B$ then $y \in A \cup B$.
Since $y \in B$ then $y \in B$ or $y \in C$. Thus, $y \in B \cup C$.
Since $y \in A$ then $y \in C$ or $y \in A$. Thus, $y \in C \cup A$.
Hence, $y \in A \cup B$ and $y \in B \cup C$ and $y \in C \cup A$, so $y \in G$.
Therefore, $y \in A \cap B \to y \in G$.
Case 2: Suppose $y \in B \cap C$.

Then $y \in B$ and $y \in C$.
Since $y \in B$ then $y \in A$ or $y \in B$. Thus, $y \in A \cup B$.
Since $y \in B$ and $y \in C$ then $y \in B \cup C$.
Since $y \in C$ then $y \in C$ or $y \in A$. Thus, $y \in C \cup A$.
Hence, $y \in A \cup B$ and $y \in B \cup C$ and $y \in C \cup A$, so $y \in G$.
Therefore, $y \in B \cap C \to y \in G$.
Case 3: Suppose $y \in C \cap A$.

Then $y \in C$ and $y \in A$.
Since $y \in A$ then $y \in A$ or $y \in B$. Thus, $y \in A \cup B$.
Since $y \in C$ then $y \in B$ or $y \in C$. Thus, $y \in B \cup C$.

Since $y \in C$ and $y \in A$ then $y \in C \cup A$.
Hence, $y \in A \cup B$ and $y \in B \cup C$ and $y \in C \cup A$, so $y \in G$.
Therefore, $y \in B \cap C \to y \in G$.
Each case implies $y \in H \to y \in G$.
Since y is arbitrary, then $\forall y. y \in H \to y \in G$.
Therefore, $H \subseteq G$. □

Indexed Sets

Exercise 78. If $\{A_k : k = 1, 2, 3, ...\}$ is a decreasing collection of sets, then $\cup_{k=1}^{\infty} A_k \subseteq A_1$.

Proof. To prove this conditional, we assume $\{A_k : k = 1, 2, 3, ...\}$ is a decreasing collection of sets.
To prove $\cup_{k=1}^{\infty} A_k \subseteq A_1$, we assume $x \in \cup_{k=1}^{\infty} A_k$.
We must prove $x \in A_1$.
Let $A = \{A_k : k = 1, 2, 3, ...\}$.
Since $x \in \cup_{k=1}^{\infty} A_k$, then there exists a natural number k such that $x \in A_k$.
Let k be a natural number such that $x \in A_k$.
Then either $k > 1$ or $k = 1$.
We consider these cases separately.
Case 1: Suppose $k = 1$.
Then $x \in A_1$.
Case 2: Suppose $k > 1$.
Then $1 < k$.
Since A is a decreasing family of sets and $1 < k$, then $A_1 \supseteq A_k$.
Hence, $A_k \subseteq A_1$.
Since $x \in A_k$ and $A_k \subseteq A_1$, then $x \in A_1$.
Each of the cases exhaustively show $x \in A_1$, as desired. □

Proof. To prove this conditional, we assume $\{A_k : k = 1, 2, 3, ...\}$ is a decreasing collection of sets.
To prove $\cup_{k=1}^{\infty} A_k \subseteq A_1$, we assume $x \in \cup_{k=1}^{\infty} A_k$.
We must prove $x \in A_1$.
Let $A = \{A_k : k = 1, 2, 3, ...\}$.
We use proof by contradiction.
Suppose $x \notin A_1$.
Since $x \in \cup_{k=1}^{\infty} A_k$ and $x \notin A_1$, then there exists a $k \in \mathbb{N}$ that is greater than 1 such that $x \in A_k$.
Let k be a natural number that is greater than 1 such that $x \in A_k$.
Since A is a decreasing family of sets and $1 < k$, then $A_1 \supseteq A_k$.
Hence, $A_k \subseteq A_1$.
Since $x \in A_k$ and $A_k \subseteq A_1$, then $x \in A_1$.
Thus, we have $x \notin A_1$ and $x \in A_1$, a contradiction.
Therefore, $x \in A_1$, as desired. □

Exercise 79. For all sets A, if $\{B_i : i \in \mathbb{N}\}$ is an infinite collection of sets indexed by $\mathbb{N}$, then $A - \cap_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty}(A - B_i)$.

Solution. This statement is of the form $\forall x.p(x)$.

Hence, we assume A is an arbitrary set.

To prove $A - \cap_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty}(A - B_i)$, we assume $\{B_i : i \in \mathbb{N}\}$ is an infinite collection of sets indexed by $\mathbb{N}$.

To prove $A - \cap_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty}(A - B_i)$, we could use the definition of set equality, but that approach doesn't seem to go anywhere. We could try proving each set is a subset of the other, but that doesn't seem fruitful either. Instead, we'll use algebraic set identities and properties to prove this. □

Proof. Let A be an arbitrary set.

Let $B = \{B_i : i \in \mathbb{N}\}$ be an infinite collection of sets indexed by $\mathbb{N}$.

Then

$$\begin{aligned} A - \cap_{i=1}^{\infty} B_i &= A \cap (\overline{\cap_{i=1}^{\infty} B_i}) \\ &= A \cap (\cup_{i=1}^{\infty} \overline{B}_i) \\ &= (\cup_{i=1}^{\infty} \overline{B}_i) \cap A \\ &= \cup_{i=1}^{\infty} (\overline{B}_i \cap A) \\ &= \cup_{i=1}^{\infty} (A \cap \overline{B}_i) \\ &= \cup_{i=1}^{\infty} (A - B_i) \end{aligned}$$

□

Exercise 80. Let $A = \{A_k : k \in \mathbb{N}\}$.

Let $A_k = \{1, 2, 3, ..., k\}$.

Discuss A and compute $\cup_{k=1}^{\infty} A_k$ and $\cap_{k=1}^{\infty} A_k$.

Solution. Observe that $A_k = \{n \in \mathbb{N} : n \leq k\}$.

Hence, $A_k \subseteq \mathbb{N}$.

We compute $A = \{A_1, A_2, A_3, ...\} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, ...\}$.

We observe that as k increases, each succeeding A_k contains the previous set.

More precisely, if k, m are arbitrary natural numbers and $k < m$, then $A_k \subseteq A_m$.

Hence, A is an increasing collection of sets.

We compute $\cap_{k=1}^{\infty} A_k$.

Observe that $\cap_{k=1}^{\infty} A_k = A_1 \cap A_2 \cap ... = \{x : (\forall k \in \mathbb{N})(x \in A_k)\} = \{1\} = A_1$. We could also graph each A_k on a number line and easily see this result.

Conjecture: If A is an increasing collection of sets, is it always true that $\cap_{k=1}^{\infty} A_k = A_1$?

We could try to prove this conjecture.

We now compute $\cup_{k=1}^{\infty} A_k$.

We know $\cup_{k=1}^{\infty} A_k = A_1 \cup A_2 \cup ... = \{x : (\exists k \in \mathbb{N})(x \in A_k)\}$.

Intuitively, we see that as k gets larger, no matter how large, A_k seems to get larger approaching $\mathbb{N}$.

So, we conjecture that the union of all the A_k is $\mathbb{N}$ itself.
Let's try to prove $\cup_{k=1}^{\infty} A_k = \mathbb{N}$.
We must prove both $\cup_{k=1}^{\infty} A_k \subset \mathbb{N}$ and $\mathbb{N} \subset \cup_{k=1}^{\infty} A_k$.
We prove $\cup_{k=1}^{\infty} A_k \subset \mathbb{N}$.
Clearly, $\cup_{k=1}^{\infty} A_k$ is not empty, so let $x \in \cup_{k=1}^{\infty} A_k$.
Is $x \in \mathbb{N}$?
Since $x \in \cup_{k=1}^{\infty} A_k$, then $x \in A_k$ for some $k \in \mathbb{N}$.
Since $x \in A_k$ and $A_k \subseteq \mathbb{N}$, then $x \in \mathbb{N}$.
Thus, $x \in \cup_{k=1}^{\infty} A_k \Rightarrow x \in \mathbb{N}$, so $\cup_{k=1}^{\infty} A_k \subset \mathbb{N}$.
We prove $\mathbb{N} \subset \cup_{k=1}^{\infty} A_k$.
Suppose $x \in \mathbb{N}$. Is $x \in \cup_{k=1}^{\infty} A_k$?
No matter how large x is, x is always in A_x.
Thus, $x \in A_k$ for $k = x$.
Hence, $(\exists k \in \mathbb{N})(x \in A_k)$, so $x \in \cup_{k=1}^{\infty} A_k$.
Therefore, $x \in \mathbb{N} \Rightarrow x \in \cup_{k=1}^{\infty} A_k$, so $\mathbb{N} \subset \cup_{k=1}^{\infty} A_k$.
Since $\cup_{k=1}^{\infty} A_k \subset \mathbb{N}$ and $\mathbb{N} \subset \cup_{k=1}^{\infty} A_k$, then $\cup_{k=1}^{\infty} A_k = \mathbb{N}$. □

Exercise 81. Let $A = \{A_k : k \in \mathbb{N}\}$.
Let $A_k = \{k, k+1, k+2, ...\}$.
Discuss A and compute $\cup_{k=1}^{\infty} A_k$ and $\cap_{k=1}^{\infty} A_k$.

Solution. Observe that $A_k = \{n \in \mathbb{N} : n \geq k\}$.
Hence, $A_k \subseteq \mathbb{N}$ and $A_k \neq \emptyset$.
We compute $A = \{A_1, A_2, A_3, ...\} = \{\{1,2,3,4,...\}, \{2,3,4,5,...\}, \{3,4,5,6,...\}, \{4,5,6,7,...\}, ...\}$.
We observe that as k increases, each succeeding A_k is contained in the previous set.
More precisely, if k, m are arbitrary natural numbers and $k < m$, then $A_k \supseteq A_m$.
Hence, A is a decreasing collection of sets.

We compute $\cup_{k=1}^{\infty} A_k$.
Observe that $\cup_{k=1}^{\infty} A_k = A_1 \cup A_2 \cup A_3 \cup ... = A_1 = \mathbb{N}$.
We could also graph each A_k on a number line and easily see this result.
We compute $\cap_{k=1}^{\infty} A_k$.
Observe that $\cap_{k=1}^{\infty} A_k = A_1 \cap A_2 \cap ... = \{x : (\forall k \in \mathbb{N})(x \in A_k)\}$.
Hence, x is an element of $\cap_{k=1}^{\infty} A_k$ iff $x \in A_1 \wedge x \in A_2 \wedge x \in A_3 \wedge ....$
Is $1 \in \cap_{k=1}^{\infty} A_k$?
We observe that $1 \in A_1$, but $1 \notin A_2$.
Hence, 1 cannot be in $\cap_{k=1}^{\infty} A_k$.
Is $2 \in \cap_{k=1}^{\infty} A_k$?
We observe that $2 \in A_1, A_2$, but $2 \notin A_3$.
Hence, 2 cannot be in $\cap_{k=1}^{\infty} A_k$.
Is $3 \in \cap_{k=1}^{\infty} A_k$?
We observe that $3 \in A_1, A_2, A_3$, but $3 \notin A_4$.
Hence, 3 cannot be in $\cap_{k=1}^{\infty} A_k$.
Is $4 \in \cap_{k=1}^{\infty} A_k$?

We observe that $4 \in A_1, A_2, A_3, A_4$, but $4 \notin A_5$.

Hence, 4 cannot be in $\cap_{k=1}^{\infty} A_k$.

Based on this pattern, we conjecture that no natural number is in $\cap_{k=1}^{\infty} A_k$.

Since each A_k is a subset of $\mathbb{N}$ and the intersection of any collection of subsets of $\mathbb{N}$ is a subset of $\mathbb{N}$, then $\cap_{k=1}^{\infty} A_k$ is a subset of $\mathbb{N}$. Hence, $\cap_{k=1}^{\infty} A_k \subset \mathbb{N}$.

Thus we conjecture that $\cap_{k=1}^{\infty} A_k$ is a subset of $\mathbb{N}$ that contains no natural numbers.

In other words, we conjecture that $\cap_{k=1}^{\infty} A_k = \emptyset$.

Let's prove $\cap_{k=1}^{\infty} A_k = \emptyset$.

We use proof by contradiction.

Suppose $\cap_{k=1}^{\infty} A_k \neq \emptyset$.

Then there exists an x in the domain of discourse $\mathbb{N}$ such that $x \in \cap_{k=1}^{\infty} A_k$.

Let x be a natural number such that $x \in \cap_{k=1}^{\infty} A_k$.

Then $x \in A_1 \wedge x \in A_2 \wedge x \in A_3 \wedge ....$

Since $x \in A_1$, then $x \geq 1$.

Since $x \in A_2$, then $x \geq 2$.

Since $x \in A_3$, then $x \geq 3$.

Since $x \in A_4$, then $x \geq 4$, and so on.

Thus, $x \geq n$ for every $n \in \mathbb{N}$.

Hence, $(\forall n \in \mathbb{N})(x \geq n)$, so x is a greatest natural number.

Therefore, there exists a greatest natural number.

But, we know that there is no greatest natural number.

Hence, we have there is a greatest natural number and there is no greatest natural number, a contradiction.

Therefore, $\cap_{k=1}^{\infty} A_k = \emptyset$. $\square$

Exercise 82. Let $A = \{A_k : k \in \mathbb{N}\}$.

Let $A_k = (-\infty, k]$.

Discuss A and compute $\cup_{k=1}^{\infty} A_k$ and $\cap_{k=1}^{\infty} A_k$.

Solution. Observe that $A_k = \{x \in \mathbb{R} : x \leq k\}$.

Hence, $A_k \subset \mathbb{R}$.

Observe that $A = \{A_1, A_2, A_3, ...\} = \{(-\infty, 1], (-\infty, 2], (-\infty, 3], (-\infty, 4], (-\infty, 5], ...\}$.

If we draw a number line representing these sets A_k , we see that A is an increasing collection of sets.

We can also prove A is increasing.

To prove A is an increasing collection of sets, we must prove $(\forall i \in \mathbb{N})(\forall j \in \mathbb{N})(i < j \rightarrow A_i \subseteq A_j)$ is true.

Let i and j be arbitrary natural numbers.

To prove $i < j \rightarrow A_i \subseteq A_j$ is true, we assume $i < j$.

To prove $A_i \subseteq A_j$, we let $x \in A_i$.

We must prove $x \in A_j$.

Observe that $A_i = \{x \in \mathbb{R} : x \leq i\}$ and $A_j = \{x \in \mathbb{R} : x \leq j\}$.

Since $x \in A_i$, then $x \in \mathbb{R}$ and $x \leq i$.

Since $x \leq i$ and $i < j$, then by the transitive property of the $<$ relation, $x < j$.

Since $x \in \mathbb{R}$ and $x < j$, then $x \in A_j$, as desired.
We compute $\cap_{k=1}^{\infty} A_k$.
Observe that $\cap_{k=1}^{\infty} A_k = A_1 \cap A_2 \cap A_3 \cap ... = A_1 = (-\infty, 1]$.
We compute $\cup_{k=1}^{\infty} A_k$.
As k gets sufficiently large, A_k covers more of the real number line.
So, we conjecture that $\cup_{k=1}^{\infty} A_k = \mathbb{R}$.
We prove $\cup_{k=1}^{\infty} A_k = \mathbb{R}$.
To prove this, we must prove $\cup_{k=1}^{\infty} A_k \subset \mathbb{R}$ and $\mathbb{R} \subset \cup_{k=1}^{\infty} A_k$.

We prove $\cup_{k=1}^{\infty} A_k \subset \mathbb{R}$.
The union of a collection of subsets of $\mathbb{R}$ is a subset of $\mathbb{R}$.
Since A_k is a subset of $\mathbb{R}$, then $\cup_{k=1}^{\infty} A_k$ is a subset of $\mathbb{R}$.
Hence, $\cup_{k=1}^{\infty} A_k \subset \mathbb{R}$.
We prove $\mathbb{R} \subset \cup_{k=1}^{\infty} A_k$.
To prove this, we let $x \in \mathbb{R}$.
To prove $x \in \cup_{k=1}^{\infty} A_k$, we must prove there exists a natural number k such that $x \in A_k$.
By the trichotomy property of $\mathbb{R}$, either $x \leq 1$ or $x > 1$.
We consider these cases separately.
Case 1: Suppose $x \leq 1$.
Then $x \in (-\infty, 1]$, so $x \in A_1$.
Since $1 \in \mathbb{N}$ and $x \in A_1$, then if we let $k = 1$, we see that there exists a natural number k such that $x \in A_k$, as desired.
Case 2: Suppose $x > 1$.
To prove there exists a natural number k such that $x \in A_k$, we prove there exists $k \in \mathbb{N}$ such that $k > x$.
Since $x > 1$, then $x \neq 0$.
Hence, x has a unique multiplicative inverse, $1/x \in \mathbb{R}$.
Since $x > 1$ and $1 > 0$, then $x > 0$.
Since $x > 0$, then $1/x > 0$.
Hence, $1/x$ is a positive real number.
By the Archimedean property of $\mathbb{N}$, we know to the positive real number $1/x$, there corresponds a positive integer k such that $1/k < 1/x$.
Let k be a positive integer such that $1/k < 1/x$.
Since $k > 0$, then $1 < k/x$.
Since $x > 0$, then $x < k$.
Hence, $k > x$.
Since $k \in \mathbb{N}$, then $A_k = (-\infty, k] = \{x \in \mathbb{R} : x \leq k\}$.
Since $x \in \mathbb{R}$ and $x < k$, then $x \in A_k$. Therefore, there exists a natural number k such that $x \in A_k$, as desired. □

Exercise 83. Let $A = \{A_k : k \in \mathbb{N}\}$.
Let $A_k = (0, k-1)$.
Discuss A and compute $\cup_{k=1}^{\infty} A_k$ and $\cap_{k=1}^{\infty} A_k$.

Solution. Observe that $A_k = \{x \in \mathbb{R} : 0 < x < k-1\}$.

Hence, $A_k \subset \mathbb{R}$.

Let x be an arbitrary element of A_k.

Then $x \in \mathbb{R}$ and $0 < x < k-1$.

Hence, $0 < x < k-1$, so $0 < x$ and $x < k-1$.

Since $k \in \mathbb{N}$, then $k \geq 1$.

Since $k \geq 1$, then each x in A_k is positive.

Hence, $A_k \subset \mathbb{R}^+$.

Observe that $A = \{A_1, A_2, A_3, ...\} = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), ...\}$.

If we draw a number line representing these sets A_k , we see that A is an increasing collection of sets.

We can also prove A is increasing.

To prove A is an increasing collection of sets, we must prove $(\forall i \in \mathbb{N})(\forall j \in \mathbb{N})(i < j \rightarrow A_i \subseteq A_j)$ is true.

Let i and j be arbitrary natural numbers.

To prove $i < j \rightarrow A_i \subseteq A_j$ is true, we assume $i < j$.

To prove $A_i \subseteq A_j$, we let $x \in A_i$.

We must prove $x \in A_j$.

Observe that $A_i = \{x \in \mathbb{R} : 0 < x < i-1\}$ and $A_j = \{x \in \mathbb{R} : 0 < x < j-1\}$.

Hence, we must prove $x \in \mathbb{R}$ and $0 < x < j-1$.

Since $x \in A_i$, then $x \in \mathbb{R}$ and $0 < x < i-1$.

Thus, $0 < x < i-1$, so $0 < x$ and $x < i-1$.

Hence, $0 < x$.

Since $x < i-1$, then $x+1 < i$.

Since $x+1 < i$ and $i < j$, then $x+1 < j$, by the transitive property of $<$ relation on $\mathbb{R}$.

Hence, $x < j-1$.

Since $0 < x$ and $x < j-1$, then $0 < x < j-1$.

Since $x \in \mathbb{R}$ and $0 < x < j-1$, then $x \in A_j$, as desired.

We compute $\cap_{k=1}^{\infty} A_k$.

Observe that $\cap_{k=1}^{\infty} A_k = A_1 \cap A_2 \cap A_3 \cap ... = A_1 = (0,0) = \emptyset$.

We compute $\cup_{k=1}^{\infty} A_k$.

As k gets sufficiently large, A_k covers more of the positive real numbers.

So, we conjecture that $\cup_{k=1}^{\infty} A_k = \mathbb{R}^+$.

We prove $\cup_{k=1}^{\infty} A_k = \mathbb{R}^+$.

Note that $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} = (0, \infty)$.

To prove this, we prove $\cup_{k=1}^{\infty} A_k \subset \mathbb{R}^+$ and $\mathbb{R}^+ \subset \cup_{k=1}^{\infty} A_k$.

We prove $\cup_{k=1}^{\infty} A_k \subset \mathbb{R}^+$.

Since the union of a collection of subsets of $\mathbb{R}^+$ is a subset of $\mathbb{R}^+$ and A_k is a subset of $\mathbb{R}^+$, then $\cup_{k=1}^{\infty} A_k$ is a subset of $\mathbb{R}^+$.

Hence, $\cup_{k=1}^{\infty} A_k \subset \mathbb{R}$, as desired.

We prove $\mathbb{R}^+ \subset \cup_{k=1}^{\infty} A_k$.

To prove this, we let $x \in \mathbb{R}^+$.

To prove $x \in \cup_{k=1}^{\infty} A_k$, we must prove there exists a natural number k such that $x \in A_k$.

Thus, we must show that there exists $k \in \mathbb{N}$ such that $x \in \mathbb{R}$ and $0 < x < k - 1$.

Hence, we must show there exists $k \in \mathbb{N}$ such that $x \in \mathbb{R}$ and $0 < x$ and $x < k - 1$.

Since $x \in \mathbb{R}^+$, then $x \in \mathbb{R}$ and $x > 0$.

Hence, $x \in \mathbb{R}$.

Since $\mathbb{R}$ is closed under addition, then $x + 1 \in \mathbb{R}$.

Since $x > 0$, then $0 < x$.

Since $0 < x$ and $x < x + 1$, then $0 < x + 1$, by transitivity of $<$ in $\mathbb{R}$.

Hence, $x + 1 \neq 0$.

Since $x+1$ is a nonzero real number, then there exists a unique real number that is the multiplicative inverse of $x + 1$, namely, $1/(x + 1) \in \mathbb{R}$.

Since $x + 1 > 0$, then $1/(x + 1) > 0$.

Since $1/(x + 1) \in \mathbb{R}$ and $1/(x + 1) > 0$, then $1/(x + 1)$ is a positive real number.

By the Archimedean property of $\mathbb{N}$, we know that to each positive $\mu \in \mathbb{R}$, there corresponds $k \in \mathbb{Z}^+$ such that $1/k < \mu$.

Hence, let k be a positive integer such that $1/k < 1/(x + 1)$.

Since $k > 0$, then $1 < k/(x + 1)$.

Since $x + 1 > 0$, then $x + 1 < k$.

Thus, $x < k - 1$.

Since $x \in \mathbb{R}$ and $0 < x$ and $x < k - 1$ for some $k \in \mathbb{Z}^+$, then we conclude $x \in A_k$ for some $k \in \mathbb{N}$, as desired. □

Exercise 84. Let $A = \{A_k : k \in \mathbb{N}\}$ be an infinite collection of subsets of $\mathbb{R}$.

Then each set $A_k \in \mathbb{R}$.

Suppose $\cap_{k=1}^{\infty} A_k = \emptyset$.

Does this imply $(\exists n \in \mathbb{N})(\cap_{k=1}^{n} A_k = \emptyset)$?

Solution. We must decide if the statement $\cap_{k=1}^{\infty} A_k = \emptyset \rightarrow (\exists n \in \mathbb{N})(\cap_{k=1}^{n} A_k = \emptyset)$ is true.

We devise several examples for A_k and conjecture that this statement appears true, so let's prove this.

Since $\cap_{k=1}^{\infty} A_k = \emptyset$, then $A_1 \cap A_2 \cap ... = \emptyset$.

We know that $A \cap B = \emptyset$ iff either A or B is empty.

Hence, $\cap_{k=1}^{\infty} A_k = \emptyset$ iff at least one of the sets A_k is empty.

Therefore, $\cap_{k=1}^{\infty} A_k = \emptyset$ iff $(\exists n \in \mathbb{N})(A_n = \emptyset)$.

Since $\cap_{k=1}^{\infty} A_k = \emptyset$ and $\cap_{k=1}^{\infty} A_k = \emptyset \Leftrightarrow (\exists n \in \mathbb{N})(A_n = \emptyset)$, then $(\exists n \in \mathbb{N})(A_n = \emptyset)$.

Thus, let n be a natural number such that $A_n = \emptyset$.

Observe that

$$\begin{aligned}
\emptyset &= \cap_{k=1}^{\infty} A_k \\
&= \cap_{k=1}^{n-1} A_k \cap A_n \cap_{k=n+1}^{\infty} A_k \\
&= \cap_{k=1}^{n-1} A_k \cap \emptyset \cap_{k=n+1}^{\infty} A_k \\
&= \emptyset \cap_{k=n+1}^{\infty} A_k
\end{aligned}$$

Hence, $\emptyset = \cap_{k=1}^{n-1} A_k \cap \emptyset$.

Thus, $\emptyset = \cap_{k=1}^{n-1} A_k \cap A_n = \cap_{k=1}^{n} A_k$, as desired. □

Proposition 85. *Let A, B be finite sets.*

Then $|A \cup B| = |A| + |B| - |A \cap B|$

Solution. We observe that $A \cup B$ is the union of 3 sets: $A - B$, $A \cap B$, and $B - A$.

We know that $|A \cup B| = |A| + |B|$ if A and B are finite disjoint sets.

Suppose $A \cap B = \emptyset$.

Then the collection $\{A, B\}$ forms a partition of $A \cup B$. □

Proof. Observe that $(A - B) \cup (A \cap B) = (A \cap \overline{B}) \cup (A \cap B) = A \cap (\overline{B} \cup B) = A \cap U = A$.

Observe that $(A - B) \cap (A \cap B) = (A \cap \overline{B}) \cap (B \cap A) = A \cap (\overline{B} \cap B) \cap A = A \cap \emptyset \cap A = \emptyset$.

Thus $A - B$ and $A \cap B$ are disjoint.

Hence $\{A - B, A \cap B\}$ form a partition of A so $|A| = |A - B| + |A \cap B|$.

Observe that $(B - A) \cup (A \cap B) = (B \cap \overline{A}) \cup (B \cap A) = B \cap (\overline{A} \cup A) = B \cap U = B$.

Observe that $(B - A) \cap (A \cap B) = (B \cap \overline{A}) \cap (A \cap B) = B \cap (\overline{A} \cap A) \cap B = B \cap \emptyset \cap B = \emptyset$.

Thus $B - A$ and $A \cap B$ are disjoint.

Hence $\{B - A, A \cap B\}$ form a partition of B so $|B| = |B - A| + |A \cap B|$.

Observe that $(A - B) \cap (B - A) = (A \cap \overline{B}) \cap (B \cap \overline{A}) = A \cap (\overline{B} \cap B) \cap \overline{A} = A \cap \emptyset \cap \overline{A} = \emptyset$.

Thus $A - B$ and $B - A$ are disjoint.

Since $A - B$, $A \cap B$, and $B - A$ are mutually disjoint and $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$ then $\{A - B, A \cap B, B - A\}$ forms a partition of $A \cup B$.

Therefore $|A \cup B| = |A - B| + |A \cap B| + |B - A|$.

Thus $|A \cup B| = (|A| - |A \cap B|) + |A \cap B| + (|B| - |A \cap B|)$.

Hence $|A \cup B| = |A| + |B| - |A \cap B|$. □

Exercise 86. A set of 20 tools is available for two people. One person uses 15 tools and the other person uses 12 tools. What is the minimum number of tools shared?

Solution. Let A be the set of tools used by person 1. Let B be the set of tools used by person 2. Then $|A| = 15$ and $|B| = 12$. Since the total number of tools is 20, then $|A \cup B| \leq 20$. Thus, $20 \geq |A \cup B| = |A| + |B| - |A \cap B| = 15 + 12 - |A \cap B|$, so $20 \geq 27 - |A \cap B|$. Hence, $|A \cap B| \geq 7$, so the minimum number of tools shared is 7. □

Exercise 87. Every multiple of 18 is a multiple of 6.

Proof. Let the set of multiples of 18 be the set $A = \{x \in \mathbb{Z} : 18|x\}$.
Let the set of multiples of 6 be the set $B = \{x \in \mathbb{Z} : 6|x\}$.

We prove $A \subseteq B$.
Suppose $a \in A$.
Then $a \in \mathbb{Z}$ and $18|a$.
By definition of divisibility, there is an integer k for which $a = 18k$.
Thus $a = 18k = 6(3k)$, and so $6|a$.
Consequently a is one of the integers that 6 divides, so $a \in B$. Thus $a \in A$ implies $a \in B$, so it follows that $A \subseteq B$. □

Proof. Let the set of even numbers be the set $A = \{x \in \mathbb{Z} : 2|x\}$.

Let the set of multiples of 9 be the set $B = \{x \in \mathbb{Z} : 9|x\}$.

The set of multiples of 18 is the set $A \cap B = \{x : x \in A \wedge x \in B\} = \{x \in \mathbb{Z} : 2|x \text{ and } 9|x\}$.

Let the set of multiples of 6 be the set $C = \{x \in \mathbb{Z} : 6|x\}$.

We must show that $A \cap B \subseteq C$.
Suppose $a \in A \cap B$.
Then $a \in \mathbb{Z}$ and $2|a$ and $9|a$.
Thus $a = 2k = 9m$ for some $k, m \in Z$ by definition of divisibility.
Since $9m = 2k$ then $9m$ is even.
Since $9m$ is even and 9 is odd, then m is even, for if m were odd, then the product $9m$ would be odd.
Hence $m = 2n$ for some $n \in \mathbb{Z}$, so $9m = 9(2n) = 18n = 6(3n)$ which implies $6|9m$, and therefore $6|a$.
Since $a \in Z$ and $6|a$, then $a \in C$.
Thus $a \in A \cap B$ implies $a \in C$, so it follows that $A \cap B \subseteq C$. □

Exercise 88. Every multiple of 12 is a multiple of both 2 and 3.

Solution. We first translate this English sentence into formal logic symbols in order to prove the truth of the assertion 'every multiple of 12 is a multiple of both 2 and 3' (which we know is obviously true). We define the predicates below.

$M(n)$: n is a multiple of 12
$T(n)$: n is a multiple of 2
$H(n)$: n is a multiple of 3
The English sentence means:

For every n , if n is a multiple of 12, then n is a multiple of 2 and n is a multiple of 3.

This translates into logic symbols as:
$\forall n, M(n) \Rightarrow (T(n) \wedge H(n))$.

Now we can use set theory to translate the predicates as follows:

Let the set of multiples of 12 be $A = \{12n : n \in \mathbb{Z}\}$.

Let the set of multiples of 2 be $B = \{2n : n \in \mathbb{Z}\}$.

Let the set of multiples of 3 be $C = \{3n : n \in \mathbb{Z}\}$.

The logic symbols now translate as:

$\forall n, n \in A \Rightarrow (n \in B \wedge n \in C)$.

By definition of subset and set intersection we know this means:

$\forall n, n \in A \Rightarrow (n \in (B \cap C))$

so this implies

$A \subseteq B \cap C$.

Thus we must prove $A \subseteq B \cap C$. □

Proof. We use direct proof.

Suppose $a \in \{12n : n \in \mathbb{Z}\}$.

Then $a = 12n$ for some $n \in \mathbb{Z}$.

Since $a = 12n = 2(6n)$, then a is a multiple of 2.

Since $a = 2(6n)$ where $6n \in \mathbb{Z}$, then $a \in \{2n : n \in \mathbb{Z}\}$.

Also, since $a = 12n = 3(4n)$ for some $n \in \mathbb{Z}$, then a is a multiple of 3.

Since $a = 3(4n)$ where $4n \in \mathbb{Z}$, then $a \in \{3n : n \in \mathbb{Z}\}$.

We now have both $a \in \{2n : n \in \mathbb{Z}\}$ and $a \in \{3n : n \in \mathbb{Z}\}$, so $a \in \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.

Since we now have $a \in \{12n : n \in \mathbb{Z}\} \Rightarrow a \in \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$, then by definition of subset, it follows that $\{12n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$. □

Exercise 89. Every factor of an integer is a factor of its square.

Solution. We first translate this into an equivalent English sentence:

For each integer k every factor of k is a factor of k^2.

We then translate this English sentence into formal logic symbols in order to prove the truth of the assertion 'for each integer k every factor of k is a factor of k^2' (which we know is obviously true). We define the predicate below.

$F(a, b)$: a is a factor of b

The English sentence means:

For each integer k, for every n, if n is a factor of k, then n is a factor of k^2.

This translates into logic symbols as:

$\forall (k \in \mathbb{Z}) \forall n, F(n, k) \Rightarrow F(n, k^2)$.

Now we can use set theory to translate the predicate as follows:
We know that a is a factor of b means a divides b.
So we let the set of integers that are factors of k be $A = \{n \in \mathbb{Z} : n|k\}$.
We let the set of integers that are factors of k^2 be $B = \{n \in \mathbb{Z} : n|k^2\}$.
The logic symbols now translate as:
$\forall(k \in \mathbb{Z})\forall n, n \in A \Rightarrow n \in B$.
By definition of subset we know this means:
$\forall(k \in \mathbb{Z}), A \subseteq B$.
This implies:
$k \in \mathbb{Z} \Rightarrow A \subseteq B$.
Thus we must prove $k \in \mathbb{Z} \Rightarrow A \subseteq B$. □

Proof. We use direct proof.
Suppose $k \in \mathbb{Z}$.
Let $a \in \{n \in \mathbb{Z} : n|k\}$.
Then $a \in \mathbb{Z}$ and $a|k$.
Since $a|k$, then $k = ma$ for some $m \in \mathbb{Z}$.
Thus $k^2 = (am)^2 = a(am^2)$, so $a|k^2$ since $am^2 \in \mathbb{Z}$.
Since $a \in \mathbb{Z}$ and $a|k^2$, then $a \in \{n \in \mathbb{Z} : n|k^2\}$.
So we have $a \in \{n \in \mathbb{Z} : n|k\} \Rightarrow a \in \{n \in \mathbb{Z} : n|k^2\}$.
It follows that $\{n \in \mathbb{Z} : n|k\} \subseteq \{n \in \mathbb{Z} : n|k^2\}$. □

Exercise 90. There is some multiple of every pair of positive integers.

Solution. We first translate this into an equivalent English sentence:
For each pair of positive integers p, q, there is some multiple of both p and q.

We then translate this English sentence into formal predicate logic symbols in order to prove the truth of the assertion 'for every pair of positive integers p, q, there is some multiple of both p and q' (which we know is obviously true; an example is the product pq). We define the predicate below.

$M(a, b)$: a is a multiple of b

The English sentence means:
For every pair of positive integers p, q, there is some natural number k such that k is a multiple of p and k is a multiple of q.

This translates into logic symbols as:
$\forall(p \in \mathbb{Z}^+)\forall(q \in \mathbb{Z}^+), \exists(k \in \mathbb{N})(M(k, p) \wedge M(k, q))$.

Now we can use set theory to translate the predicate as follows:
Let the set of positive multiples of p be $A = \{pn : n \in \mathbb{N}\}$.
Let the set of positive multiples of q be $B = \{qn : n \in \mathbb{N}\}$.
The logic symbols now translate as:
$\forall(p \in \mathbb{Z}^+)\forall(q \in \mathbb{Z}^+), \exists(k \in \mathbb{N})(k \in A \wedge k \in B)$.
By definition of set intersection we know this means:

$\forall(p \in \mathbb{Z}^+)\forall(q \in \mathbb{Z}^+), \exists(k \in \mathbb{N})(k \in A \cap B)$.
This implies:
$\forall(p \in \mathbb{Z}^+)\forall(q \in \mathbb{Z}^+), (A \cap B) \neq \emptyset$.
Thus we must prove $p, q \in \mathbb{Z}^+ \Rightarrow (A \cap B) \neq \emptyset$. □

Proof. We use direct proof.
Suppose $p, q \in \mathbb{Z}^+$.
Then we know the product $pq \in \mathbb{Z}^+$.
Since $q \in \mathbb{N}$, then $pq \in \{pn : n \in \mathbb{N}\}$.
Since $pq = qp$ and $p \in \mathbb{N}$, then $pq \in \{qn : n \in \mathbb{N}\}$.
Both of the above together imply $pq \in \{pn : n \in \mathbb{N}\} \cap \{qn : n \in \mathbb{N}\}$.
Therefore $\{pn : n \in \mathbb{N}\} \cap \{qn : n \in \mathbb{N}\} \neq \emptyset$. □

Exercise 91. The set $\{9^n : n \in \mathbb{Z}\}$ is a subset of $\{3^n : n \in \mathbb{Z}\}$, but $\{9^n : n \in \mathbb{Z}\} \neq \{3^n : n \in \mathbb{Z}\}$.

Proof. Suppose $a \in \{9^n : n \in \mathbb{Z}\}$.
Then $a = 9^n$ for some $n \in \mathbb{Z}$
Thus $a = 9^n = (3^2)^n = 3^{2n}$ where $2n \in \mathbb{Z}$, so $a \in \{3^n : n \in \mathbb{Z}\}$.
We have shown that $a \in \{9^n : n \in \mathbb{Z}\}$ implies $a \in \{3^n : n \in \mathbb{Z}\}$, so it follows that $\{9^n : n \in \mathbb{Z}\} \subseteq \{3^n : n \in \mathbb{Z}\}$.
Observe that $3 \in \{3^n : n \in \mathbb{Z}\}$ but $3 \notin \{9^n : n \in \mathbb{Z}\}$, so $\{9^n : n \in \mathbb{Z}\} \neq \{3^n : n \in \mathbb{Z}\}$. □

Exercise 92. Prove $\{12a + 4b : a, b \in \mathbb{Z}\} = \{4c : c \in \mathbb{Z}\}$.

Proof. First we prove $\{12a + 4b : a, b \in \mathbb{Z}\} \subseteq \{4c : c \in \mathbb{Z}\}$.
Suppose $x \in \{12a + 4b : a, b \in \mathbb{Z}\}$.
Then $x = 12a + 4b$ for some integers a and b.
From this we get $x = 4(3a + b)$, so $x = 4c$ where c is the integer $3a + b$.
Consequently $x \in \{4c : c \in \mathbb{Z}\}$.
So we have $x \in \{12a + 4b : a, b \in \mathbb{Z}\} \Rightarrow x \in \{4c : c \in \mathbb{Z}\}$, and it follows that $\{12a + 4b : a, b \in \mathbb{Z}\} \subseteq \{4c : c \in \mathbb{Z}\}$.

Next we show $\{4c : c \in \mathbb{Z}\} \subseteq \{12a + 4b : a, b \in \mathbb{Z}\}$.
Suppose $x \in \{4c : c \in \mathbb{Z}\}$.
Then $x = 4c$ for some $c \in \mathbb{Z}$.
Thus $x = 4c = 12c + 4(-2c)$, and since c and $-2c$ are integers, we have $x \in \{12a + 4b : a, b \in \mathbb{Z}\}$.
Consequently $x \in \{4c : c \in \mathbb{Z}\} \Rightarrow x \in \{12a + 4b : a, b \in \mathbb{Z}\}$, and it follows that $\{4c : c \in \mathbb{Z}\} \subseteq \{12a + 4b : a, b \in \mathbb{Z}\}$.
This proves that $\{12a + 4b : a, b \in \mathbb{Z}\} = \{4c : c \in \mathbb{Z}\}$. □

Exercise 93. Prove $\{12a + 25b : a, b \in \mathbb{Z}\} = \mathbb{Z}$.

Solution. Let $A = \{12a + 25b : a, b \in \mathbb{Z}\}$.
We must prove $A = \mathbb{Z}$.
Since A and $\mathbb{Z}$ are sets, we must prove equality of sets.
To prove this we use the definition of set equality: $A = \mathbb{Z}$ iff $A \subseteq \mathbb{Z} \wedge \mathbb{Z} \subseteq A$.
So we must do the following:

1) Prove $A \subseteq \mathbb{Z}$.

2) Prove $\mathbb{Z} \subseteq A$.
To prove $A \subseteq \mathbb{Z}$:
We know that $A \subseteq \mathbb{Z}$ means $\forall x(x \in A \Rightarrow x \in \mathbb{Z})$.
So, we let x be an arbitrary object.
Suppose $x \in A$.
By definition of set membership, $x = 12a + 25b$ with $a, b \in \mathbb{Z}$.
Since $a, b \in \mathbb{Z}$, then $12a + 25b \in \mathbb{Z}$, so $x \in \mathbb{Z}$.
To prove $\mathbb{Z} \subseteq A$:
We know that $\mathbb{Z} \subseteq A$ means $\forall y(y \in \mathbb{Z} \Rightarrow y \in A)$.
So, we let y be an arbitrary object.
Suppose $y \in \mathbb{Z}$.
We observe that $y = 12(-2y) + 25(y)$, with $-2y \in \mathbb{Z}$, so $y \in A$.
We now can write up a coherent proof. □

Proof. Let $A = \{12a + 25b : a, b \in \mathbb{Z}\}$.
Let x be arbitrary.
Suppose $x \in A$.
Then $x = 12a + 25b$ and $a, b \in \mathbb{Z}$.
Since $a, b \in \mathbb{Z}$, then $12a + 25b \in \mathbb{Z}$.
Hence, $x \in \mathbb{Z}$.
Therefore, $x \in A \Rightarrow x \in \mathbb{Z}$, so $A \subseteq \mathbb{Z}$.

Let y be arbitrary.
Suppose $y \in \mathbb{Z}$.
Then $y = 12(-2y) + 25(y)$ and $-2y \in \mathbb{Z}$.
Hence, $y \in A$.
Therefore, $y \in \mathbb{Z} \Rightarrow y \in A$, so $\mathbb{Z} \subseteq A$.
Since $A \subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq A$, then $A = \mathbb{Z}$.
Therefore, we conclude that $\{12a + 25b : a, b \in \mathbb{Z}\} = \mathbb{Z}$. □

Exercise 94. Suppose F, G are families of sets.
If $F \subseteq G$, then $\cup F \subseteq \cup G$.

Solution. Let $F = \{F_1, F_2, ..., F_n\}$ where each F_i is a set.
Let $G = \{G_1, G_2, ..., G_n\}$ where each G_i is a set.
We can prove the proposition by working backwards, applying definitions and logical inference.

Our hypothesis is: $F \subseteq G$.

Our conclusion is: $\cup F \subseteq \cup G$.
The conclusion means(by definition of subset):

$\forall a(a \in \cup F \Rightarrow a \in \cup G)$.
So, we let a be an arbitrary object.
In the implication, we consider the meaning of the consequent $a \in \cup G$.

We know that $\cup G = G_1 \cup G_2 \cup ... \cup G_n = \{x : x \in G_1 \vee x \in G_2 ... \vee x \in G_n\} = \{x : x \in G_i \text{ for at least one } G_i\}$.
Hence, if $a \in \cup G$, then $a \in G_i$ for at least one G_i.
This means there exists $G_i \in G$ such that $a \in G_i$.
In logic symbols this is $\exists G_i \in G(a \in G_i)$.
Let $b = G_i$ to simplify, so we get $\exists b \in G(a \in b)$.
Now, let's consider the meaning of the antecedent in the implication $a \in \cup F$.

We know that $\cup F = F_1 \cup F_2 \cup ... \cup F_n = \{x : x \in F_1 \vee x \in F_2 ... \vee x \in F_n\} = \{x : x \in F_i \text{ for at least one } F_i\}$.
Hence, if $a \in \cup F$, then $a \in F_i$ for at least one F_i.
This means there exists $F_i \in F$ such that $a \in F_i$.
In logic symbols this is $\exists F_i \in F(a \in F_i)$.
Let $b = F_i$ to simplify, so we get $\exists b \in F(a \in b)$.
Now, to prove the consequent, we must have a concrete b in the antecedent.
Thus, since $\exists b \in F(a \in b)$, then we apply existential instantiation to get:
We choose X such that $X \in F \wedge a \in X$.(We're simply substituting X for b).
By hypothesis we have $F \subseteq G$, so by definition of subset this means:

$\forall w(w \in F \rightarrow w \in G)$.
We can apply universal instantiation with $X = w$ to get:
$X \in F \rightarrow X \in G$ since we have chosen a concrete X.
So, now we have $X \in F$ and $X \in F \rightarrow X \in G$.
So, by modus ponens, we conclude $X \in G$.
So, now we have the existential generalization $\exists X \in G(a \in X)$.
By definition of set union of G, this means $a \in \cup G$.
We can now write up a coherent proof. □

Proof. Let a be arbitrary.
Suppose $a \in \cup F$.
We can choose X such that $a \in X$ and $X \in F$.
Since $F \subseteq G$, then $X \in F \rightarrow X \in G$.
Since $X \in F$ and $X \in F \rightarrow X \in G$, then $X \in G$.
Since $X \in G$ and $a \in X$, then $a \in \cup G$.
Therefore, $a \in \cup F \rightarrow a \in \cup G$.
Since a was arbitrary, we conclude that $\cup F \subseteq \cup G$. □

Cardinality of Sets

Exercise 95. Prove $|\{0,1\} \times \mathbb{N}| = |\mathbb{N}|$.

Solution. Let $S = \{0,1\} \times \mathbb{N}$.

We must prove $|S| = |\mathbb{N}|$, so we need to devise a bijective function(one to one correspondence) between S and $\mathbb{N}$.

We can devise a bijection either from S onto $\mathbb{N}$ or from $\mathbb{N}$ onto S.

To make things simpler let's try to devise a bijection from S onto $\mathbb{N}$.

We can write out set S and $\mathbb{N}$ in a table form as shown below:

(Note that we could come up with other ways of tabulating,but this one works.)

S	ℕ
(1,1)	1
(0,1)	2
(1,2)	3
(0,2)	4
(1,3)	5
(0,3)	6
(1,4)	7
(0,4)	8
(1,5)	9
(0,5)	10
...	

The table suggests a function:

$$f(a,b) = \begin{cases} 2b-1 & \text{if } a = 1 \\ 2b & \text{if } a = 0 \end{cases}$$

In other words, $f(a,b) = 2b - a$ regardless if $a = 1$ or $a = 0$.

Let's see if we can prove if $f(a,b)$ is one to one and onto. □

Proof. Let $S = \{0,1\} \times \mathbb{N}$.

Let function $f : S \Rightarrow \mathbb{N}$ be defined by $f(a,b) = 2b - a$ where $(a,b) \in S$.

We use proof by contrapositive to prove f is one to one.

Suppose $f(a,b) = f(c,d)$ where (a,b) and $(c,d) \in S$.

Then $2b - a = 2d - c$, so $a - c = 2(b - d)$.

Thus, $a - c$ is even.

Suppose for the sake of contradiction that $a \neq c$.

We know $a, c \in \{0,1\}$, so this means one of a and c is 0 and the other is 1.

Hence, the difference $|a - c| = 1$, so it follows that $a - c$ must be odd.

But, this contradicts the fact that $a - c$ is even, so it cannot be true that $a \neq c$.

Therefore, $a = c$.

Hence, $2b - a = 2d - a$, so it follows that $b = d$.

Consequently, $(a, b) = (c, d)$.
Since $f(a, b) = f(c, d) \Rightarrow (a, b) = (c, d)$, then f is one to one.

We prove f is onto.
Suppose c is any natural number.
Then c is either even or odd.
We consider these cases separately.
Case 1: Suppose c is even.

Then $c = 2k$ for some integer k.
Observe that $f(0, k) = 2k - 0 = 2k = c$.
Case 2: Suppose c is odd.

Then $c = 2k + 1$ for some integer k.
Observe that $f(1, k + 1) = 2(k + 1) - 1 = 2k + 2 - 1 = 2k + 1 = c$.
Both of these cases show that each natural number c is the image of at least one element of S, so f is onto.
Since f is both one to one and onto, then f is bijective, so $|S| = |\mathbb{N}|$.
Therefore, $|\{0, 1\} \times \mathbb{N}| = |\mathbb{N}|$. □

Exercise 96. Prove $\{0, 1\} \times \mathbb{N}$ is countably infinite.

Solution. Let $S = \{0, 1\} \times \mathbb{N}$.
We previously showed in exercise 95 that $|S| = |\mathbb{N}| = \aleph_0$, so we know S is countably infinite. We can also write S as an infinite linear sequence. □

Proof. Let $S = \{0, 1\} \times \mathbb{N}$.
We can write the elements of S as an infinite linear sequence: $(0, 1), (1, 1), (0, 2), (1, 2), (0, 3), (1, 3,), (0, 4), (1,$
Hence, S is countably infinite. □

Exercise 97. Prove $\{\ln(n) : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is countably infinite.

Solution. Let $S = \{\ln(n) : n \in \mathbb{N}\}$.
We know that the domain of $f(x) = \ln(x)$ is dom(f) $= (0, \infty)$ and the range(f) $= \mathbb{R}$.
If we restrict the domain to $\mathbb{N}$, then each $\ln(n) \in \mathbb{R}$, so $S \subseteq \mathbb{R}$.
We can prove S is countably infinite by writing out each element of S in an infinite sequence.
The n^{th} term of the sequence is $a_n = \ln(n)$ where $n \in \mathbb{N}$. □

Proof. Let $S = \{\ln(n) : n \in \mathbb{N}\}$.
Then we can write the elements of S as an infinite sequence: $\ln(1), \ln(2), \ln(3), \ln(4), \ln(5), ...$
Hence, S is countably infinite. □

Exercise 98. Prove $\{(5n, -3n) : n \in \mathbb{Z}\}$ is countably infinite.

Solution. Let $S = \{(5n, -3n) : n \in \mathbb{Z}\}$.

We note that $\mathbb{Z}$ is countably infinite, so $\mathbb{Z} \times \mathbb{Z}$ is also countably infinite.

We suspect that every subset of a countably infinite set is also countably infinite and we know that $S \subseteq \mathbb{Z} \times \mathbb{Z}$ (so S is a binary relation on $\mathbb{Z}$).

We need to devise a bijective function from $\mathbb{Z}$ onto S (or, from S onto $\mathbb{Z}$).

Let's devise a function $f : \mathbb{Z} \mapsto S$ using the table below:

$\mathbb{Z}$	S
0	(0,0)
1	(5,-3)
-1	(-5,3)
2	(10,-6)
-2	(-10,6)
3	(15,-9)
-3	(-15,9)
4	(20,-12)
-4	(-20,12)
5	(25,-15)
-5	(-25,15)
6	(30,-18)
...	

The table suggests the function:

$f(k) = (5k, -3k)$ where $k \in \mathbb{Z}$.

We need to prove f is truly bijective. □

Proof. Let $S = \{(5n, -3n) : n \in \mathbb{Z}\}$.

Let function $f : \mathbb{Z} \mapsto S$ be defined by $f(k) = (5k, -3k)$ where $k \in \mathbb{Z}$.

We prove f is bijective.

We use proof by contrapositive to prove f is one to one.

Suppose $a, b \in \mathbb{Z}$ such that $f(a) = f(b)$.

Then $(5a, -3a) = (5b, -3b)$.

Hence, $5a = 5b$ and $-3a = -3b$, so $a = b$.

Therefore, $f(a) = f(b)$ implies $a = b$, so f is one to one.

We prove f is onto S.

Let $c \in S$.

Let $k \in \mathbb{Z}$.

Observe that $f(k) = (5k, -3k) = c$.

Hence, each element of S is an image of at least one element of $\mathbb{Z}$, so f is onto.

Since f is one to one and onto, then f is bijective, so $|\mathbb{Z}| = |S|$.

Since $|\mathbb{Z}| = |\mathbb{N}| = \aleph_0$, then $|S| = \aleph_0$, so S is countably infinite. □

Exercise 99. Suppose B is an uncountable set and A is a set.

Let $f : A \mapsto B$ be a surjection. What can we deduce about the cardinality of A?

Solution. Since f is a function onto B, then each $b \in B$ is the image of at least one $a \in A$.

Let a_b be an element of A for which $f(a_b) = b$.
Let $C = \{a_b : b \in B\}$ where $C \subseteq A$.

We use proof by contrapositive to prove $f : C \mapsto B$ is one to one.
Suppose $b_1, b_2 \in B$ such that $f(a_{b_1}) = f(a_{b_2})$.
Then $b_1 = b_2$, by definition of f, so $a_{b_1} = a_{b_2}$.
Hence, $f(a_{b_1}) = f(a_{b_2})$ implies $a_{b_1} = a_{b_2}$, so $f : C \mapsto B$ is one to one.
Since f is onto and one to one, then f is bijective.
Therefore, $|C| = |B|$.
Since B is uncountable, then this means C is uncountable.
Since $C \subseteq A$, then A is a superset of C.
We know that any superset of an uncountable set is uncountable.
Hence, A is uncountable. □

Exercise 100. Prove or disprove: If A is uncountable, then $|A| = |\mathbb{R}|$.

Solution. We can let A be the powerset of $\mathbb{R}$ because we know that $|\mathbb{R}| < |\mathscr{P}(\mathbb{R})|$. □

Proof. This is false. We produce the counterexample below.
Let $A = \mathscr{P}(\mathbb{R})$.
Then $|A| = |\mathscr{P}(\mathbb{R})|$ and $|\mathbb{R}| < |\mathscr{P}(\mathbb{R})|$.
Thus, $|\mathbb{R}| < |A|$, so $|\mathbb{R}| \neq |A|$. □