Set Theory Notes

Jason Sass

July 16, 2023

Sets

Definition 1. equality of objects

Let a and b be objects.
The statement 'a is the same as b' is denoted $a = b$.

Let a and b be objects.
Then a is the same as b iff $a = b$.
Therefore, a is not the same as b iff $a \neq b$.

Definition 2. set

A **set** is a well-defined collection of distinct objects.

Well-defined means it is possible to determine unambiguously whether an object is a member of a set.
Distinct means each member of a set is unique, so no two members of a set are the same. Therefore, no duplicates are allowed in a set.
The order of objects listed in a set does not matter.

Since an object either is a member of a set or is not a member of a set, but not both, we define a notation that expresses this concept.
An element of a set is an object that is a member of the set.

Definition 3. element of a set

Let S be a set.
Let x be an object.
The predicate 'x is an element of S', denoted $x \in S$, means 'x is a member of S'.

Let S be a set.
Let x be an object.
Then $x \in S$ iff x is an element of S iff x is a member of S.
Therefore, $x \notin S$ iff x is not an element of S iff x is not a member of S.
Since $x \in S \vee x \notin S$ is a tautology, then either $x \in S$ or $x \notin S$.
Since $x \in S \wedge x \notin S$ is a contradiction, then it cannot be the case that $x \in S$ and $x \notin S$.

Example 4. Let S be the set containing only the following objects: $\square$, $\triangle$, $\bigstar$, $\bigcirc$.

Then $S = \{\square, \triangle, \bigstar, \bigcirc\}$.

Since the order of objects listed in a set does not matter, then $S = \{\bigstar, \triangle, \bigcirc, \square\}$.

Since $\bigstar$ is a member of S, we write $\bigstar \in S$.

Since 14 is not a member of S, we write $14 \notin S$.

Example 5. Let S be the set of numbers $1, 2, 3, 4, 5$.

Then $S = \{1, 2, 3, 4, 5\}$ and $4 \in S$, but $6 \notin S$.

Example 6. The set of all natural numbers, denoted $\mathbb{N}$, is the set $\{1, 2, 3, ...\}$.

Therefore, $\mathbb{N} = \{1, 2, 3, ...\}$.

Observe that $1 \in \mathbb{N}$, but $\frac{1}{2} \notin \mathbb{N}$.

Example 7. The set of all integers, denoted $\mathbb{Z}$, is the set $\{0, \pm 1, \pm 2, \pm 3, ...\}$.

Therefore, $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$.

Observe that $5 \in \mathbb{Z}$, but $\frac{-1}{3} \notin \mathbb{Z}$.

Set Builder Notation

We use **set builder notation** to specify a collection of objects that satisfy some condition.

The condition that each object in the set satisfies is a predicate.

Therefore, the predicate is the condition that holds true for each object in the set.

The notation $S = \{expression : predicate\}$ means S is a set and the predicate is the condition that holds true for each object in S.

The domain of discourse is the collection of objects that is under discussion.

Definition 8. universal set

A **universal set** is the set of all of the elements in the **domain of discourse**.

Let U be the domain of discourse.

Then U is the universal set and $U = \{x : x \in U\}$.

Therefore, U is the set of all x such that $x \in U$.

Hence, for each x in the domain of discourse, $x \in U$.

Therefore, for all x, $x \in U$, so $(\forall x)(x \in U)$.

Hence, if U is a universal set, then $(\forall x)(x \in U)$.

Example 9. Let S be the set of all natural numbers greater than 7.

Then $\mathbb{N}$ is the universal set.

Let $n \in \mathbb{N}$.

Define a predicate $p(n) : n > 7$ over $\mathbb{N}$.

Then $S = \{n \in \mathbb{N} : n > 7\}$.

Hence, a natural number n is an element of S iff $n > 7$.

Thus, if $n \in \mathbb{N}$, then $n \in S$ iff $n > 7$.

Each element of S is a natural number greater than 7.
Thus, for each $s \in S$, $s \in \mathbb{N}$ and $s > 7$, so $(\forall s \in S)(s \in \mathbb{N} \wedge s > 7)$.
Since $8 \in \mathbb{N}$ and $8 > 7$, then $8 \in S$.
Since $7 > 7$ is false, then $7 \notin S$.

Example 10. A rational number is a quotient of integers.
The set of all rational numbers, denoted $\mathbb{Q}$, is the set $\{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$.
Therefore, $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$.

Let $\mathbb{Q}$ be the set of all rational numbers.
Since $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$, then the domain of discourse is $\mathbb{Z}$ and the expression is $\frac{m}{n}$ and the predicate defined over $\mathbb{Z}$ is $(\exists m \in \mathbb{Z})(\exists n \in \mathbb{Z})(n \neq 0)$.
Thus, if $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, then $\frac{m}{n} \in \mathbb{Q}$ iff $n \neq 0$.
Hence, if $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, then $\frac{m}{n} \notin \mathbb{Q}$ iff $n = 0$.

Let q be a rational number.
Then $q \in \mathbb{Q}$, so there exists $m \in \mathbb{Z}$ and there exists $n \in \mathbb{Z}$ with $n \neq 0$ such that $q = \frac{m}{n}$.
Since $1 \in \mathbb{Z}$ and $2 \in \mathbb{Z}$ and $2 \neq 0$, then $\frac{1}{2} \in \mathbb{Q}$.
Since $1 \in \mathbb{Z}$ and $0 \in \mathbb{Z}$ and $0 = 0$, then $\frac{1}{0} \notin \mathbb{Q}$.

Example 11. The set of all real numbers is denoted $\mathbb{R}$.
Therefore, $\mathbb{R} = \{x : x \text{ is a real number}\}$.

Example 12. The set of all nonzero real numbers is denoted $\mathbb{R}^*$.
The domain of discourse is $\mathbb{R}$.
Let $x \in \mathbb{R}$.
Let $p(x) : x \neq 0$ be a predicate defined over $\mathbb{R}$.
Then $\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\}$.

Example 13. The set of all positive real numbers is denoted $\mathbb{R}^+$.
The domain of discourse is $\mathbb{R}$.
Let $x \in \mathbb{R}$.
Let $p(x) : x > 0$ be a predicate defined over $\mathbb{R}$.
Then $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.

Definition 14. truth set of a predicate
Let U be a universal set.
Let $x \in U$.
Let $p(x)$ be a predicate defined over U.
The set of all $x \in U$ such that $p(x)$ is true, denoted $\{x \in U : p(x)\}$, is called the **truth set** of $p(x)$.

Let U be a universal set.
Let $x \in U$.
Let $p(x)$ be a predicate defined over U.
Let P be the truth set of $p(x)$.
Then $P = \{x \in U : p(x)\}$.

Let a be an arbitrary element of P.
Then $a \in P$, so $a \in U$ and $p(a)$ is true.
Since $a \in P$ and $a \in U$, then the conditional $a \in P \rightarrow a \in U$ is true.
By universal generalization, $(\forall x)(x \in P \rightarrow x \in U)$.
Therefore, $P \subset U$.

Russell's Paradox
Let X be a set.
Let $S = \{X : X \notin X\}$.
Suppose $S \in S$.
Then $S \notin S$, by defn of S.
Thus, $S \in S$ and $S \notin S$, a contradiction.
Suppose $S \notin S$.
Then $S \in S$, by defn of S.
Thus, $S \notin S$ and $S \in S$, a contradiction.
Therefore, no such set S exists.
Thus, S has the form $\{expression : predicate\}$, but S is not well defined, so S is not a set.

Definition 15. subset
Let A and B be sets.
Then **A is a subset of B**, denoted $A \subset B$, iff $(\forall x)(x \in A \rightarrow x \in B)$.

Let A and B be sets.
Then A is a subset of B iff $(\forall x)(x \in A \rightarrow x \in B)$ iff every element of A is an element of B.
Observe that $(\forall x)(x \in A \rightarrow x \in B) \Leftrightarrow (\forall x \in A)(x \in B)$.

Observe that

$$\begin{aligned}
\neg(\forall x)(x \in A \rightarrow x \in B) &\Leftrightarrow (\exists x)\neg(x \in A \rightarrow x \in B) \\
&\Leftrightarrow (\exists x)(x \in A \wedge x \notin B) \\
&\Leftrightarrow (\exists x \in A)(x \notin B)
\end{aligned}$$

Therefore, A is not a subset of B, denoted $A \not\subset B$, iff 'not every element of A is in B'
iff
$(\exists x \in A)(x \notin B)$
iff
'there is at least one element of A that is not an element of B'
iff
'some element of A is not an element of B'

Therefore $A \not\subset B$ iff $(\exists x \in A)(x \notin B)$.

Suppose $A \subset B$.
Then the statement $(\forall x)(x \in A \to x \in B)$ is true.
Thus, $x \in A \to x \in B$ is true for every x.
Hence, $x \in A$ logically implies $x \in B$.
Therefore, $x \in A \Rightarrow x \in B$.

Suppose $x \in A \Rightarrow x \in B$.
Then $x \in A$ logically implies $x \in B$, so $x \in A \to x \in B$ is true for every x.
Hence, the statement $(\forall x)(x \in A \to x \in B)$ is true.
Therefore, $A \subset B$.
Therefore, $A \subset B$ iff $x \in A \Rightarrow x \in B$.

B is a **superset** of A, denoted $B \supset A$, iff $A \subset B$.

Proposition 16. *Every set is a subset of itself.*

Therefore $S \subset S$ for every set S.

Two sets are equal iff they contain the same elements.

Definition 17. equal sets
Let A and B be sets.
Then A **equals** B, denoted $A = B$, iff $(\forall x)(x \in A \leftrightarrow x \in B)$.

Let A and B be sets.
Then $A = B$ iff $(\forall x)(x \in A \leftrightarrow x \in B)$ iff for every x, $x \in A$ iff $x \in B$.
Observe that

$$\begin{aligned}
\neg(\forall x)(x \in A \leftrightarrow x \in B) &\Leftrightarrow (\exists x)\neg(x \in A \leftrightarrow x \in B) \\
&\Leftrightarrow (\exists x)[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)] \\
&\Leftrightarrow (\exists x)(x \in A \wedge x \notin B) \vee (\exists x)(x \in B \wedge x \notin A) \\
&\Leftrightarrow (\exists x \in A)(x \notin B) \vee (\exists x \in B)(x \notin A)
\end{aligned}$$

Therefore, A does not equal B, denoted $A \neq B$, iff 'A does not contain the same elements as B'
iff
$(\exists x \in A)(x \notin B) \vee (\exists x \in B)(x \notin A)$
iff
'either there is at least one element of A that is not an element of B, or there is at least one element of B that is not an element of A'
iff
'either some element of A is not an element of B, or some element of B is not an element of A'
iff
'either some element is in A, but not in B, or some element is in B, but not in A.'

Therefore $A \neq B$ iff $(\exists x \in A)(x \notin B) \vee (\exists x \in B)(x \notin A)$.

Suppose $A = B$.
Then the statement $(\forall x)(x \in A \leftrightarrow x \in B)$ is true.
Thus, $x \in A \leftrightarrow x \in B$ is true for every x.
Hence, $x \in A$ is logically equivalent to $x \in B$.
Therefore, $x \in A \Leftrightarrow x \in B$.

Suppose $x \in A \Leftrightarrow x \in B$.
Then $x \in A$ is logically equivalent to $x \in B$, so $x \in A \leftrightarrow x \in B$ is true for every x.
Hence, the statement $(\forall x)(x \in A \leftrightarrow x \in B)$ is true.
Therefore, $A = B$.
Therefore, $A = B$ iff $x \in A \Leftrightarrow x \in B$.

Proposition 18. ***necessary and sufficient conditions for set equality***
Let A and B be sets.
Then $A = B$ iff $A \subset B$ and $B \subset A$.

Therefore two sets are equal if and only if each is a subset of the other.

Definition 19. proper subset
Let A and B be sets.
Then **A is a proper subset of B**, denoted $A \subsetneq B$, iff $A \subset B$ and $A \neq B$.

A is not a proper subset of B, denoted $A \not\subsetneq B$, iff either $A \not\subset B$ or $A = B$.
Observe that

$$\begin{aligned} A \subsetneq B &\Leftrightarrow A \subset B \wedge A \neq B \\ &\Leftrightarrow A \subset B \wedge \neg(A = B) \\ &\Leftrightarrow A \subset B \wedge \neg(A \subset B \wedge B \subset A) \\ &\Leftrightarrow A \subset B \wedge (A \not\subset B \vee B \not\subset A) \\ &\Leftrightarrow (A \subset B \wedge A \not\subset B) \vee (A \subset B \wedge B \not\subset A) \\ &\Leftrightarrow A \subset B \wedge B \not\subset A \end{aligned}$$

Therefore, A is a proper subset of B iff $A \subset B$ and $B \not\subset A$.
$A \subseteq B$ means $A \subset B \vee A = B$

Proposition 20. *Every set equals itself.*

Therefore $S = S$ for every set S.

Definition 21. empty set
Let S be a set.
Then S **is empty** iff $\neg(\exists x)(x \in S)$.

Let S be a set.

Then S is empty iff $\neg(\exists x)(x \in S)$ iff 'there is no x such that $x \in S$' iff 'there is no element in S'.

Therefore, S is empty iff $\neg(\exists x)(x \in S)$ iff there is no element in S.

Observe that $\neg(\exists x)(x \in S) \Leftrightarrow (\forall x)(x \notin S)$.

Therefore, S is empty iff $(\forall x)(x \notin S)$.

Thus, S is not empty iff $(\exists x)(x \in S)$ iff 'there exists x such that $x \in S$' iff 'there is some element in S' iff 'there is at least one element in S'.

Therefore, S is not empty iff $(\exists x)(x \in S)$ iff there is at least one element in S.

The **empty set** is a set that contains no elements.

Therefore, an empty set is a set S such that S is empty.

Lemma 22. *There is at most one empty set.*

Proposition 23. *The empty set is unique.*

The **empty set** is denoted $\emptyset$.

$\emptyset = \{\}$.

Since the empty set contains no elements, then $\emptyset$ is empty, so $\neg(\exists x)(x \in \emptyset)$.

Therefore, $(\forall x)(x \notin \emptyset)$.

$|\emptyset| = 0$

Proposition 24. *The empty set is a subset of every set.*

Therefore, $\emptyset \subset S$ for every set S.

Theorem 25. *The subset relation is a partial order.*

1. reflexive $A \subset A$ for every set A.
2. antisymmetric if $A \subset B$ and $B \subset A$, then $A = B$ for all sets A, B.
3. transitive if $A \subset B$ and $B \subset C$, then $A \subset C$ for all sets A, B, C.

Theorem 26. *The set equality relation is an equivalence relation.*

1. reflexive $A = A$ for any set A.
2. symmetric if $A = B$, then $B = A$ for any sets A, B.
3. transitive if $A = B$ and $B = C$, then $A = C$ for any sets A, B, C.

Definition 27. power set

Let S be a set.

The **power set of** S, denoted $\mathscr{P}$, is the set of all subsets of S.

Therefore, $\mathscr{P} = \{X : X \subset S\}$.

Let $\mathscr{P}$ be the power set of a set S.

Then $X \in \mathscr{P}$ iff $X \subset S$.

Since $S \subset S$, then $S \in \mathscr{P}$.

Since $\emptyset \subset S$, then $\emptyset \in \mathscr{P}$.

Relationship between Sets

Let A, B be sets.

Then exactly one of the 5 below relationships is true.

1. A and B are **disjoint**. $A \cap B = \emptyset$
2. A and B intersect but are not subsets of each other. $A \cap B \neq \emptyset$
3. A is a proper subset of B. $A \subset B$
4. B is a proper subset of A. $B \subset A$
5. A and B are identical. $A = B$

Set Operations and Algebraic properties of Sets

Definition 28. set union

Let A and B be sets.

The **union** of A and B, denoted $A \cup B$, is the set of elements that are in A or B (or both).

Therefore $A \cup B = \{x : x \in A \vee x \in B\}$.

Set union corresponds to **logical inclusive disjunction**.

Proposition 29. ***Inclusion Exclusion Principle***

If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

If A and B are disjoint finite sets, then $|A \cup B| = |A| + |B|$.

Let A and B be finite sets.

Then $|A \cup B| \leq |A| + |B|$.

Definition 30. set intersection

Let A and B be sets.

The **intersection** of A and B, denoted $A \cap B$, is the set of elements that are in both A and B.

Therefore $A \cap B = \{x : x \in A \wedge x \in B\}$

Set intersection corresponds to **logical conjunction**.

Definition 31. relative complement

Let A and B be sets.

The **relative complement** of B in A, denoted $\overline{B}_A$, is the set of all elements of A not contained in B.

Therefore $\overline{B}_A = \{x \in A : x \notin B\} = A - B$.

Definition 32. absolute complement

Let U be a universal set.

The **absolute complement** of A in U, denoted $\overline{A}$, is the set of all elements of U not contained in A.

Therefore $\overline{A} = \{x \in U : x \notin A\} = U - A$.

Therefore, if U is a universal set, then $\overline{A} = \{x \in U : x \notin A\}$.

Set complement corresponds to **logical negation**.

Let $A \subset U$.
Observe that $A = \{x \in U : x \in A\}$ and $\overline{A} = \{x \in U : x \notin A\}$.
Let $x \in U$.
Suppose $x \notin A$.
Since $x \in U$ and $x \notin A$ then $x \in \overline{A}$ by defn of $\overline{A}$.
Thus $x \notin A \to x \in \overline{A}$.
Conversely, suppose $x \in \overline{A}$.
Then $x \notin A$ by defn of $\overline{A}$.
Thus $x \in \overline{A} \to x \notin A$.
Since $x \notin A \to x \in \overline{A}$ and $x \in \overline{A} \to x \notin A$ then $x \notin A$ iff $x \in \overline{A}$.

Therefore $x \notin A \Leftrightarrow x \in \overline{A}$.

Proposition 33. *Let U be a universal set and $A \subset U$.*
If U is finite, then $|\overline{A}| = |U| - |A|$.

Algebraic Properties of Sets

Let U be a universal set.
Let A, B, C be sets contained in U.

Identity

$$A \cap U = A$$
$$A \cup \emptyset = A$$

Involution

$$\overline{\overline{A}} = A$$

Domination

$$A \cup U = U$$
$$A \cap \emptyset = \emptyset$$

Idempotent

$$A \cup A = A$$
$$A \cap A = A$$

Commutative

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associative

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

DeMorgan

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Complement

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \emptyset$$
$$\overline{\emptyset} = U$$
$$\overline{U} = \emptyset$$

Absorption

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

Principle of Duality

Interchange $\cup$ and $\cap$ and interchange $\emptyset$ and U to transform one of the pair of identities into the other.

We can draw a Venn diagram of sets A, B to visualize the propositions below.

Proposition 34. *Let A and B be sets.*
Then $A \subset B$ iff $A \cup B = B$.

Proposition 35. *Let A and B be sets.*
Then $A \subset B$ iff $A \cap B = A$.

Proposition 36. *Let A and B be sets.*
Then $(A \cap B) \subset A \subset (A \cup B)$.

Corollary 37. *Let A and B be sets.*
Then $A \cap B \subset A \cup B$.

Definition 38. set difference

Let A and B be sets.

The **difference** of A and B, denoted $A - B$, is the set of elements that are in A but not in B.

Therefore $A - B = \{x : x \in A \wedge x \notin B\} = \{x \in A : x \notin B\}$

Let A, B be sets.
Let $x \in A - B$.
Then $x \in A$ and $x \notin B$, so $x \in A$.
Thus, $A - B \subset A$.
Therefore, if A and B are sets, then $A - B \subset A$.

Proposition 39. *Let A and B be sets.*
Then
1. $A - B = A \cap \overline{B} = A - (A \cap B)$.
2. $A - A = \emptyset$.
3. $A - \emptyset = A$.
4. $A \subset B$ *iff* $A - B = \emptyset$.

Proposition 40. $A \cup B$ ***is a union of 3 disjoint sets.***
Let A and B be sets. Then
$A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$.

Proposition 41. ***counting set difference***
If A and B are finite sets, then $|A - B| = |A| - |A \cap B|$.

Note: We need to prove this!

Definition 42. symmetric difference

Let A and B be sets.

The **symmetric difference** between A and B, denoted $A \triangle B$, is the set of elements that are in A or B, but not both.

Therefore

$$\begin{aligned} A \triangle B &= \{x : (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\} \\ &= \{x : (x \in A \cup B) \wedge \neg(x \in A \cap B)\} \\ &= \{x : (x \in A \cup B) \wedge (x \notin A \cap B)\} \\ &= \{x : x \in A \cup B - A \cap B\} \\ &= (A \cup B) - (A \cap B). \end{aligned}$$

Symmetric difference corresponds to **logical exclusive disjunction**.

Proposition 43. ***symmetric difference is logic exclusive OR***
Let A and B be sets. Then
$(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

Proposition 44. ***Properties of symmetric difference***
1. $A \triangle B = B \triangle A$ *for all sets A, B. (commutative)*
2. $(A \triangle B) \triangle C = A \triangle (B \triangle C)$ *for all sets A, B, C.(associative)*
3. $A \triangle A = \emptyset$ *for every set A.*
4. $A \triangle \emptyset = A$ *for every set A.*

Let A and B be finite sets.

Then $|A \triangle B| = |A| + |B| - 2|A \cap B|$.

We need to prove this.

Cartesian Product

Definition 45. ordered pair

An **ordered pair** (a, b) is the set $\{\{a\}, \{a, b\}\}$.

Therefore $(a, b) = \{\{a\}, \{a, b\}\}$.

Proposition 46. *Let (a, b) and (c, d) be ordered pairs.*

Then $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

Definition 47. equality of ordered pairs

Let (a, b) and (c, d) be ordered pairs.

Then $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

Definition 48. Cartesian product

Let A and B be sets.

The **Cartesian product of** A **and** B, denoted $A \times B$, is the set of all ordered pairs whose first component is an element of A and whose second component is an element of B.

Therefore $A \times B = \{(a, b) : a \in A \wedge b \in B\} = \{(a, b) : a \in A, b \in B\}$.

An object x is in $A \times B$ iff $(\exists a \in A)$ and $(\exists b \in B)$ such that $x = (a, b)$.

Proposition 49. *Let A, B be finite sets.*

Then $|A \times B| = |A||B|$.

Proposition 50. ***Domination law for cartesian product***

$A \times \emptyset = \emptyset \times A = \emptyset$ for every set A.

Proposition 51. ***Distributive properties of cartesian product***

Let A, B, C be sets. Then

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$. (left distributive $\times$ over $\cup$)

2. $(B \cup C) \times A = (B \times A) \cup (C \times A)$. (right distributive $\times$ over $\cup$)

3. $A \times (B \cap C) = (A \times B) \cap (A \times C)$. (left distributive $\times$ over $\cap$)

4. $(B \cap C) \times A = (B \times A) \cap (C \times A)$. (right distributive $\times$ over $\cap$)

5. $(B - C) \times A = (B \times A) - (C \times A)$. (right distributive $\times$ over $-$)

Example 52. cartesian product is not commutative

Let $A = \{1, 2\}$ and $B = \{3\}$.

Then $A \times B = \{(1, 3), (2, 3)\}$ and $B \times A = \{(3, 1), (3, 2)\}$.

Since $(1, 3) \in A \times B$, but $(1, 3) \notin B \times A$, then $A \times B \neq B \times A$.

Example 53. cartesian product is not associative

Let $A = \{1, 2\}$ and $B = \{3, 4\}$ and $C = \{5\}$.

Then $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$ and $B \times C = \{(3, 5), (4, 5)\}$.

Thus, $(A\times B)\times C = \{((1,3),5),((1,4),5),((2,3),5),((2,4),5)\}$ and $A\times(B\times C) = \{(1,(3,5)),(1,(4,5)),(2,(3,5)),(2,(4,5))\}$.

Since $((1,3),5) \in (A\times B)\times C$, but $((1,3),5) \notin A\times(B\times C$, then $(A\times B)\times C \neq A\times(B\times C)$.

Theorem 54. *Let A and B be sets.*

Then $(a,b) \in A\times B$ iff $(b,a) \in B\times A$ for all $(a,b)\in A\times B$.

If $A\times B = \emptyset$, then either $A=\emptyset$ or $B=\emptyset$.

If $A\times B = B\times A$, then either $A=\emptyset$ or $B=\emptyset$ or $A=B$.

If $A\times B = A\times C$ and $A\neq\emptyset$, then $B=C$. (left cancellation law holds)

If $B\times A = C\times A$ and $A\neq\emptyset$, then $B=C$. (right cancellation law holds)

Definition 55. n **tuple**

Let $n\in\mathbb{Z}, n\geq 0$.

An **ordered n-tuple** is a finite sequence of n elements.

Let A be a finite set.

Then there exists $n\in\mathbb{Z}, n\geq 0$ such that $|A|=n$.

Let $A=\{a_1,a_2,...,a_n\}=\{a_i : i\in\mathbb{N}_n\}$, where $\mathbb{N}_n=\{1,2,...,n\}$.

Then the sequence $(a_1,a_2,...,a_n)$ is an ordered n tuple of the elements of A.

Note that this implies the existence of a function $f:\mathbb{N}_n\to A$ defined by $f(k)=a_k$.

Suppose $x,y\in\mathbb{N}_n$ such that $x\neq y$.

Then $a_x\neq a_y$, so $f(x)\neq f(y)$.

Hence, f is 1-1.

Suppose $y\in A$.

Then $y=a_x$ for some $x\in\mathbb{N}_n$.

Hence, $f(x)=a_x=y$, so f is onto A.

Therefore, f is a one to one correspondence between $\mathbb{N}_n$ and A.

Suppose $n=0$.

Then $|A|=0$, so $A=\emptyset$.

Hence, the ordered zero tuple is empty, so $()=\emptyset$.

Definition 56. Cartesian product of n sets

Let $n\in\mathbb{Z}, n\geq 0$.

Let $A_1,A_2,...,A_n$ be sets.

The **Cartesian product of** n sets, denoted $A_1\times A_2\times...\times A_n$, is the set of all ordered n tuples whose i^{th} term is in A_i.

$$\begin{aligned}\prod_{i=1}^{n} A_i = A_1\times A_2\times...\times A_n &= \{(a_1,a_2,...,a_n) : a_1\in A_1\wedge a_2\in A_2\wedge...\wedge a_n\in A_n\}\\ &= \{(a_1,a_2,...,a_n) : a_i\in A_i \text{ for each } i\in\{1,2,...,n\}\}\\ &= \{(a_1,a_2,...,a_n) : (\forall i\in\{1,2,...,n\})(a_i\in A_i)\}.\end{aligned}$$

Each $(a_1,a_2,...,a_n)$ is an **n-tuple**.

An object x is in $A_1 \times A_2 \times ... \times A_n$ if and only if $(\exists a_1 \in A_1)$ and $(\exists a_2 \in A_2)$ and ... and $(\exists a_n \in A_n)$ such that $x = (a_1, a_2, ..., a_n)$.

Let each A_i be finite.

Then $|A_1 \times A_2 \times ... \times A_n| = |A_1| \times |A_2| \times ... \times |A_n|$.

Definition 57. Cartesian power of a set

Let $n \in \mathbb{Z}, n \geq 0$.

Let A be a set.

The **Cartesian power of** A, denoted A^n, is the Cartesian product of A with itself n times.

$$\begin{aligned} A^n &= A \times A \times ... \times A \\ &= \{(a_1, a_2, ..., a_n) : a_1 \in A \wedge a_2 \in A ... \wedge a_n \in A\} \\ &= \{(a_1, a_2, ..., a_n) : a_i \in A \text{ for each } i \in \{1, 2, ..., n\}\} \\ &= \{(a_1, a_2, ..., a_n) : (\forall i \in \{1, 2, ..., n\})(a_i \in A)\} \end{aligned}$$

An object x is in A^n if and only if $(\exists a_1 \in A)$ and $(\exists a_2 \in A)$ and ... and $(\exists a_n \in A)$ such that $x = (a_1, a_2, ..., a_n)$.

Example 58. $\mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R} = \mathbb{R}^n =$ real n-space

$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$.

$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$.

$\mathbb{R}^2 \times \mathbb{R} = \{((x, y), z) : x, y, z \in \mathbb{R}\}$.

$\mathbb{R} \times \mathbb{R}^2 = \{(x, (y, z)) : x, y, z \in \mathbb{R}\}$.

Thus, $\mathbb{R}^2 \times \mathbb{R} \neq \mathbb{R} \times \mathbb{R}^2$.

Indexed Sets

Definition 59. indexed family of sets

Let I be a set.

Then $\{S_i : i \in I\}$ is a **collection of sets (family of sets)** indexed by I.

The subscript i that labels a set S_i in the collection is an **index**.

The set I is an **index set**.

The indexed family of sets is also denoted $\{S_i\}_{i \in I}$.

Example 60. family of sets indexed by $\mathbb{N}$

Let $\mathcal{S} = \{S_i : i = 1, 2, 3, ...\} = \{S_1, S_2, S_3, ...\} = \{S_i : i \in \mathbb{N}\} = \{S_i\}_{i \in \mathbb{N}}$.

Then $\mathcal{S}$ is a family of sets indexed by $\mathbb{N}$ and each S_i is a set in the collection.

$\mathcal{S}$ is a countably infinite collection of sets.

Definition 61. Let $\mathcal{S} = \{S_k : k \in \mathbb{N}\}$ be a collection of sets indexed by $\mathbb{N}$.

$\mathcal{S}$ is **increasing** iff $(\forall m \in \mathbb{N})(\forall n \in \mathbb{N})(m < n \rightarrow S_m \subset S_n)$.

$\mathcal{S}$ is **decreasing** iff $(\forall m \in \mathbb{N})(\forall n \in \mathbb{N})(m < n \rightarrow S_m \supset S_n)$.

$\mathcal{S}$ is **mutually disjoint (pairwise disjoint)** iff $(\forall m \in \mathbb{N})(\forall j \in \mathbb{N})(m \neq n \rightarrow S_m \cap S_n = \emptyset)$.

Example 62. Let $A_i = \{i\}$ for $i = 1, 2, 3, ...$

Then $A = \{A_i : i \in \mathbb{N}\} = \{\{1\}, \{2\}, \{3\}, ...\}$ is a family of sets indexed by $\mathbb{N}$.

Each A_i is a **singleton set**.

Let $i, j \in \mathbb{N}$ such that $i \neq j$.

Then $\{i\} \cap \{j\} = \emptyset$, so $A_i \cap A_j = \emptyset$.

Therefore, A is a mutually disjoint family of sets.

Example 63. Let $B_k = \{1, 2, 3, ..., k\}$ for each $k = 1, 2, 3, ...$

Then $B_k = \{n \in \mathbb{N} : 1 \leq n \leq k\}$ for all $k \in \mathbb{N}$ and

$B = \{B_k : k \in \mathbb{N}\} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, ...\}$ is a family of sets indexed by $\mathbb{N}$.

Let $i, j \in \mathbb{N}$ such that $i < j$.

Observe that $B_i = \{n \in \mathbb{N} : 1 \leq n \leq i\}$ and $B_j = \{n \in \mathbb{N} : 1 \leq n \leq j\}$.

Let $x \in B_i$.

Then $x \in \mathbb{N}$ and $1 \leq x \leq i$, so $1 \leq x$ and $x \leq i$.

Since $x \leq i$ and $i < j$, then $x < j$, so $x \leq j$.

Thus, $1 \leq x$ and $x \leq j$, so $1 \leq x \leq j$.

Since $x \in \mathbb{N}$ and $1 \leq x \leq j$, then $x \in B_j$.

Therefore, $x \in B_i$ implies $x \in B_j$, so $B_i \subset B_j$.

Hence, B is an increasing family of sets.

Example 64. Let $C_i = [i, \infty)$ for each $i = 1, 2, 3, ...$

Then $C_i = [i, \infty) = \{x \in \mathbb{R} : i \leq x\}$ for all $i \in \mathbb{N}$.

Let $C = \{C_i : i \in \mathbb{N}\} = \{[1, \infty), [2, \infty), [3, \infty), [4, \infty), ...\}$ be a family of intervals indexed by $\mathbb{N}$.

Let $i, j \in \mathbb{N}$ such that $i < j$.

Observe that $C_i = [i, \infty) = \{x \in \mathbb{R} : i \leq x\}$ and $C_j = [j, \infty) = \{x \in \mathbb{R} : j \leq x\}$.

Let $x \in C_j$.

Then $x \in \mathbb{R}$ and $j \leq x$.

Since $i < j$ and $j \leq x$, then $i < x$.

Thus, $x \in \mathbb{R}$ and $i < x$, so $x \in C_i$.

Hence, $x \in C_j$ implies $x \in C_i$, so $C_j \subset C_i$.

Therefore, $C_i \supset C_j$.

Hence, C is a decreasing family of intervals, or **family of nested intervals**.

Definition 65. finite union and intersection of sets

Let n be a fixed natural number.

Let $\mathcal{S} = \{S_1, S_2, S_3, ..., S_n\} = \{S_k : k \in \mathbb{N}, 1 \leq k \leq n\}$ be a finite collection of sets.

Then

$$\begin{aligned}
\bigcup_{k=1}^{n} S_k &= S_1 \cup S_2 \cup ... \cup S_n \\
&= \text{the union of the finite collection } \mathcal{S} \\
&= \text{the set that consists of everything that is in at least one of the sets } S_k \\
&= \{x : x \in S_k \text{ for some } k \in \mathbb{N} \text{ with } 1 \leq k \leq n\} \\
&= \{x : (\exists k \in \mathbb{N}, 1 \leq k \leq n)(x \in S_k)\}
\end{aligned}$$

$$\begin{aligned}
\bigcap_{k=1}^{n} S_k &= S_1 \cap S_2 \cap ... \cap S_n \\
&= \text{the intersection of the finite collection } \mathcal{S} \\
&= \text{the set that consists of everything that is in common to each of the sets } S_k \\
&= \{x : x \in S_k \text{ for every } k \in \mathbb{N} \text{ with } 1 \leq k \leq n\} \\
&= \{x : (\forall k \in \mathbb{N}, 1 \leq k \leq n)(x \in S_k)\}
\end{aligned}$$

Definition 66. infinite countable union and intersection of sets

Let $\mathcal{S} = \{S_1, S_2, S_3, ...\} = \{S_n : n \in \mathbb{N}\}$ be an infinite collection of sets indexed by $\mathbb{N}$.

Then

$$\begin{aligned}
\bigcup_{n=1}^{\infty} S_n &= S_1 \cup S_2 \cup ... \\
&= \text{the union of the countably infinite collection } \mathcal{S} \\
&= \text{the set that consists of everything that is in at least one of the sets } S_n \\
&= \{x : x \in S_n \text{ for some } n \in \mathbb{N}\} \\
&= \{x : (\exists n \in \mathbb{N})(x \in S_n)\}
\end{aligned}$$

$$\begin{aligned}
\bigcap_{n=1}^{\infty} S_n &= S_1 \cap S_2 \cap ... \\
&= \text{the intersection of the countably infinite collection } \mathcal{S} \\
&= \text{the set that consists of everything that is in common to each of the sets } S_n \\
&= \{x : x \in S_n \text{ for every } n \in \mathbb{N}\} \\
&= \{x : (\forall n \in \mathbb{N})(x \in S_n)\}
\end{aligned}$$

Definition 67. arbitrary union of a collection of sets

Let $\mathcal{S}$ be a collection of sets.

The union of the sets in $\mathcal{S}$, denoted $\bigcup \mathcal{S}$, is the set of all x such that $x \in S$ for some $S \in \mathcal{S}$.

Therefore, $\bigcup \mathcal{S} = \{x : x \in S \text{ for some } S \in \mathcal{S}\} = \{x : (\exists S \in \mathcal{S})(x \in S)\}$.

Definition 68. arbitrary union of an indexed collection of sets

Let $\mathcal{S} = \{S_i : i \in I\}$ be a collection of sets indexed by a set I.

The union of the sets in $\mathcal{S}$, denoted $\bigcup_{i\in I} S_i$, is the set of all x such that $x \in S_i$ for some $i \in I$.

Therefore, $\bigcup_{i\in I} S_i = \{x : x \in S_i \text{ for some } i \in I\} = \{x : (\exists i \in I)(x \in S_i)\}$.

Definition 69. arbitrary intersection of a collection of sets

Let $\mathcal{S}$ be a collection of sets.

The intersection of the sets in $\mathcal{S}$, denoted $\bigcap \mathcal{S}$, is the set of all x such that $x \in S$ for all $S \in \mathcal{S}$.

Therefore, $\bigcap \mathcal{S} = \{x : x \in S \text{ for all } S \in \mathcal{S}\} = \{x : (\forall S \in \mathcal{S})(x \in S)\}$.

Definition 70. arbitrary intersection of an indexed collection of sets

Let $\mathcal{S} = \{S_i : i \in I\}$ be a collection of sets indexed by a set I.

The intersection of the sets in $\mathcal{S}$, denoted $\bigcap_{i\in I} S_i$, is the set of all x such that $x \in S_i$ for all $i \in I$.

Therefore, $\bigcap_{i\in I} S_i = \{x : x \in S_i \text{ for all } i \in I\} = \{x : (\forall i \in I)(x \in S_i)\}$.

Proposition 71. ***DeMorgan law for relative complements***

For all sets A, B, C,

1. $A - (B \cup C) = (A - B) \cap (A - C)$. (Complement of union = intersection of complements).

2. $A - (B \cap C) = (A - B) \cup (A - C)$. (Complement of intersection = union of complements).

Theorem 72. ***Generalized DeMorgan law***

Let S be a set.

Let $\{A_i : i \in I\}$ be a family of sets indexed by a set I.

Then

1. $S - \cup_{i\in I} A_i = \cap_{i\in I}(S - A_i)$. (Complement of union = intersection of complements)

2. $S - \cap_{i\in I} A_i = \cup_{i\in I}(S - A_i)$. (Complement of intersection = union of complements)

Theorem 73. ***Generalized DeMorgan law***

If $A_1, A_2, ...A_n$ are sets in some universal set U, then $\overline{\bigcap_{i=1}^n A_i} = \bigcup_{i=1}^n \overline{A_i}$ for $n \geq 2$.

Complement of intersections = union of complements.

Theorem 74. *Let $\{A_n : n \in \mathbb{N}\}$ be an infinite collection of sets from a universal set U.*

Let $k \in \mathbb{N}$. Then

1. $A_k \subset \cup_{n=1}^{\infty} A_n$.

2. $\cap_{n=1}^{\infty} A_n \subset A_k$.

3. $\overline{\cap_{n=1}^{\infty} A_n} = \cup_{n=1}^{\infty} \overline{A}_n$. (Generalized DeMorgan law)

4. $\overline{\cup_{n=1}^{\infty} A_n} = \cap_{n=1}^{\infty} \overline{A}_n$.

5. if $B \subset U$, then $\cup_{n=1}^{\infty} A_n \cap B = \cup_{n=1}^{\infty}(A_n \cap B)$.(Generalized distributivity)

6. if $B \subset U$, then $\cap_{n=1}^{\infty} A_n \cup B = \cap_{n=1}^{\infty}(A_n \cup B)$.

Let $k \in \mathbb{N}$.
Then $\cup_{n=1}^{\infty} A_n$ is a superset of each A_k and $\cap_{n=1}^{\infty} A_n$ is a subset of each A_k.

Theorem 75. *Let $\mathcal{S} = \{S_k : k \in \mathbb{N}\}$ be an infinite collection of subsets of a universal set U.*
Then $\cup_{k=1}^{\infty} S_k$ is the least upper bound of $\mathcal{S}$.

$\cup_{k=1}^{\infty} S_k$ is the **least upper bound** of $\mathcal{S}$.
This means $(\forall B \in U)[(\forall n \in \mathbb{N})(B \supset S_n) \to (\cup_{k=1}^{\infty} S_k \subset B)]$.
Hence, $\cup_{k=1}^{\infty} S_k$ is the smallest such B that satisfies the above statement.

Theorem 76. *Let $\mathcal{S} = \{S_k : k \in \mathbb{N}\}$ be an infinite collection of subsets of a universal set U.*
Then $\cap_{k=1}^{\infty} S_k$ is the greatest lower bound of $\mathcal{S}$.

$\cap_{k=1}^{\infty} S_k$ is the **greatest lower bound** of $\mathcal{S}$.
This means $(\forall B \in U)[(\forall n \in \mathbb{N})(B \subset S_n) \to (B \subset \cap_{k=1}^{\infty} S_k)]$.
Hence, $\cap_{k=1}^{\infty} S_k$ is the largest such B that satisfies the above statement.

Theorem 77. *Let $\mathcal{S} = \{S_k : k \in \mathbb{N}\}$ be an infinite collection of sets.*
1. *If $\mathcal{S}$ is increasing, then $\cap_{k=1}^{\infty} S_k = S_1$.*
2. *If $\mathcal{S}$ is decreasing, then $\cup_{k=1}^{\infty} S_k = S_1$.*
3. *If $\mathcal{S}$ is mutually disjoint, then $\cap_{k=1}^{\infty} S_k = \emptyset$.*

Definition 78. union of an arbitrary collection of sets
Let $\mathcal{S}$ be an arbitrary collection of sets.
The union of $\mathcal{S}$, denoted $\cup\mathcal{S}$, is the set $\{x : (\exists X \in \mathcal{S})(x \in X)\}$.

Therefore, $\cup\mathcal{S} = \{x : (\exists X \in \mathcal{S})(x \in X)\}$.

Definition 79. intersection of an arbitrary collection of sets
Let $\mathcal{S}$ be an arbitrary collection of sets.
The intersection of $\mathcal{S}$, denoted $\cap\mathcal{S}$, is the set $\{x : (\forall X \in \mathcal{S})(x \in X)\}$.

Therefore, $\cap\mathcal{S} = \{x : (\forall X \in \mathcal{S})(x \in X)\}$.

Proposition 80. *Let $\mathcal{S}$ be an arbitrary collection of sets. Then*
1. *$X \subset \cup\mathcal{S}$ for each $X \in \mathcal{S}$.*
2. *If A is a set and $S \subset A$ for all $S \in \mathcal{S}$, then $\cup\mathcal{S} \subset A$.*

Sets of Numbers

$$
\begin{aligned}
n\mathbb{Z} &= \{nk : k \in \mathbb{Z}\} = \text{set of all multiples of natural number n} \\
\mathbb{Z}^+ &= \{n \in \mathbb{Z} : n > 0\ \} = \{1, 2, 3, ...\} = \text{ set of positive integers} = \mathbb{N} \\
\mathbb{Z}^* &= \{n \in \mathbb{Z} : n \neq 0\ \} = \{..., -3, -2, -1, 1, 2, 3, ...\} = \text{ set of nonzero integers} \\
\mathbb{Q}^+ &= \{\frac{m}{n} : m, n \in \mathbb{Z}^+\} = \text{ set of positive rational numbers} \\
\mathbb{C} &= \{a + bi : a, b \in \mathbb{R}, i^2 = -1\} = \{|z| \text{ cis } \theta\ : z, \theta \in \mathbb{R}\} = \text{set of complex numbers} \\
\mathbb{C}^* &= \{|z| \text{ cis } \theta\ : |z| \neq 0, \theta \in \mathbb{R}\} = \text{ set of nonzero complex numbers}
\end{aligned}
$$

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Cardinality of Sets

Definition 81. numeric equivalence
A set A is **numerically equivalent** to a set B, denoted $A \sim B$, iff there exists a bijective map $f : A \to B$.

Therefore, a set A is **numerically not equivalent** to a set B, denoted $A \not\sim B$, iff there does not exist a bijective map $f : A \to B$.

Proposition 82. ***numeric equivalence is an equivalence relation.***
Let $\sim$ be the numeric equivalence relation defined on the collection of all subsets of a given universal set U.
Let A, B, C be sets. Then
1. $A \sim A$. (reflexive)
2. If $A \sim B$, then $B \sim A$. (symmetric)
3. If $A \sim B$ and $B \sim C$, then $A \sim C$. (transitive)

Therefore, the relation of numerical equivalence is an equivalence relation on the collection of all subsets of a given universal set U.

Definition 83. Let n be a nonnegative integer.
Define $\mathbb{N}_n = \{k \in \mathbb{N} : 1 \leq k \leq n\}$.

Example 84. $\mathbb{N}_0 = \emptyset$.

Proof. Suppose for the sake of contradiction that $\mathbb{N}_0 \neq \emptyset$.
Then there exists $m \in \mathbb{N}_0$, so $m \in \mathbb{N}$ and $1 \leq m \leq 0$.
Since $1 \leq m \leq 0$, then $1 \leq 0$, a contradiction.
Therefore, $\mathbb{N}_0 = \emptyset$. □

Example 85. $\mathbb{N}_5 = \{k \in \mathbb{N} : 1 \leq k \leq 5\} = \{1, 2, 3, 4, 5\}$.

Proposition 86. *Let S be a set.*
Then $S \sim \emptyset$ iff $S = \emptyset$.

Definition 87. finite set
A set S is **finite** iff there exists a nonnegative integer n such that $S \sim \mathbb{N}_n$.

Proposition 88. *Let S be a set.*
Then S is finite iff either $S = \emptyset$ or there is a bijection $f : S \to \{1, ..., n\}$ for some $n \in \mathbb{N}$.

A finite set of n elements has 2^n subsets.
Therefore $|2^S| = 2^{|S|} = 2^n$.

Lemma 89. *For every $n \in \mathbb{N}$, $n \in \mathbb{N}_n$.*

Proposition 90. *For every $n \in \mathbb{N}$, $\mathbb{N}_n$ is finite.*

Lemma 91. *For every $n \in \mathbb{N}$, if $n > 1$, then $\mathbb{N}_2 \subset \mathbb{N}_n$.*

Lemma 92. *Let A be a set such that $a \in A$.*
Then for all $n \in \mathbb{N}$, if $A \sim \mathbb{N}_n$, then $A - \{a\} \sim \mathbb{N}_{n-1}$.

Theorem 93. ***counting theorem***
Let $m, n \in \mathbb{N}$.
Then $\mathbb{N}_m \sim \mathbb{N}_n$ iff $m = n$.

Definition 94. cardinality of a finite set
Let $n \in \mathbb{N}$.
A set S is said to have **cardinality** n iff $S \sim \mathbb{N}_n$.
We say that a set S has n elements iff S has cardinality n.
The cardinality of the empty set is defined to be zero.

Theorem 95. ***cardinality of a finite set is well defined.***
The cardinality of a finite set is unique.

The cardinality of a finite set is the number of elements in the set.
Let S be a finite set.
Then the cardinality of S is unique.
The cardinality of S is denoted $|S|$.
Therefore, $|S|$ denotes the number of elements in a finite set S.
Since the cardinality of the empty set is zero, then $|\emptyset| = 0$.

Let S be a finite set.
Then either $S = \emptyset$ or $S \neq \emptyset$.
We consider these cases separately.
Case 1: Suppose $S = \emptyset$.
Then $|S| = |\emptyset| = 0$.
Case 2: Suppose $S \neq \emptyset$.
Since S is finite and not empty, then there exists $n \in \mathbb{N}$ such that $S \sim \mathbb{N}_n$.
Thus, $|S| = |\mathbb{N}_n| = n$, so the cardinality of a nonempty finite set S is a positive integer n such that $S \sim \mathbb{N}_n$.
We say that S has finite cardinal number n.
Observe that $|S| = n$ iff $S \sim \mathbb{N}_n$ iff there exists a bijective function $f : S \to \mathbb{N}_n$.
A nonempty finite S with n elements may be denoted as $S = \{s_1, s_2, ..., s_n\}$.

Example 96. The cardinality of the singleton set $\{a\}$ is 1 since $\{a\} \sim \mathbb{N}_1 = \{1\}$.

Lemma 97. *Let S be a finite set.*
If $x \in S$, then $|S - \{x\}| = |S| - 1$.

Theorem 98. *Every subset of a finite set is finite.*

Let S be a finite set.
If $T \subset S$, then T is finite.

A set that is not finite is infinite.
An infinite set has infinitely many elements.

Definition 99. infinite set
A set S is **infinite** iff S is not finite.

A set is either finite or infinite.

Theorem 100. ***characterization of infinite sets***
*A set S is **infinite** iff S is numerically equivalent to some proper subset of S.*

Therefore, a set S is infinite iff there exists some proper subset T of S such that $S \sim T$.
Therefore, a set S is infinite iff there exists T such that $T \subset S$ and $T \neq S$ and $S \sim T$.
Therefore, a set S is not infinite iff there is no proper subset T such that $S \sim T$.

Theorem 101. ***equivalent characterization of infinite sets***
*A set S is **infinite** iff there exists a function $f : S \to S$ such that f is injective, but not surjective.*

Example 102. $\mathbb{N}$ is infinite.
The set of natural numbers is infinite.

Proof. Define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(k) = 2k$ for all $k \in \mathbb{N}$.
We prove f is injective.
Let $k, m \in \mathbb{N}$ such that $f(k) = f(m)$.
Then $2k = 2m$, so $k = m$.
Therefore, f is injective.
We prove f is not surjective by contradiction.
Suppose f is surjective.
Then for each $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $f(k) = m$.
Let $m = 1$.
Then there exists $k \in \mathbb{N}$ such that $f(k) = 1$.
Thus, $2k = 1$, so $k = \frac{1}{2}$.
But, $\frac{1}{2} \notin \mathbb{N}$, so $k \notin \mathbb{N}$.
Therefore, we have $k \in \mathbb{N}$ and $k \notin \mathbb{N}$, a contradiction.
Hence, f is not surjective.
Since $f : \mathbb{N} \to \mathbb{N}$ is injective, but not surjective, then $\mathbb{N}$ is infinite. □

Let $\sim$ be the numeric equivalence relation on a collection of sets.
Since $\sim$ is an equivalence relation, then any collection of sets is partitioned into equivalence classes.
Any two sets in the same class are numerically equivalent and any two sets in different classes are numerically nonequivalent.

The cardinality of a set describes the size of a set(i.e. the number of elements in a set).

Definition 103. cardinal number of a set

Two numerically equivalent sets are said to have the **same cardinal number**.

The cardinal number of a set is simply a symbol associated with sets in an equivalence class of numerically equivalent sets.

We denote the cardinal number of a set S by $|S|$.

The cardinal number of a set measures the 'size of a set' (i.e. the number of elements in a set).

Two numerically equivalent sets are said to have the same cardinality.

Definition 104. sets with the same cardinality

A set A has the same cardinality as a set B, denoted $|A| = |B|$, iff $A \sim B$.

Therefore two sets with the same cardinality have the same cardinal number.

Therefore two sets with the same cardinality have the same size (i.e. same number of elements).

Therefore, a set A does not have the same cardinality as a set B, denoted $|A| \neq |B|$, iff $A \not\sim B$.

Therefore, two sets with different cardinality have different cardinal numbers.

Therefore, two sets with different cardinality have different sizes (i.e. different number of elements).

Let A, B be sets.

Then A and B have the same cardinality iff

the cardinal number of A equals the cardinal number of B iff

$|A| = |B|$ iff

$A \sim B$ iff

there exists a bijective function $f : A \to B$.

Example 105. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$.

Define a function $f : A \to B$ by $f(1) = c$, $f(2) = a$, $f(3) = d$, $f(4) = b$.

Then clearly f is a bijection, so $A \sim B$.

Therefore, $|A| = |B|$.

Definition 106. comparing cardinalities of sets

The cardinal number of a set A is less than or equal to the cardinal number of a set B, denoted $|A| \leq |B|$, iff there exists an injective function $f : A \to B$.

The cardinal number of a set A is strictly less than the cardinal number of a set B, denoted $|A| < |B|$, iff $|A| \leq |B|$, but $A \not\sim B$.

Let A, B be sets.

Then $A \prec B$ iff $A \preceq B$, but $A \not\sim B$

iff there exists an injective function $f : A \to B$ and $A \not\sim B$

iff there exists an injective function $f : A \to B$ and f is not bijective iff there exists an injective function $f : A \to B$, and either f is not injective or f is not surjective

iff there exists an injective function $f : A \to B$, and f is not surjective

Therefore, $|A| < |B|$ means there exists function $f : A \to B$ that is one to one, but not onto.

Definition 107. cardinality of $\mathbb{N}$

The cardinality of $\mathbb{N}$ is denoted $\aleph_0$.

Therefore, $|\mathbb{N}| = \aleph_0$.

To count the elements of a set is to establish a one to one correspondence between a set and some subset of $\mathbb{N}$.

Definition 108. countable set

A set S is **countable** iff there exists T such that $T \subset \mathbb{N}$ and $S \sim T$.

A set S is **uncountable** iff S is not countable.

Theorem 109. ***characterization of countable sets***

A set S is countable iff there exists an injective function $f : S \to \mathbb{N}$.

Example 110. $\mathbb{N}$ is countable

The set of natural numbers is countable.

Proposition 111. ***equivalent characterization of countable sets***

A set S is countable iff either S is finite or $S \sim \mathbb{N}$.

Definition 112. countably infinite set

A set S is countably infinite iff S is countable and infinite.

Example 113. $\mathbb{N}$ is countably infinite

Since $\mathbb{N}$ is countable and $\mathbb{N}$ is infinite, then $\mathbb{N}$ is countably infinite.

Theorem 114. ***characterization of countably infinite sets***

A set S is countably infinite iff $S \sim \mathbb{N}$.

Let S be a countably infinite set.

Then $S \sim \mathbb{N}$, so $|S| = |\mathbb{N}| = \aleph_0$.

Therefore, every countably infinite set has cardinality $\aleph_0$.

Since a set S is countable iff either S is finite or $S \sim \mathbb{N}$, and S is countably infinite iff $S \sim \mathbb{N}$, then a set S is countable iff either S is finite or S is countably infinite.

Example 115. $\mathbb{Z}$ is countable

$|\mathbb{Z}| = |\mathbb{N}|$.

Therefore the set of integers is countable.

Example 116. $\mathbb{Q}$ is countable

$|\mathbb{Q}| = |\mathbb{N}|$.

Therefore the set of rational numbers is countable.
Since $\mathbb{Q}$ can be arranged in an infinite sequence, then $\mathbb{Q}$ is countable.

Example 117. $\mathbb{R}$ is uncountable
$|\mathbb{R}| = |\mathscr{P}(\mathbb{N})|$.
$|\mathbb{R}| = |\mathscr{P}(\mathbb{N})| = 2^{\aleph_0}$.
Since there are no surjections onto $\mathbb{R}$ then there are no bijections $f : \mathbb{N} \to \mathbb{R}$.
Thus $|\mathbb{N}| \neq |\mathbb{R}|$. Hence, $\mathbb{R}$ is uncountable.
We know $|S| < |\mathscr{P}(S)|$ for any set S.
Thus $|\mathscr{P}(\mathbb{N})| > |\mathbb{N}|$ so $|\mathbb{R}| > |\mathbb{N}|$.
Therefore the set of real numbers is larger than the set of natural numbers.
Show that every set has a cardinality(ie a size).
even infinite sets have a cardinality.

Theorem 118. *A set is countably infinite if and only if its elements can be arranged in an infinite sequence.*

S is countably infinite iff its elements can be arranged in an infinite sequence, so $S = \{s_1, s_2, s_3, s_4, ...\}$.

Theorem 119. *The cartesian product of two countable sets is countable.*

Corollary 120. *The cartesian product of n countable sets is countable.*

Theorem 121. *The union of two countable disjoint sets is countable.*

Theorem 122. *A subset of a countable set is countable.*

Theorem 123. *A superset of an uncountable set is uncountable.*

Let S be an arbitrary set.
Let $f : S \to 2^S$ be a map.
Then for each $x \in S$, $f(x) \in 2^S$ so $f(x) \subset S$.
Either $x \in f(x)$ or $x \notin f(x)$.
Let $T = \{x \in S : x \notin f(x)\}$.
Then $T \subset S$ so $T \in 2^S$.
Suppose f is surjective.
Then $\exists a \in S$ such that $f(a) = T$.
Choose $a \in S$ such that $f(a) = T$.
Is $a \in T$?
Suppose $a \in T$.
Then $a \notin f(a)$.
Since $f(a) = T$ then $a \notin T$.
Hence we have $a \in T$ and $a \notin T$, a contradiction.
Suppose $a \notin T$.
Since $a \in S$ and $a \notin T$ then $a \in f(a)$.
Since $f(a) = T$ then $a \in T$.
Hence we have $a \notin T$ and $a \in T$, a contradiction.
Thus f cannot be surjective.
Therefore no surjective map exists from S to 2^S for any set S.
Therefore there is no bijective map $f : S \mapsto 2^S$ for any set S.

Theorem 124. ***Cantor's Theorem***

The power set of a set is strictly larger than the set itself.

Let S be a set.

Then $|S| < |\mathscr{P}(S)|$.

$\aleph_0 = |\mathbb{N}| < |\mathscr{P}(\mathbb{N})| < |\mathscr{P}(\mathscr{P}(\mathbb{N}))| < |\mathscr{P}(\mathscr{P}(\mathscr{P}(\mathbb{N})))| < ...$

Continuum Hypothesis

There is no set S such that $\aleph_0 < |S| < 2^{\aleph_0}$.

Thus, every infinite subset of the continuum either has the same cardinality as $\mathbb{Z}$ or as the continuum itself.

It is not known whether the continuum hypothesis is true or false.

Finite Sets

Theorem 125. ***Pigeonhole Principle***

Let $m, n \in \mathbb{N}$.

Let $A = \{1, 2, ..., m\}$.

Let $B = \{1, 2, ..., n\}$.

1. If $m > n$, then there is no injective function $f : A \to B$. (one hole has at least 2 pigeons flying into it)

2. If $m < n$, then there is no surjective function $f : A \to B$. (at least one pigeonhole remains empty)

Therefore, if f is one to one, then $|A| \leq |B|$.

Similarly, if f is onto, then $|A| \geq |B|$.

Thus, if f is one to one and onto, then $|A| \leq |B|$ and $|A| \geq |B|$, so $|A| = |B|$.

Therefore, if f is bijective, then $|A| = |B|$.

Let A, B be finite sets.

Suppose $\exists$ a bijection between A and B.

Since A is finite then either $A = \emptyset$ or $A \neq \emptyset$.

Suppose $A = \emptyset$.

Then $|A| = |\emptyset| = 0$.

Since a bijection exists between A and B then $|A| = |B|$.

Hence $0 = |B|$.

Thus $B = \emptyset$.

Suppose $A \neq \emptyset$.

Since A is finite then $|A| = k$ for some $k \in \mathbb{Z}^+$ and $\exists$ a one to one correspondence between A and $\{1, 2, ..., k\}$.

Thus the elements of A form a sequence $\langle a \rangle = \{a_1, a_2, ..., a_k\}$.

Let f be a bijection defined by $f(a_i) = b_i$ for all $a_i \in \langle a \rangle$.

Then we have a sequence $\langle b \rangle = \{b_1, b_2, ..., b_k\}$.

Since f is bijective then there exists a bijection between $\langle b \rangle$ and B.

Hence $|B| = k$ so $|A| = |B|$.

Proposition 126. *Let A and B be finite sets and $|A| = |B|$.*
Let $f : A \to B$ be a function.
Then f is injective iff f is surjective.

Let S be a set.

$$|S| = \begin{cases} 0 & \text{if } S = \emptyset \\ n & \text{if } |S| = |\{1, 2, 3, ...n\}| \\ \infty & \text{otherwise} \end{cases}$$

Let S be a finite set.
Then $\exists$ a bijection between S and $\mathbb{N}_k = \{1, 2, ..., k\}$ or $S = \emptyset$.
Thus $\exists$ a bijection between S and any subset of $\mathbb{N}$ so S is countable.
Therefore any finite set is countable.