Topology Notes

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Topology is the study of topological spaces.

Major ideas are: open and closed sets interior and closure neighborhood and closeness compactness and connectedness continuous functions convergence of sequences, nets, filters separation axioms countability axiom

Specialized branches: algebraic topology, geometric topology, differential topology

Point Set Topology

If one geometric object can be continuously transformed into another, then the two objects are topologically the same.

Example: A circle and square are topologically equivalent.

A figure eight curve is not topologically the same as a circle. When a point is removed from a circle, a single connected arc remains. However, when the point of contact of the two circles of a figure eight is removed, then two distinct connected pieces remain.

Two geometric objects that are topologically equivalent are **homeomorphic**.

Example: A circle and square are homeomorphic.

Place circle C inside square S with the same center point. Define function $f : C \mapsto S$ by project the circle radially outward to the square. Then f is continuous and f is invertible. Therefore $f^{-1} : S \mapsto C$ is the inverse of f and projects the square radially inward to the circle.

Hence, f is a homeomorphism between C and S.

Definition 1. Open Set

Let $S \subset \mathbb{R}$.

S is an **open subset of** $\mathbb{R}$ iff to each point of S there corresponds an interval (a, b) that contains x and is contained in S.

Therefore S is open iff $(\forall x \in S)(\exists (a, b))[x \in (a, b) \wedge (a, b) \subset S]$.

Examples: Every open interval is an open set.

$\mathbb{R}$ is an open set.

half open intervals (a, ∞) and $-\infty, a)$ are open.

The complement of a finite set in $\mathbb{R}$ is open. (we should prove all of these statements!)

If S is the union of the infinite sequence $x_n = 1/n, n = 1, 2, ...,$ together with its limit 0, then $\mathbb{R} - S$ is open.

The union of open intervals is an open set.

Every open set is a union of open intervals. (because for each $x \in S$, there exists an open interval (a_x, b_x) with $x \in (a_x, b_x) \subset S$ and S is the union of all these intervals (a_x, b_x).

The empty set ϕ is open. (vacuously true because there are no points x where the condition could fail to hold)

Examples of set that are not open:

A closed interval $[a, b]$ is not open.(because there is no open interval about either a or b that is contained in $[a, b]$.

half-open intervals $[a, b)$ and $(a, b]$ are not open sets when $a < b$.

A nonempty finite set is not open.

Definition 2. Continuity at a point

Let $A \subset \mathbb{R}$.

Let $f : A \mapsto \mathbb{R}$ be a function.

Let $a \in A$.

Then f **is continuous at** a iff for every positive number ϵ, there exists a positive δ such that, for all $x \in A \cap (a - \delta, a + \delta), |f(x) - f(a)| < \epsilon$.

Definition 3. Continuous Function

Let $f : A \mapsto \mathbb{R}$ be a function.

Then f is **continuous** iff for each open set S in $\mathbb{R}$, the preimage of S is an open set.

The preimage(inverse image) of S in $\mathbb{R}$ is $f^{-1}(S) = \{x \in A : f(x) \in S\}$.

Let A, B be topological spaces. Let $f : A \mapsto B$ be a function.

Then f is **continuous** iff the inverse image of every open set is open.

Therefore f is not continuous iff the inverse image of some open set is not open.

Definition 4. Homeomorphism

Let A, B be topological spaces.

Let $f : A \mapsto B$ be a bijective map such that

1. f is continuous
2. f^{-1} is continuous

Then f is a **homeomorphism**.

Therefore a homeomorphism is a bijection that is continuous and whose inverse map is also continuous.

Homeomorphic topological spaces are essentially the same(topologically equivalent).

Two topological spaces are homeomorphic iff there exists a homeomorphism between them.

A homeomorphism preserves topological properties.

Definition 5. topology on a set

Let X be a nonempty set.

A **topology** τ is a collection of subsets of X such that the following axioms hold:

T1. $\emptyset \in \tau$ and $X \in \tau$.

T2. The union of any collection of sets in τ is in τ. (τ is closed under arbitrary union)

T3. The intersection of any finite collection of sets in τ is in τ. (τ is closed under finite intersection)

T1 means both the empty set $\emptyset$ and X are open sets.

T2 means the union of open sets is an open set.

T3 means the finite intersection of open sets is an open set.

Definition 6. topological space

A **topological space**(X, τ) is a nonempty set X with a topology τ defined on X.

Let (X, τ) be a topological space.

X is the underlying(ground) set.

The elements of X are the **points** of the topological space.

The elements of τ are subsets of X and τ is a collection of open sets of X that satisfy the axioms above.

Therefore, $\tau = \{S : S \subset X\}$ and τ satisfies the axioms $T1, T2, T3$.

Let $S \in \tau$.

Then $S \subset X$ and S is an **open set** of X.

The complement of S in X is a **closed set**.

Therefore $\overline{S}_X = \{x \in X : x \notin S\} = X - S$, so $X - S$ is a closed set.

A subset of X may be either

1. open
2. closed
3. both open and closed
4. neither open nor closed

Let X be a nonempty set.

Example 7. trivial topology

The smallest,coarsest topology is the collection $\tau_0 = \{\emptyset, X\}$.

Example 8. discrete topology

The largest, finest topology is the power set of X.

Example 9. Let $X = \{a, b, c\}$.

Then $\tau_1 = \{\emptyset, \{a\}, \{a, c\}, X\}$ is a topology of X.